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ON COMPACT CONVEX SUBSETS OF $D[0, 1]$

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ABSTRACT. It is proved that a subset K of $D[0, 1]$ which is convex and conditionally compact with respect to the Skorokhod topology is conditionally compact with respect to the uniform topology on $D[0, 1]$. Consequences of this result are indicated for limit laws for generalized random variables in $D[0, 1]$ which use tightness of measures as a hypothesis. A characterization of the convex, conditionally compact subsets of $D[0, 1]$ is given in terms of the modulus of continuity and finitely many jump points.

1. Introduction. In considering sequences of generalized random variables (X_n) the structure of the infinite-dimensional linear space where they take their values is crucial in determining which limit laws are valid. For example, there are few laws of large numbers which are valid for sequences of random variables taking values in an arbitrary (separable) Banach space. Tightness of a sequence (X_n) of random variables (that is, tightness of the corresponding sequence of induced probability measures) is a condition which has recently been used, in conjunction with moment conditions, to obtain laws of large numbers for random variables taking values in an arbitrary separable Fréchet space ([3], [7]). In this paper results are obtained which determine the usefulness of tightness as a hypothesis in obtaining laws of large numbers in the space $D[0, 1]$ (defined in §2).

Let E denote a topological space, made into a measurable space $(E, \mathscr{B})$ by providing it with the σ-field $\mathscr{B}$ of Borel sets: the σ-field generated by the open sets in E. Let (Ω, F, P) be a probability space. A random variable X in E is a measurable map from Ω into E.

DEFINITION 1.1. A sequence (X_n) (finite or infinite) of random variables taking values in E (or the sequence of corresponding probability measures on E) is said to be *tight* if, to every $\varepsilon > 0$, there is a compact subset K of E such that $P[X_n \notin K] \leqq \varepsilon$, uniformly in n.

The concept of tightness thus makes use of the compact sets in E. Laws of large numbers concern sequences of Cesàro sums $\bar{X}_n = n^{-1}\sum_{k=1}^{n} X_k$ of random variables (X_n). If we assume a linear structure on E such that the

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operations of addition and scalar multiplication are measurable with respect to the Borel field $\mathscr{B}$ on E, it makes sense to consider sequences of Cesàro averages $(\bar{X}_n)$ of random variables taking values in E. $D[0, 1]$ with the Skorokhod topology is just such a space E. For tightness to be useful here, it is in general necessary that the compact sets K involved in its definition also be convex, in order that $\bar{X}_n \in K$ whenever $X_i \in K$, $i = 1, \ldots, n$. If E is a Fréchet space, this requirement of convexity is no restriction and need not be stated explicitly in the definition of tightness ([**5**], p 72). In $D[0, 1]$ with the Skorokhod topology ([**1**], Chapter 3) however, the requirement of convexity presents a serious restriction, since the convex hull of a conditionally compact set in $D[0, 1]$ need not be conditionally compact [**2**].

This situation motivated the study of those conditionally compact subsets of $D[0, 1]$ having the property that their convex hulls are again conditionally compact. The collection of such subsets will be denoted by $\mathscr{K}$. In [**4**], $\mathscr{K}$ was characterized in terms of sets of jump points in $[0, 1]$. In Theorem 3.6 of §3, another characterization of $\mathscr{K}$ is obtained in terms of the modulus of continuity $w_x(\delta)$, which looks more like the characterization given in ([**1**], p. 116) of the conditionally compact subsets of $D[0, 1]$ in terms of the modulus $w'_x(\delta)$. In this paper it is shown that the requirement that a conditionally compact set in $D[0, 1]$ be convex is somewhat restrictive and consequently that tightness is of limited usefulness as a hypothesis for laws of large numbers. Specifically, Corollary 3.2 states that any conditionally compact set in $D[0, 1]$, with the Skorokhod topology, which has the property that its convex hull is again conditionally compact, is conditionally compact in the uniform topology on $D[0, 1]$— the topology generated by the supremum norm which makes $D[0, 1]$ a Banach space. It follows that any law of large numbers in $D[0, 1]$ which uses tightness as a hypothesis can be obtained in the Banach space obtained by providing $D[0, 1]$ with the supremum norm, bypassing the Skorokhod topology and leaving open the question of good sufficient conditions for laws of large numbers in $D[0, 1]$ (for existing results see [**2**], [**4**]).

