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THREE DIMENSIONAL HYPERBOLIC SPACES

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1. Introduction. Euclidean 3-space is a useful local model of the physical world. For two millennia from Euclid to Saccheri, mathematicians tried to prove that Euclidean geometry was the only consistent geometry of space. The effort continued until early in the 19th century when Bolyai, Lobachevsky, and Gauss independently investigated hyperbolic geometry. Later Riemann recognized spherical geometry as another non-Euclidean geometry and developed Riemannian geometry. By 1900, the axiomatic method and the role of a model in geometry were reasonably well understood. This viewpoint was spread by Hilbert's *Foundations of Geometry* and played an important part in the development of Einstein's theory of relativity. (See **[2]**, **[4]**, and **[11]**.) A homogeneous and isotropic 3-space with a curvature or scale factor which varies with time is a model for the large-scale spatial universe.

The use of the hyperbolic plane to study Riemann surfaces and Fuchsian groups was initiated by Poincaré and remains an active research area. Three dimensional spaces of constant negative curvature are less familiar. The recent work of W. P. Thurston **[16]** and others in this area is likely to have considerable impact in 3-manifold theory. In this paper we intend to give an account of some of the geometrical properties of negatively curved spaces. A complete Riemannian manifold of constant curvature is the quotient of a simply-connected manifold under the action of a discrete, fixed-point free, group of isometries. We will discuss several models for the covering space, the action of the isometries, volumes, some theorems about the quotients, and some examples.

2. Hyperbolic models and volume. The basic facts of riemannian geometry will be assumed. Let $\mathbf{R}^n$ denote Euclidean n-space and let $|x|^2 = x_1^2 + \cdots + x_n^2$. There are several useful models for hyperbolic n-space of curvature $-K < 0$. (See Wolf **[19]**.) The Poincaré disk model is $B^n(K) = \{x \in \mathbf{R}^n \colon |x|^2 < 1/K\}$ with the metric

$$\frac{4 \sum_{i=1}^{n} dx_i^2}{(1 - K|x|^2)^2}.$$

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The Poincaré upper half space model is $H^n(K) = \{x \in \mathbf{R}^n\colon x_n > 0\}$ with the metric

$$\frac{\sum_{i=1}^{n} dx_i^2}{Kx_n^2}.$$

These metrics are conformally equivalent to the Euclidean metric. That is, the inner product on the tangent space at a given point is a constant multiple of the Euclidean inner product. So angles in these models agree with Euclidean angles. Hyperbolic length is, of course, quite different from Euclidean length. Each point of the sphere bounding $B^n(K)$ can be considered as a point at infinity for $B^n(K)$, with a similar statement for $H^n(K)$.

The geodesics for $B^2(K)$ are arcs of circles meeting the boundary circle orthogonally or straight lines through the origin. The geodesics for $H^2(K)$ are semi-circles with center in $\mathbf{R} = \{(x_1, x_2) \in \mathbf{R}^2\colon x_2 = 0\}$ or half-lines perpendicular to $\mathbf{R}$. Euclid's parallel postulate is replaced by the property that given a geodesic ℓ and a point $x \notin \ell$, there are infinitely many geodesics through x not meeting ℓ. Hyperbolic space is not compact and geodesics have infinite length.

A third model for hyperbolic space is

$$\{(x_0, x_1, \ldots, x_n)\colon x_i \in \mathbf{R},\ -x_0^2 + \sum_{i=1}^{n} x_i^2 = -K\}$$

with metric induced by the pseudo-riemannian metric $-dx_0^2 + \sum_{i=1}^{n} dx_i^2$. This model might be regarded as a sphere with imaginary radius $i\sqrt{K}$.

A ball centered at the origin in $B^n(K)$ with Euclidean radius s has hyperbolic radius r with

$$r = \int_0^s \frac{2}{1 - Kt^2}\,dt = \frac{1}{\sqrt{K}} \log\left(\frac{1 + \sqrt{K}\,s}{1 - \sqrt{K}\,s}\right).$$

so that

$$s = \frac{1}{\sqrt{K}} \tanh\left(\frac{\sqrt{K}\,r}{2}\right).$$

The volume element in $B^3(K)$ is

$$dV = \frac{8}{(1 - K|x|^2)^3}\,dx_1 dx_2 dx_3$$

so the volume of a ball of radius r is

$$V(r) = \int_0^s \frac{4\pi t^2}{(1 - Kt^2)^3}\,dt = 2\pi K^{-3/2}\left[\frac{1}{2}\sinh(2\sqrt{K}\,r) - \sqrt{K}\,r\right].$$

This formula and others were obtained by Taurinus [2] in about 1830 by replacing the real radius $\sqrt{K}$ in the formulas for spherical trigonometry with the imaginary radius $i\sqrt{K}$.

