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ON GENERALIZED MID-POINT CONVEXITY

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ABSTRACT. A function f is said to be convex with respect to a T-system $\{u_0, u_1, \ldots, u_{n-1}\}$ on the interval (a, b) if

$$U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1}, f \\ t_0, t_1, \ldots, t_{n-1}, t_n \end{bmatrix} \geqq 0$$

for all $a < t_0 < t_1 < \cdots < t_n < b$. It is shown that if f is bounded and if the above inequality is satisfied for equally spaced points, then f is convex.

Introduction and background. A function f is said to be mid-point convex if the inequality

$$f((x_1 + x_2)/2) \leqq (1/2)(f(x_1) + f(x_2))$$

holds for every pair of real points x_1 and x_2 belonging to the interval of definition of the function f. In **[1]** Blumberg shows that every bounded mid-point convex function is continuous. Later, Popoviciu **[4]** noticed that every generalized mid-point convex function with respect to $1, x, x^2, \ldots, x^{n-1}$ (see equivalent definition 2), is convex with respect to these functions, if f and its n-th divided differences are bounded. We generalize these results, showing that every bounded, generalized mid-point convex function with respect to a continuous Tchebycheff-system (T-system) is convex with respect to this T-system.

2. Generalized mid-point convexity. Let $u_0, u_1, \ldots, u_{n-1}$ be n continuous functions on the closed interval $[a, b]$. We say that they form a (continuous) T-system on $[a, b]$ if for every choice of n points: $a \leqq t_0 < t_1 < \cdots < t_{n-1} \leqq b$, the determinant

$$U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1} \\ t_0, t_1, \ldots, t_{n-1} \end{bmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \ldots & u_0(t_{n-1}) \\ u_1(t_0) & u_1(t_1) & \ldots & u_1(t_{n-1}) \\ \vdots & \vdots & & \vdots \\ u_{n-1}(t_0) & u_{n-1}(t_1) & \ldots & u_{n-1}(t_{n-1}) \end{vmatrix}$$

is positive.

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DEFINITION 1.[2]. A function f, defined on the interval I, is said to be convex (with respect to the T-system $\{u_i\}_{i=0}^{n-1}$) if the determinants

$$(1)\qquad U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1}, f \\ t_0, t_1, \ldots, t_{n-1}, t_n \end{bmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \ldots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \ldots & u_1(t_n) \\ \vdots & \vdots & & \vdots \\ u_{n-1}(t_0) & u_{n-1}(t_1) & \ldots & u_{n-1}(t_n) \\ f(t_0) & f(t_1) & \ldots & f(t_n) \end{vmatrix}$$

are non-negative whenever $t_0 < t_1 < \cdots < t_n$ are $n + 1$ points in the interval I.

The set of all convex functions is called the convexity cone and is denoted by $C_I(u_0, u_1, \ldots, u_{n-1})$ (or $C(u_0, u_1, \ldots, u_{n-1})$ if no ambiguity arises).

DEFINITION 2. A function f, defined on an interval I is said to be generalized mid-point convex (GMPC) (with respect to the T-system $\{u_0, u_1, \ldots, u_{n-1}\}$) if the determinants (1) are non-negative whenever $t_j = t_0 + jh$ $j = 1, 2, \ldots, n-1, n$ and $h > 0$, and both t_0 and $t_0 + nh$ are in I. In this case we denote the determinants (1) by $U(t_0, t_n; f)$.