2. Definitions and preliminaries. Definitions and known results which will be needed to prove the results in §3 on properties of sets in $\mathscr{K}$ are collected here.

Denote by $D = D[0, 1]$ the linear space of functions on $[0, 1]$ which are right continuous and possess left-hand limits at each point of $(0, 1)$ and are right and left continuous at $t = 0$ and $t = 1$, respectively. The supremum norm $\|x\|_\infty = \sup_{0 \leq t \leq 1} |x(t)|$ turns D into a Banach space but this space is non-separable. Separability is a requirement which guarantees that the sum of two random variables in again a random variable (that is, measurable). Skorokhod [**6**] introduced a topology on D which makes it a separable metric space. The metric d generating this topology is needed for the proofs in §3. Let Λ denote the group of all continuous, strictly increasing maps λ of [0, 1] onto itself. We denote by id the identity map $\mathrm{id}(t) = t$, and by $x \circ \lambda$ the composition: $x \circ \lambda(t) = x(\lambda(t))$.

DEFINITION 2.1. For $x, y \in D$, $d(x, y) = \inf\{\varepsilon: \|x \circ \lambda - y\|_\infty \leqq \varepsilon$ and $\|\lambda - \mathrm{id}\|_\infty \leqq \varepsilon$ for some $\lambda \in \Lambda\}$.

This definition is that of [1]; for a detailed account of the convergence of probability measures defined on the Borel field of the Skorokhod topology, see Chapter 3 of [1].

The conditionally compact subsets of D are characterized in terms of a "modulus of continuity" $w'_x(\delta)$ and this characterization will be needed.

DEFINITION 2.2. For $T \subset [0, 1]$ and $x \in D$, $w_x(T) = \sup_{s,t\in T} |x(s) - x(t)|$. For $\delta > 0$ and $s, t \in [0, 1]$, $w_x(\delta) = \sup_{|s-t|<\delta} |x(s) - x(t)|$.

This is the familiar modulus of continuity $w_x(\delta)$.

DEFINITION 2.3. A finite partition $\mathscr{P} = \{t_i\}_{i=0}^N$, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$, of [0, 1] is said to be *δ-coarse* if $\min_{1\leqq i\leqq N}\{t_i - t_{i-1}\} > \delta$.

DEFINITION 2.4. $w'_x(\delta) = \inf\{\max_{1\leqq i\leqq N} w_x([t_{i-1}, t_i))\}$, where the infimum is taken over all δ-coarse partitions $\mathscr{P}$ of [0, 1].

The familiar modulus of continuity for functions in $C[0, 1]$ is too stringent for D; $w'_x(\delta)$ allows for a finite number of "large" jumps to occur and provides the analogue in D of the Arzelà-Ascoli characterization, in terms of equicontinuity, of conditional compactness in $C[0, 1]$. This characterization, proved in ([1], p. 116) will be needed and is formulated now.

THEOREM 2.5. *A set $K \subset D$ is conditionally compact if and only if*

$$\text{(i)} \quad \sup_{x\in K} \|x\|_\infty < \infty$$

and

$$\text{(ii)} \quad \lim_{\delta\downarrow 0} \sup_{x\in K} w'_x(\delta) = 0.$$

In D the convex hull of a conditionally compact set need not be conditionally compact. Indeed, the closure of a convex set need not be convex [2]. For $A \subset D$, let co(A) denote the convex hull of A. Let $\mathscr{K}$ denote the collection of conditionally compact subsets K of D which have the property that co(K) is again conditionally compact. (These are precisely the conditionally compact subsets which would be used in a definition of tightness in D to get laws of large numbers). In order to characterize $\mathscr{K}$ we define the following sets: for $A \subset D$ and $\varepsilon > 0$, let

$$S_\varepsilon(A) = \{t \in [0, 1]: \sup_{x\in A}|x(t) - x(t - 0)| > \varepsilon\}.$$

THEROEM 2.6. *A set $K \subset D$ is in $\mathscr{K}$ if and only if $S_\varepsilon(K)$ is finite, for each $\varepsilon > 0$.*

Theorem 2.6 is proved in **[4]**. Using Theorem 2.6 it is easy to show that $\mathscr{K}$ is a ring of subsets of D.