As K approaches 0, the volume of a ball approaches Euclidean volume. As K gets large, the volume of a ball of a given radius goes to infinity. A tourist from Euclidean space visiting a hyperbolic space would find that his immediate neighborhood has a rather large volume and that the scenery changes rapidly as he moves about.

3. Isometry groups. If F is a subfield or a subring of the complex numbers $\mathbf{C}$, then PSL(2, F) denotes the projective special linear group of 2×2 matrices with entries from F. That is,

$$\mathrm{PSL}(2,\ F) = \left\{ \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\ b,\ c,\ d \in F,\ ad - bc = 1 \right\}$$

A subgroup of PSL(2, $\mathbf{C}$) is discrete if no sequence of distinct elements converges in the usual topology to an element of PSL(2, $\mathbf{C}$). If F is discrete as a subset of $\mathbf{C}$, then PSL(2, F) and all of its subgroups are discrete groups.

We will deal only with orientation-preserving isometries. The isometries of the hyperbolic plane are the linear fractional transformations of the complex plane which leave (one of the models of) the hyperbolic plane invariant. For the upper half-plane model these are of the form $z \mapsto (az + b)/(cz + d)$ where $a, b, c, d \in \mathbf{R}$ and $ad - bc = 1$. Since composition of these mappings agrees with matrix multiplication, the isometry group is the Lie group PSL(2, $\mathbf{R}$). A Fuchsian group is a discrete group of isometries of the hyperbolic plane. The quotient space is a Riemann surface.

By a theorem of Liouville, the only conformal mappings of a domain in $\mathbf{R}^n$ for $n \geqq 3$ are Moebius transformations, i.e., products of translations, rotations, inversions in spheres, reflections in hyperplanes, and dilatations. It follows that the isometry group of $H^3(K)$ is the Lie group PSL(2, $\mathbf{C}$).

To understand this action we write $A(z) = (az + b)/(cz + d)$ where $a, b, c, d \in \mathbf{C}$ and $ad - bc = 1$ as

$$A(z) = \frac{a}{c} + \left(\frac{1}{|c|}\right)^2 T_c(z + d/c)^*$$

where $T_c(z) = (-\bar{c}/c)\bar{z}$ and $w^* = w/|w|^2$. If $x = (x_1,\ x_2,\ x_3)$ and $y = x/|x|^2$, then a tedious but elementary calculation shows that

$$\sum_{i=1}^{3} dy_i^2/y_3^2 = \sum_{i=1}^{3} dx_i^2/x_3^2$$

where

$$dy_i = \sum_{j=1}^{3} \frac{\partial y_i}{\partial x_j} dx_j.$$

That is, this inversion in a sphere is a hyperbolic isometry. It is convenient to write a point of $H^3(K)$ as $z + tj$ where $j = (0, 0, 1)$ and z is a complex number. Defining $T_c(z + tj) = T_c(z) + tj$ and $(z + tj)^* = (z + tj)/|z + tj|^2$ then extends A to an isometry of $H^3(K)$ by replacing z with $z + tj$. Furthermore, A consists of inversion in the sphere $|(z + tj) + d/c| = 1/|c|$, a rotation of $\mathbf{R}^3$ leaving $\mathbf{C}$ invariant and fixing $-d/c$, a reflection in a plane perpendicular to $\mathbf{C}$ through $-d/c$, and a translation of $-d/c$ to a/c. The sphere $|(z + tj) + d/c| = 1/|c|$ is usually called the isometric sphere of A. The isometric sphere of A^{-1} is given by $|(z + tj) - a/c| = 1/|c|$. The action of A takes the exterior (interior) of the isometric sphere of A to the interior (exterior) of the isometric sphere of A^{-1}.