LEMMA 1. *Let $\{u_i\}_{i=0}^{n-1}$ be continuous functions on $[a, b]$, forming a T-system on it, and let f be defined on (a, b). If*

$$(2)\qquad U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1}, f \\ t_0, t_1, \ldots, t_{n-1}, t_n \end{bmatrix} \geqq 0$$

and

$$(3)\qquad U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1}, f \\ t_1, t_2, \ldots, t_n, t_{n+1} \end{bmatrix} \geqq 0$$

with $a < t_0 < t_1 < \cdots < t_n < t_{n+1} < b$, then for every $1 \leqq i \leqq n$,

$$(4)\qquad U\begin{bmatrix} u_0, \ldots & \ldots, u_{n-1}, f \\ t_0, \ldots, t_{i-1}, t_{i+1}, & \ldots, t_{n+1} \end{bmatrix} \geqq 0.$$

PROOF. Assume that (4) does not hold for some i, i.e.,

$$(5)\qquad U\begin{bmatrix} u_0, \ldots & \ldots, u_{n-1}, f \\ t_0, \ldots, t_{i-1}, t_{i+1}, & \ldots, t_{n+1} \end{bmatrix} < 0$$

Let u be the element of $\operatorname{span}\{u_0, u_1, \ldots, u_{n-1}\}$ that interpolates f at t_j for $j = 0, 1, \ldots, i-1, i+1, \ldots, n$. The last row in the determinant

$$(5')\qquad U\begin{bmatrix} u_0, \ldots & \ldots, u_{n-1}, f - u \\ t_0, \ldots, t_{i-1}, t_{i+1}, & \ldots, t_{n+1} \end{bmatrix}$$

is $(0, 0, \ldots, 0, (f-u)(t_{n+1}))$, and since the determinant is negative, $(f-u)(t_{n+1}) < 0$. But $(f-u)(t_i) \neq 0$ since otherwise the determinant (3) would be negative. Consider now the determinant

$$(2') \qquad U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1}, f-u \\ t_0, t_1, \ldots, t_{n-1}, t_n \end{bmatrix} \geqq 0.$$

The only non-zero term in the last row of (2') is $(f-u)(t_i)$ and since (2') is non-negative, $(-1)^{n+i}(f-u)(t_i) > 0$.

Finally,

$$(3') \qquad U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1}, f-u \\ t_1, t_2, \ldots, t_n, t_{n+1} \end{bmatrix} = (-1)^{n+i-1}(f-u)(t_i)U_i + (f-u)(t_{n+1})U_{n+1}.$$

Since both U_i and U_{n+1} are positive, (3') would be negative and thus (3) would be contradicted. Hence (4) holds for every i.

DEFINITION 3. For $\alpha < \beta$, two points in the interval (a, b), $R(\alpha, \beta)$ denotes the set $\{x; a < x < b \text{ and } (x-\alpha)/(\beta-\alpha) \text{ is a rational number}\}$.

COROLLARY 1. *Let f be a* GMPC *with respect to the T-system* $\{u_i\}_{i=0}^{n-1}$. *Then*

$$U\begin{bmatrix} u_0, u_1, \ldots, u_{n-1}, f \\ t_0, t_1, \ldots, t_{n-1}, t_n \end{bmatrix}$$

is non-negative whenever $t_0 < t_1 < \cdots < t_n$ are elements of $R(\alpha, \beta)$, for some α, β.

LEMMA 2. *Let* $\{u_i\}_{i=0}^{n-1}$ *be as in Lemma* 1. *If f is a bounded* GMPC *function with respect to this T-system, then f is continuous in (a, b).*

PROOF. Let M be a common bound for $|u_0|, |u_1|, \ldots, |u_{n-1}|$ and $|f|$ (in (a, b)). Assume that f has a discontinuity at some point x, $a < x < b$. There exists a sequence $\{t_k\}$, converging to x and such that $\lim_{k\to\infty} f(t_k)$ exists and is not equal to $f(x)$. We discuss the case $t_k \to x$ from the left and $f(t_k) > f(x)$, i.e., $f(t_k) = f(x) + h_k$, $h_k > 0$ and $\lim_{k\to\infty} h_k = h > 0$. For other cases, the proof follows a similar line.