3. The main results.

THEOREM 3.1 *If K is a convex and conditionally compact subset of D and (x_n) is a sequence of elements of K, then* $\lim_{n\to\infty} d(x_n, x_0) = 0$ *if and only if* $\lim_{n\to\infty} \|x_n - x_0\|_\infty = 0$.

PROOF. The "if" part is trivial. To prove the "only if" part, assume that $\lim_{n\to\infty}\|x_n - x_0\|_\infty > 0$. Then there is a subsequence $\{x_{n'}\}$ and $\varepsilon > 0$ such that $\|x_{n'} - x_0\|_\infty > \varepsilon$, for all n'. Then, by compactness of [0, 1], there is a further subsequence $\{x_{n''}\}$ and a corresponding sequence $\{t_{n''}\}$ of points in [0, 1] such that $\lim t_{n''} = t_0 \in [0, 1]$ and $|x_{n''}(t_{n''}) - x_0(t_{n''})| > \varepsilon$, for all n''. For notational convenience we denote these latter subsequences again by $\{x_n\}$ and $\{t_n\}$. We thus have $\lim t_n = t_0$, $\|x_n - x_0\| > \varepsilon$, for every n, and in particular,

$$|x_n(t_n) - x_0(t_n)| > \varepsilon, \text{ for every } n. \tag{3.1}$$

We assume that $\lim_{n\to\infty} d(x_n, x_0) = 0$ and obtain a contradiction.
Given $\eta > 0$, choose $\delta > 0$ such that

$$\begin{aligned} &w_{x_0}([t_0, t_0 + 2\delta)) < \eta \\ &w_{x_0}((t_0 - 2\delta, t_0)) < \eta \end{aligned} \tag{3.2}$$

By $\lim_{n\to\infty} d(x_n, x_0) = 0$, find n_0 such that $n \geqq n_0$ implies that $d(x_n, x_0) < \eta$. Thus, given $n \geqq n_0$, there is $\lambda_n \in \Lambda$ such that

$$\sup_{0\leqq t\leqq 1} |x_n(\lambda_n t) - x_0(t)| \leqq \eta \tag{3.3}$$

and

$$\sup_{0\leqq t\leqq 1} |\lambda_n(t) - t| \leqq \delta. \tag{3.4}$$

Let $s_n = \lambda_n(t_0)$. Then, by (3.4), $n \geqq n_0$ implies that $t_0 - \delta < s_n < t_0 + \delta$. There are now two cases to consider: either $s_n = t_0$ for almost all n (i.e., for all $n \geqq n_1$, a fixed integer), or not. The second case is further broken up into two cases according to whether x_0 is continuous at $t = t_0$, or not.

Case I. If $s_n = t_0$ for almost all n, let n_1 be such that $s_n = t_0$ for $n \geqq n_1$. Suppose that $t_0 \leqq s, t < t_0 + \delta$. In(3.2) and (3.3) take $\eta = \varepsilon/8$, choose n_0 and $\delta > 0$ accordingly, and put $n_2 = \max\{n_0, n_1\}$. Then for $n \geqq n_2$, $t_0 \leqq \lambda_n(s), \lambda_n(t) < t_0 + 2\delta$, so that, using (3.2) and (3.3), we have

$$\begin{aligned} &|x_n(s) - x_n(t)| \\ &\quad \leqq |x_n(s) - x_0(\lambda_n s)| + |x_0(\lambda_n s) - x_0(\lambda_n t)| + |x_0(\lambda_n t) - x_0(t)| \\ &\quad < \varepsilon/8 + \varepsilon/8 + \varepsilon/8 = 3\varepsilon/8, \end{aligned}$$

for $n \geqq n_2$.