Since similarity mappings of hyperbolic space are also isometries, two figures are similar if and only if they are congruent. In particular, the lengths of the sides of a triangle are determined by its angles.

An isometry of H^3 without a fixed point in H^3 has either one or two fixed points on $\partial H^3 = \mathbf{C} \cup \{\infty\}$. These two cases can be distinguished by the trace $a + d$ of the matrix. An isometry with two fixed points is called either (i) hyperbolic or (ii) loxodromic and is characterized respectively by (i) $a + d$ is real and $|a + d| > 2$ or (ii) the imaginary part of $a + d$ is not zero. An isometry with one fixed point is called parabolic and is characterized by $a + d = \pm 2$. A hyperbolic or loxodromic transformation is conjugate in PSL(2, $\mathbf{C}$) to a map $z \mapsto \rho z$ where $\rho \in \mathbf{C}$ and $|\rho| \neq 0, 1$. A parabolic transformation is conjugate to the translation $z \mapsto z + 1$. (See [5] or [9].)

An elementary argument with fixed points shows that two parabolic, hyperbolic, or loxodromic transformations commute if and only if they have the same fixed points.

4. Ideal polyhedra. In the hyperbolic plane the Gauss-Bonnet Theorem gives the area of a convex polygon with geodesic sides in terms of the angles at the vertices. Since hyperbolic surfaces can be obtained by glueing edges of convex polygons in pairs, this also gives formulas for the area of hyperbolic surfaces [9]. In a similar way, hyperbolic 3-manifolds can be realized by glueing sides of convex polyhedra together in pairs. If all of the vertices of the polyhedron are at infinity, it will be called an ideal polyhedron. Note that at infinity in $H^3(K)$ means either the usual point at infinity or a point in $\mathbf{C}$.

The totally geodesic surfaces in $H^3(K)$ are half-planes perpendicular to **C** and hemispheres with center in **C**. Ideal polyhedra with totally geodesic sides are shown in Figure 1. The volume element in $H^3(K)$ is

$$dV = K^{-3/2} \frac{dxdydt}{t^3}.$$

In Figure 1 the dihedral angles between the vertical planes and the hemispheres at the bottom are $\pi/4$. The volume of one of these polyhedra is

$$K^{-3/2} \int_{-1/2}^{1/2} dx \int_{-1/2}^{1/2} dy \int_{f(x,y)}^{\infty} t^{-3}\, dt,$$

where $f(x, y) = (1/2 - x^2 - y^2)^{1/2}$. After some calculations, this reduces to $2K^{-3/2} \int_0^{\pi/4} \log \cot u\, du$.

A systematic approach to the volume of ideal polyhedra is developed by J. Milnor in [**16**]. Bolyai and Lobachevsky had both developed formulas for the volume of tetrahedra in hyperbolic space [**2**]. Lobachevsky's formulas for hyperbolic volume were expressed in terms of the function $\int_0^\theta \log \sec u\, du$. Milnor gives the name Lobachevsky function to $L(\theta) = -\int_0^\theta \log|2 \sin t|dt$ and develops its remarkable properties. Its relationship to volumes is given in the following theorem of Milnor.

THEOREM 1. *Consider an infinite hyperbolic cone with base an n-gon on a hemisphere and with all vertices at infinity. Let $\alpha_1, \ldots, \alpha_n$ be the dihedral angles between the vertical planes and the hemisphere. Then*

(i) $\sum_{i=1}^n \alpha_i = \pi$, *and*

(ii) *The volume of this ideal polyhedron is* $K^{-3/2} \sum_{i=1}^n L(\alpha_i)$.

The proof is by induction after establishing the result for $n = 3$. An ideal tetrahedron can be subdivided into tetrahedra with three right dihedral angles. The integral of dV over each piece can be written in terms of L. Using the identities satisfied by the Lobachevsky function then gives the theorem. Note that as K approaches zero, the volume of the polyhedron approaches infinity, while as K approaches infinity, the volume approaches zero. The same will be true of a hyperbolic manifold obtained by glueing the sides of finitely many such polyhedra together in pairs.