Let $a < x_0 < x_1 < \cdots < x_{n-2} < x < b$. We may assume that $x_{n-2} < t_1 < t_2 < \cdots < x$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

(i) $x_i < x_{i+1} - \delta$, $i = 0, 1, \ldots, n-3$,
(ii) $x_{n-2} < x - \delta$, and
(iii) for every choice of n points $y_i \in (x_i - \delta, x_i)$ for $i = 0, 1, \ldots, n-2$ and $y_{n-1} \in (x - \delta, x)$, the difference

$$\left| U\begin{bmatrix} u_0, & u_1, & \ldots, & u_{n-1} \\ y_0, & y_1, & \ldots, & y_{n-1} \end{bmatrix} - U\begin{bmatrix} u_0, & u_1, & \ldots, & u_{n-2}, & u_{n-1} \\ x_0, & x_1, & \ldots, & x_{n-2}, & x \end{bmatrix} \right|$$

is less than ε. Now, for every k, let $x_0^k < x_1^k < \cdots < x_{n-2}^k < x_{n-1}^k = t_k < x_n^k = x$, with $x_i^k \in (x_i - \delta, x_i)$ for $i = 0, 1, \ldots, n-2$. By Corollary 1,

$$U_{k,\varepsilon} = U\begin{bmatrix} u_0, & u_1, & \ldots, & u_{n-1}, & f \\ x_0^k, & x_1^k, & \ldots, & x_{n-1}^k, & x_n^k \end{bmatrix} \geqq 0. \tag{6}$$

Let

$$M_j^k = U\begin{bmatrix} u_0, & \cdots & & & \cdots, & u_{n-1} \\ x_0^k, & \ldots, & x_{j-1}^k, & x_{j+1}^k, & \ldots, & x_n^k \end{bmatrix}$$

Clearly $M_j^k \leqq n!\, M^{n-1}\varepsilon$ for $j = 0, 1, \ldots, n-2$ and for large k, such that $|u_i(t_k) - u_i(x)| < \varepsilon$ for $i = 0, 1, \ldots, n-1$. Expanding (6) along the last row we have

$$\begin{aligned} U_{k,\varepsilon} &\leqq (n-1)n!M^n\varepsilon - f(t_k)M_{n-1}^k + f(x)M_n^k \\ &\leqq A\varepsilon - (f(t_k) - f(x))U\begin{bmatrix} u_0, u_1, \ldots, u_{n-2}, u_{n-1} \\ x_0, x_1, \ldots, x_{n-2}, x \end{bmatrix} + B\varepsilon. \end{aligned}$$

Since $f(t_k) > f(x) + h/2$, $U_{k,\varepsilon}$ can be made negative for some small $\varepsilon > 0$ and large k in contradiction to (6).

Combining Corollary 1 and Lemma 2 we have the following theorem.

THEOREM 1. *Every bounded* GMPC *function with respect to a continuous T-system* $\{u_i\}_{i=0}^{n-1}$ *belongs to* $C(u_0, u_1, \ldots, u_{n-1})$.

COROLLARY 2. *If all the determinants* (1) *with equally spaced points vanish, then f is a polynomial in the* u_i*'s*, i.e., $f = \sum_{i=0}^{n-1} a_i u_i$.

In the following section we give an application of the generalized midpoint convexity property.

3. **Union of T-systems.** The determinant (1) is a function defined on the simplex $S = \{(t_0, t_1, \ldots, t_n);\ a < t_0 < t_1 < \cdots < t_n < b\}$ (or $a \leqq t_0$, $t_n \leqq b$ if f is defined on the closed interval $[a, b]$). This set is not closed in $\mathbf{R}^{n+1}$ and hence a sequence of points in S need not converge to a point in S even if it does converge in $\mathbf{R}^{n+1}$. In view of Theorem 1, we consider the points $t = (t_0, t_0 + h, \ldots, t_0 + nh)$ with $h > 0$. A converging sequence $\{t^k\}$ of such point will converge in S if $t_n^k - t_0^k \geqq \varepsilon$ for some $\varepsilon > 0$ and all $[t_0^k, t_n^k]$ are contained in a closed subinterval of (a, b). Let $\{u_0, u_1, \ldots, u_{n-1}\}$ be a continuous T-system on the interval $[a, b]$ and let f and g be two continuous functions such that

(i) $\{u_0, u_1, \ldots, u_{n-1}, f\}$ is a T-system on $[a, d]$,

(ii) $\{u_0, u_1, \ldots, u_{n-1}, g\}$ is a T-system on $[c, b]$, and

(iii) $a < c < d < b$ and $f\|[c, d] = g\|[c, d]$.