Similarly, for $t_0 - \delta < s$, $t < t_0$, we get $|x_n(s) - x_n(t)| < 3\varepsilon/8$, for $n \geqq n_2$. We thus have

$$(3.5) \qquad \begin{aligned} w_{x_n}([t_0, t_0 + \delta)) &< \frac{3}{8}\varepsilon \\ w_{x_n}((t_0 - \delta, t_0)) &< \frac{3}{8}\varepsilon. \end{aligned}$$

Next, using (3.5), (3.3) and (3.2) for $t_0 \leqq s$, $t < t_0 + \delta$ and $n \geqq n_2$, we have

$$\begin{aligned} &|x_n(t) - x_0(s)| \\ &\quad \leqq |x_n(t) - x_n(s)| + |x_n(s) - x_n(t_0)| + |x_n(t_0) - x_0(t_0)| + |x_0(t_0) - x_0(s)| \\ &\quad < 3\varepsilon/8 + 3\varepsilon/8 + \varepsilon/8 + \varepsilon/8 = \varepsilon, \end{aligned}$$

and so

$$(3.6) \qquad \sup_{t_0 \leqq s, t < t_0 + \delta} |x_n(t) - x_0(s)| < \varepsilon.$$

In a similar manner, we get

$$(3.7) \qquad \sup_{t_0 - \delta < s, t < t_0} |x_n(t) - x_0(s)| < \varepsilon.$$

In particular, it follows from (3.6) and (3.7) that

$$(3.8) \qquad \sup_{t_0 - \delta < t < t_0 + \delta} |x_n(t) - x_0(t)| < \varepsilon.$$

But for all sufficiently large n, $t_0 - \delta < t_n < t_0 + \delta$, and (3.8) contradicts the initial assumption (3.1).

Hence $s_n = t_0$ for almost all n is impossible.

Case II. If $x_0(t_0) = x_0(t_0 - 0)$, find $\delta > 0$ such that

$$(3.9) \qquad w_{x_0}((t_0 - 2\delta, t_0 + 2\delta)) < \varepsilon/2.$$

In (3.3) take $\eta = \varepsilon/2$, and choose n_0 and $\delta > 0$ accordingly. Let n_1 be such that $n \geqq n_1$ implies that

$$(3.10) \qquad t_0 - \delta < t_n < t_0 + \delta.$$

Put $n_2 = \max\{n_0, n_1\}$.

Now for any $\lambda \in \Lambda$ satisfying $\sup_t |\lambda(t) - t| < \delta$,

$$\sup_t |x(\lambda t) - x_0(t)| \geqq |x_n(\lambda\lambda^{-1}t_n) - x_0(\lambda^{-1}t_n)|$$
$$\geqq |x_n(t_n) - x_0(t_n)| - |x_0(t_n) - x_0(\lambda^{-1}t_n)|$$
$$> \varepsilon - |x_0(t_n) - x_0(\lambda^{-1}t_n)|, \qquad \text{by (3.1)}$$
$$> \varepsilon - \varepsilon/2 = \varepsilon/2,$$

for all $n \geqq n_2$, by (3.9) and (3.10).

Thus $\sup_t |x_n(\lambda t) - x_0(t)| > \varepsilon/2$ and this implies that $d(x_n, x_0) \geqq \varepsilon/2$, for $n \geqq n_2$. Since this holds for all $n \geqq n_2$, we have $\liminf_{n\to\infty} d(x_n, x_0) \geqq \varepsilon/2$, which contradicts the hypothesis that $\lim_{n\to\infty} d(x_n, x_0) = 0$. Thus x_0 continuous at $t = t_0$ is impossible.