5. Quotient spaces. Henceforth, take the curvature to be -1 and delete K from the notation. Let G be a discrete, fixed-point free, group of isometries of H^3. The quotient space H^3/G under the action of G is also a space of constant curvature, locally isometric to H^3, and it is geodesically complete. Its fundamental group is isomorphic to G and H^3 is its universal covering space. (See Wolf [**19**].) We discuss some theorems for such spaces.

THEOREM 2. (Mostow). *Suppose H^n/G_1 and H^n/G_2 have finite volume,*

G_1 and G_2 are isomorphic, and $n \geqq 3$. Then there is an isometry ϕ: $H^n/G_1 \to H^n/G_2$.

This result, known as Mostow's Rigidity Theorem, says that the abstract group determines the topology and geometry of the manifold in the finite volume case. This is in contrast to fuchsian groups, which have non-trivial deformation spaces. The deformation spaces are manifolds whose points represent Riemann surfaces of the same topological type but which are not necessarily isometric to each ohter. Riemann surfaces are the exception rather than the rule in this regard as is shown in Mostow **[12]**.

The next theorem says that discrete subgroups of PSL(2, **C**) are in some way uniformly discrete. It also has implications for the topology of the quotient manifold.

THEOREM 3. (Jørgensen **[8]**). *Suppose transformations A and B in* PSL (2, **C**) *generate a discrete, torsion-free group and A and B do not have the same fixed points. Then*

$$|\tau(A)^2 - 4| + |\tau(ABA^{-1}B^{-1}) - 2| \geqq 1$$

where τ is the trace function.

A horoball B in H^3 is a Euclidean ball tangent to **C** at a point or a region $\{z + tj: t > t_0\}$ for some fixed $t_0 > 0$. (The latter region is tangent to ∂H^3 at infinity.) A horoball is invariant under the action of a group of parabolic transformations with fixed point at the point of tangency.

A consequence of Theorem 3 is that there is a horoball H at each parabolic fixed point x which is precisely invariant under the stabilizer $G_x = \{A \in G: A(x) = x\}$ of the point. That is, $A(H) = H$ for $A \in G_x$ and $A(H) \cap H = \varnothing$ for $A \in G - G_x$. So the image of H in the quotient space is determined by G_x. To see this, take the parabolic fixed point to be infinity and suppose it is fixed by $A = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. If B does not fix infinity, then $B = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ with $c \neq 0$. The trace of A is 2 and the trace of $ABA^{-}B^{1-1}$ is $c^2 + 2$; so the inequality says that $|c|^2 \geqq 1$. The radius of the isometric spheres of B and B^{-1} is $1/|c|$, which is less than or equal to 1. Consequently, $\{z + tj: t > 1\}$ is precisely invariant under the stabilizer of infinity.

Consider a discrete parabolic group which is free abelian on two generators. The quotient of the horoball is homeomorphic to $S^1 \times \{z: 0 < |z| < 1\}$ where S^1 is a circle. This can be seen by considering the "infinite chimney" where $t > t_0$ in one of the ideal polyhedra in Figure 1. If the four sides of the chimney are equivalent in pairs by two translations which fix infinity, then each horizontal cross-section projects into the quotient space as a square with opposite edges identified, i.e., a torus. These tori can be viewed as fitting continuously inside of one another to make a solid torus with $S^1 \times \{0\}$ deleted.

THEOREM 4. [**16, 17**]. *If H^3/G has finite volume, then H^3/G is homeomorphic to the complement of finitely many disjoint simple closed curves in a closed 3-manifold.*

The idea of the proof in [**17**] is as follows. If H^3/G has finite volume but is not compact, then there is a nested sequence of horoballs at a point of ∂H^3 such that the corresponding sequence of volumes of their images in H^3/G approaches zero. Using Jørgensen's Inequality, it can be shown that the point of tangency of the horoballs is a parabolic fixed point. Now there is a precisely invariant horoball at this point; so its stabilizer is free abelian on two generators. By considering a fundamental polyhedron for the action of G, there can be only finitely many conjugacy classes of such parabolic fixed points.

The proof in [**16**] is similar with a more general property for certain discrete groups called the Margulis Lemma playing the role of Jørgensen's Inequality. In any case the problem is to associate the non-compact portions of the manifold with a horoball modulo a parabolic group. This is how the complement of finitely many disjoint closed curves arises. In general, the curves are knotted or linked with one another.