Define u_n by $u_n = f$ on $[a, d]$ and $u_n = g$ on $[c, b]$. We show the following theorem.

THEOREM 2. *Let* $u_0, u_1, \ldots, u_{n-1}, u_n, f$ *and* g *be defined as above, satisfying* (i)–(iii). *The functions* $u_0, u_1, \ldots, u_n$ *form a T-system on* $[a, b]$.

We first prove the following lemma.

LEMMA 3. [3]. *Let* f *be a continuous function on* $[a, b]$ *which belongs to* $C(u_0, u_1, \ldots, u_{n-1})$. *If*

$$U\begin{bmatrix} u_0, & u_1, & \ldots, & u_{n-1}, & f \\ \bar{t}_0, & \bar{t}_1, & \ldots, & \bar{t}_{n-1}, & \bar{t}_n \end{bmatrix} = 0$$

for some $a \leqq \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_n \leqq b$, *then* $f\|[\bar{t}_0, \bar{t}_n] = u\|[\bar{t}_0, \bar{t}_n]$ *where* $u \in \mathrm{Span}\{u_0, u_1, \ldots, u_{n-1}\}$.

PROOF. We may assume that $f(\bar{t}_j) = 0, j = 0, 1, \ldots, n$. Let $\bar{t}_j < t < \bar{t}_{j+1}$ for some j, $0 \leqq j \leqq n-1$, since the two determinants (1) based or the points $(\bar{t}_0, \ldots, \bar{t}_{j-1}, t, \bar{t}_{j+1}, \ldots, \bar{t}_n)$ $((t, \bar{t}_1, \ldots, \bar{t}_n)$ if $j = 0)$ and $(\bar{t}_0, \ldots, \bar{t}_j, t, \bar{t}_{j+2}, \ldots, \bar{t}_n)$ $((\bar{t}_0, \ldots, \bar{t}_{n-1}, t)$ if $j = n-1)$ are non-negative, $f(t)$ must be zero, i.e., $f\|[\bar{t}_0, \bar{t}_n] = 0$.

PROOF OF THEOREM 2. Since u_n is not a polynomial in $u_0, u_1, \ldots, u_{n-1}$ in any subinterval of $[a, b]$, it will be sufficient to prove that $u_n \in C_{[A,B]}(u_0, u_1, \ldots, u_{n-1})$ for all intervals $[c, d] \subset [A, B] \subset [a, b]$. If $u_n \notin C_{[A,B]}(u_0, u_1, \ldots, u_{n-1})$, then there exist maximal intervals $[A, y]$ and $[x, B]$ such that $u_n \in C_{[A,y]}(u_0, u_1, \ldots, u_{n-1})$ and $u_n \in C_{[x,B]}(u_0, u_1, \ldots, u_{n-1})$. Given D, $y < D < B$, since $u_n \notin C_{[A,D]}(u_0, u_1, \ldots, u_{n-1})$, there exists an interval $[\alpha, \beta]$ with $A \leqq \alpha < x$ and $y < \beta \leqq D$ such that $U(\alpha, \beta; u_n) < 0$. Since this is true for all $D > y$, there exists a sequence of closed intervals $[\alpha_k, \beta_k] \subset [A, B]$, with $\beta_k \downarrow y$ and $\lim_{k\to\infty}\alpha_k = \alpha_0 < x$ such that $U(\alpha_k, \beta_k; u_n) < 0$. By a continuity argument $U(\alpha_0, y; u_n) \leqq 0$, which is in contradiction to the maximality of $[A, y]$. So $\{u_0, u_1, \ldots, u_n\}$ is a T-system on (a, b) and by the continuity of u_n on $[a, b]$, using Lemma 3, it is a T-system on $[a, b]$.

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