Case III. If $s_n \neq t_0$ infinitely often and $|x_0(t_0) - x_0(t_0 - 0)| = c > 0$, then by choosing a further subsequence, assume that $s_n \neq t_0$ for all n. In (3.3) we put $\eta = c/4$ and choose n_0 and δ accordingly. We now show that

$$(3.11) \qquad |x_n(s_n) - x_n(s_n - 0)| > c/2, \text{ for all } n \geqq n_0.$$

We have

$$c = |x_0(t_0) - x_0(t_0 - 0)|$$
$$\leqq |x_0(t_0) - x_n(s_n)| + |x_n(s_n) - x_n(s_n - 0)| + |x_n(s_n - 0) - x_0(t_0 - 0)|$$
$$< \frac{c}{4} + |x_n(s_n) - x_n(s_n - 0)| + \frac{c}{4},$$

for $n - n_0$, by (3.3). Thus $|x_n(s_n) - x_n(s_n - 0)| > c/2$, for every $n \geqq n_0$.

But since $s_n \neq t_0$ and $\lim_{n\to\infty} s_n = t_0$, and by hypothesis $x_n \in K$, for each n, we have by (3.11) that

$$S_{c/2}(\mathrm{I}) = \{t \in [0, 1]: \sup_{x \cdot K} |x(t) - x(t - 0)| > c/2\}$$
$$\supseteqq \{t \in [0, 1]: \sup_n |x_n(t) - x_n(t - 0)| > c/2\}$$
$$\supseteqq \{s_1, s_2, \ldots\}, \text{ which is infinite,}$$

contradicting, by Theorem 2.6, the hypothesis that K is convex and conditionally compact.

Thus $\limsup_{n\to\infty} \|x_n - x_0\|_\infty = 0$.

COROLLARY 3.2. *Let $K \subset D$. If $K \in \mathscr{K}$, then K is conditionally compact in the uniform topology on D.*

PROOF. If $\{x_n\} \subset K$, then $\{x_n\} \subset \mathrm{co}(K)$, and there is a subsequence $\{x'_n\}$ such that $\lim x'_n = x_0 \in D$. But then, by Theorem 3.1. $\lim \|x'_n - x_0\|_\infty = 0$.

The following lemmas are needed in the proof of Theorem 3.6. Lemma 3.4 has a useful corollary.

The "Arzelà—Ascoli" characterization of the conditionally compact sets in $D[0, 1]$ in terms of the modulus $w'_x(\delta)$ can be sharpened considerably if attention is restricted to $\mathscr{K}$. Indeed, $\mathscr{K}$ can be characterized using the modulus of continuity $w_x(\delta)$ since, given $K \in \mathscr{K}$ and $\varepsilon > 0$, $w'_x(\delta) > \varepsilon$ can be achieved uniformly on K by means of a single partition $\mathscr{P}$. To establish this result, Theorem 3.6, we first prove two lemmas on convergent sequences in $D[0, 1]$.

LEMMA 3.3. *Let (x_n) be a sequence in D converging to $x_0 \in D$. Suppose (s_n) and (t_n) are sequences in* $[0, 1]$ *such that* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = t_0 \in [0, 1]$ *and that $|x_n(s_n) - x_n(t_n)| > \varepsilon > 0$ for each n. Then*

$$|x_0(t_0) - x_0(t_0 - 0)| \geqq \varepsilon.$$

PROOF. Suppose $|x_0(t_0) - x_0(t_0 - 0)| = \eta < \varepsilon$.
Let $a = (\varepsilon - \eta)/4$. Find $0 < \delta < a$ such that

$$w_{x_0}((t_0 - 2\delta, t_0)) < a \tag{3.12}$$

and

$$w_{x_0}([t_0, t_0 + 2\delta)) < a.$$

Find n such that $|t_n - t_0| < \delta$, $|s_n - t_0| < \delta$, and $s(x_n, x_0) < \delta$. Then there is $\lambda_n \in \Lambda$ such that

$$\|x_0 \circ \lambda_n - x_n\|_\infty < \delta \tag{3.13}$$

and

$$\|\lambda_n - \mathrm{id}\|_\infty < \delta. \tag{3.14}$$

Then

$$\begin{aligned}
&|x_n(s_n) - x_n(t_n)| \\
&\quad \leqq |x_n(s_n) - x_0(\lambda_n s_n)| + |(x_0(\lambda_n s_n) - x_0(\lambda_n t_n)| + |x_0(\lambda_n t_n) - x_n(t_n)| \\
&\quad < \delta + \eta + 2a + \delta \\
&\quad \leqq a + \eta + 2a + a = \eta + 4a = \eta + (\varepsilon - \eta) = \varepsilon,
\end{aligned}$$

using (3.19), and then (3.12) together with (3.14), yielding $|x_n(s_n) - x_n(t_n)| < \varepsilon$, a contradiction. Thus $\eta \geqq \varepsilon$ proving the lemma.