A disjoint union L of piecewise linear simple closed curves in a 3-sphere S^3 is called a (tame) link and $S^3 - L$ is a link complement. There is a standard way of presenting the fundamental group $\pi_1(S^3 - L)$ with meridians and longitudes where these correspond to loops which go once around a component of the link or run along parallel to it, respectively.

THEOREM 5. (Riley [**14**]). If *H^3/G has finite volume and G is isomorphic to $\pi_1(S^3 - L)$ for some link L with the meridians corresponding to parabolic transformations in G, then H^3/G is homeomorphic to $S^3 - L$.*

The next theorem is another way of saying that discrete subgroups of PSL(2, **C**) are uniformly discrete. It also says there is a uniform lower bound for the volume of hyperbolic manifolds.

THEOREM 6. [**17**]. *There exists $\rho > 0$ such that H^3/G contains a region isometric to a ball of radius greater than or equal to ρ for all discrete torsion-free subgroups of* PSL(2, **C**).

In fact there is a smallest volume. The following results are given in [**16**] as corollaries of theorems of Jørgensen and Gromov.

THEOREM 7. *The set of volumes of hyperbolic 3-manifolds is well-ordered. The volume is a finite-to-one function of hyperbolic manifolds.*

We note that W. Thurston has announced profound existence theorems characterizing 3-manifolds which have a hyperbolic structure. Approximately, they say that a 3-manifold which could have a hyperbolic structure does have a hyperbolic structure. Some examples of topological reasons why a 3-manifold could not have a hyperbolic structure would be if its fundamental group was finite, or had free abelian subgroups of rank greater than two, or had a non-trivial center. The reader is referred to **[16]** for details and further developments.

Finally, if G also acts discontinuously somewhere in **C**, then G is called a kleinian group. In this case the volume of H^3/G is infinite and the theory is somewhat different than in the finite volume case. This area was also investigated by Poincaré and has again been active since about 1964. See **[5]**, **[6]**, **[9]**, **[16]**.

6. Examples. One of the first nice examples was given by Gieseking **[10]** though it was not immediately recognized as (containing) the figure-eight knot group. This example was rediscovered algebraically by Riley **[13]** and geometrically by Thurston **[16]**. Many other examples have been given by Best **[1]**, Jørgensen **[7]**, Riley **[14, 15]**, Thurston **[16]** and Wielenberg **[18]**. A hyperbolic structure on the complement of the Whitehead link is described here.

Let $Z(i) = \{m + ni : m \text{ and } n \text{ are integers}\}$, the ring of Gaussian integers. A subgroup of the Picard group PSL(2, $Z(i)$) is discrete and a finite-index subgroup will act on H^3 with a finite-volume quotient.

The union P of the two polyhedra in Figure 1 is a fundamental polyhedron for a subgroup G of the Picard group. That is, the images of P under G tesselate H^3 in the sense that their union is H^3 and they intersect each other only in their sides. The generators of G are

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, u = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix} \text{ and } a = \begin{pmatrix} 1 & 0 \\ -1+i & 1 \end{pmatrix}.$$

The first two generators act as commuting translations which fix infinity. The third is a parabolic mapping which consists of an inversion in the sphere $|(z + tj) - (1 + i)/2| = 1/\sqrt{2}$, a reflection across a plane through $(1 + i)/2$ and perpendicular to the line between the centers of the spheres, and a translation of $(1 + i)/2$ to $(-1 - i)/2$. The sides of P are equivalent in pairs under the action of (products of) the generators of G. The edges of P also fall into equivalence classes called edge cycles. The dihedral angles for an edge cycle sum to 2π and the ideal vertices are parabolic fixed points of G. The quotient space H^3/G is obtained by glueing corresponding sides of P. By Milnor's Theorem, the volume of H^3/G is $8L(\pi/4) = 3.66386 \ldots$.

A presentation for G can be read from the edge cycles and G can be shown to be isomorphic (See [18]) to $\pi_1(S^3 - W)$ where W is the Whitehead link shown in Figure 3. So H^3/G is homeomorphic to $S^3 - W$. A way to see this directly is due to Thurston and is indicated in Figures 2 and 3. The image under a of the polyhedron on the right in Figure 1 attaches along the hemisphere on the bottom of the polyhedron on the left to form an ideal regular octahedron in H^3. The top and bottom halves of the octahedron are shown in Figure 2 with the lateral sides labeled in pairs and with cycles of edges labeled with one, two, or three-headed arrows. The projection of the sides into the quotient is a 2-complex which spans the Whitehead link. The 2-complex has four 2-cells and three 1-cells and is shown in two pieces in Figure 3. The edge cycles of the octahedron correspond to the arrows, with the 3-headed arrow hidden where one component of the link crosses itself in the center. At the center over-crossing in Figure 3(a) and at each of the overcrossings in Figure 3(b), the surface has a "twist" as it passes through. One can think of the components of the deleted link as being produced by the deleted vertices of the octahedron.

It is important to note that the group determines the link complement but not the link itself. This is true because different links may have homeomorphic complements. For example, consider an open disk which spans the circular component of W, lies over the over-crossing in the second component, and is twice-punctured by the second component. Cutting along this disk, making finitely many full rotations, and glueing it back is a homeomorphism of link complements. There might be no self-homeomorphism of S^3 which takes W to the other link since the second component in the other link may be knotted. So the two links themselves need not be equivalent.

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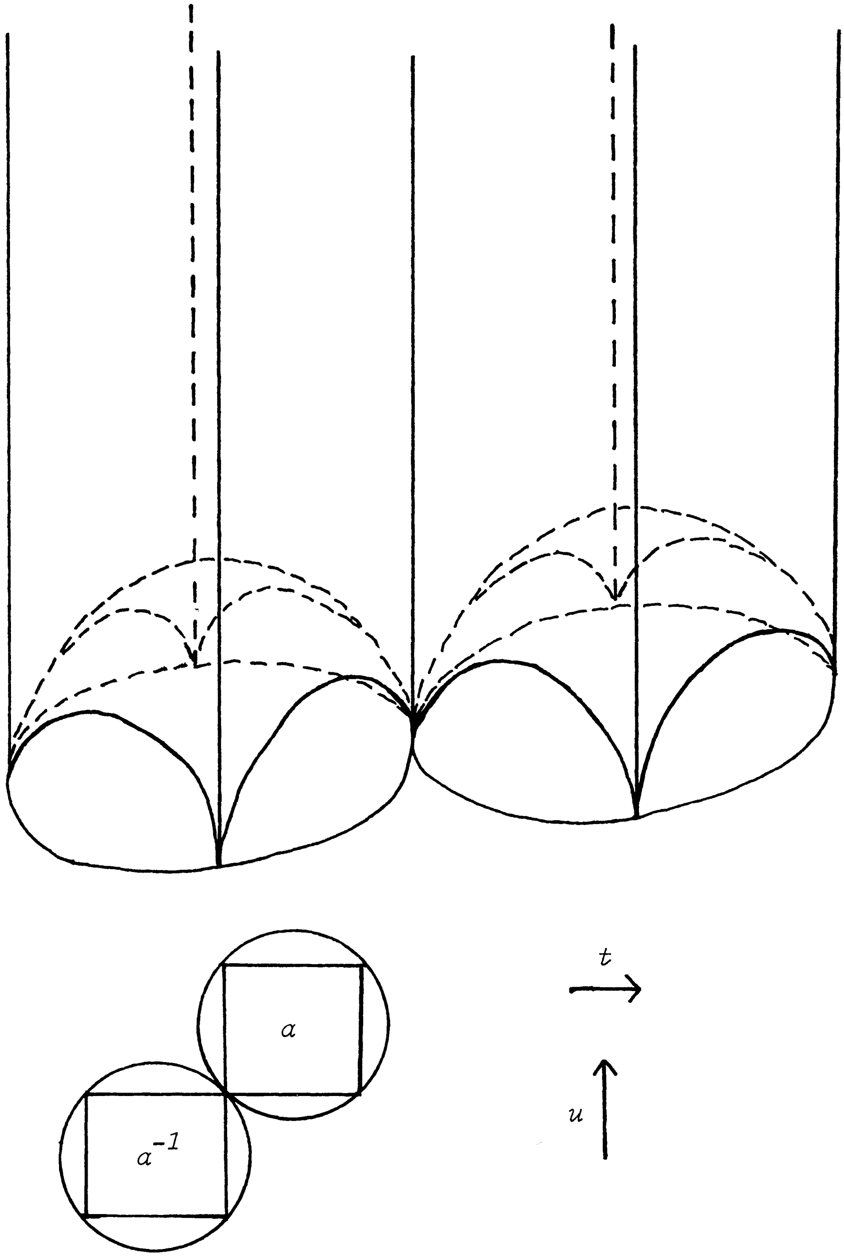

Figure 1. Side view and top view of a fundamental polyhedron.

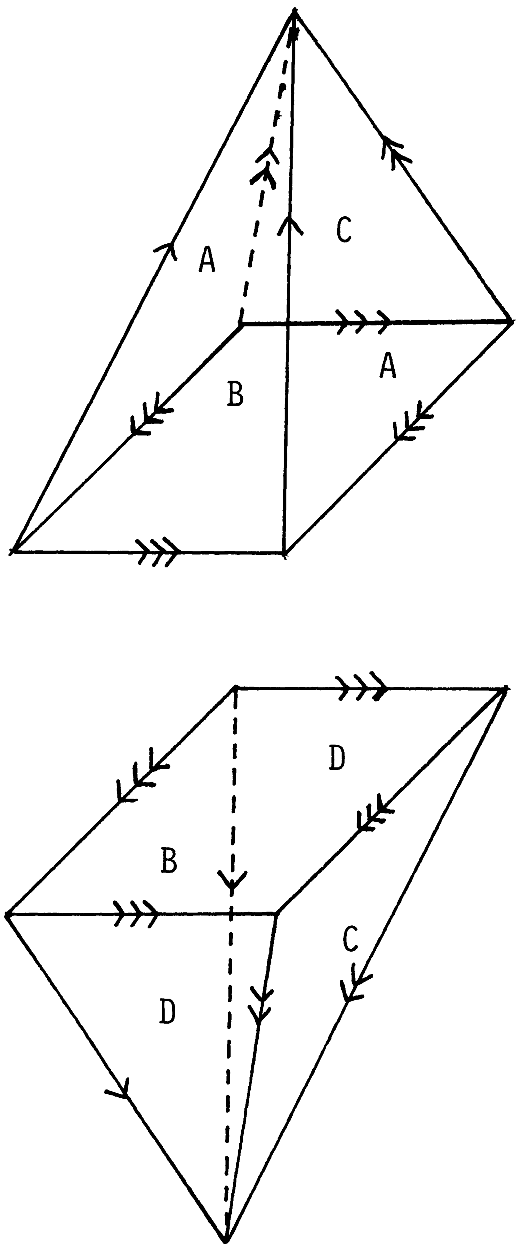

Figure 2. Halves of octahedron with side and edge identifications indicated.

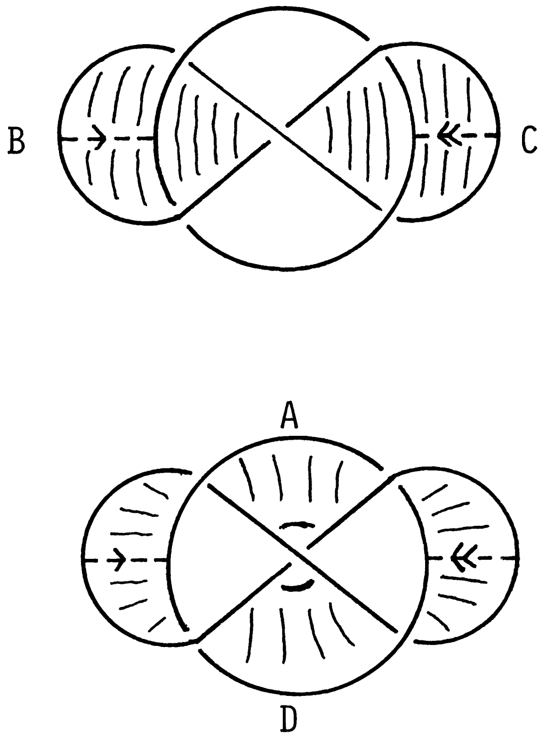

Figure 3. Whitehead link with spanning complex.