LEMMA 3.4. *Let $K \in \mathscr{K}$, and let (x_n) be a sequence of elements of K converging to an element x_0 of D. Then $|x_0(t) - x_0(t - 0)| > \varepsilon$ implies $t \in S_\varepsilon(K)$.*

PROOF. Suppose $|x_0(t_0) - x_0(t_0 - 0)| = \alpha > \varepsilon$ and that $t_0 \notin S_\varepsilon(K)$. Let $a = \min\{|t - t_0| : t \in S_\varepsilon(K)\}$. By hypothesis $a > 0$ (recall that $S(K)$ is finite by Theorem 2.6). Let $\eta = (\alpha - \varepsilon)/2$. Choose n_0 large enough that $d(x_n, x_0) < \min\{a, \eta\}$. Then to $b \geqq n_0$, there is $\lambda_n \in \Lambda$ such that

$$\|x_n \circ \lambda_n - x_0\|_\infty < \eta \tag{3.15}$$

and

$$\|\lambda_n - \mathrm{id}\|_\infty < a. \tag{3.16}$$

Thus, by (3.15), putting $t_n = \lambda_n(t_0)$.

$$\begin{aligned} \alpha &= |x_0(t_0) - x_0(t_0 - 0)| \\ &< |x_0(t_0) - x_n(t_n)| + |x_n(t_n) - x_n(t_n - 0)| + |x_n(t_n - 0) - x_0(t_0 - 0)| \\ &< \eta + |x_n(t_n) - x_n(t_n - 0)| + \eta, \end{aligned}$$

yielding

$$\begin{aligned} |x_n(t_n) - x_n(t_n - 0)| &> \alpha - 2\eta \\ &= \alpha - 2(\alpha - \varepsilon)/2 = \varepsilon. \end{aligned}$$

But this is a contradiction since, by (3.16), $|t_n - t_0| < a$ and so $t_n \notin S_\varepsilon(K)$. Hence, $|x_0(t_0) - x_0(t_0 - 0)| \leqq \varepsilon$ and $t_0 \notin S_\varepsilon(K)$, or else $t_0 \in S_\varepsilon(K)$.

For $A \subset D$ let $\mathrm{Cl}(A)$ denote the closure of A in the Skorokhod topology.

COROLLARY 3.5. *For $K \subset D$, $K \in \mathscr{K}$ if and only if $\mathrm{Cl}(K) \in \mathscr{K}$.*

PROOF. By Lemma 3.4, $S_\varepsilon(K) = S_\varepsilon \mathrm{Cl}(K)$ and the corollary follows by Theorem 2.6.

THEOREM 3.6. *Let $K \subset D$. Then $K \in \mathscr{K}$ if and only if to every $\varepsilon > 0$, there is a finite partition $\mathscr{P}$: $0 = t_0 < t_1 < \cdots < t_N = 1$ of $[0, 1]$ such that $\max_{1 \leqq i \leqq N} w_x([t_{i-1}, t_i)) \leqq \varepsilon$ for every $x \in K$.*

PROOF. We first show necessity. If the conclusion is false, then there is $\varepsilon > 0$ such that, to any partition $\mathscr{P} = \{t_i\}_{i=0}^N$ of $[0, 1]$, there is some $x \in K$, and two points s and t, such that $s, t \in [t_{i-1}, t_i)$ for some i and $|x(s) - x(t)| > \varepsilon$.

Consider a sequence $\mathscr{P}_1, \mathscr{P}_2, \ldots$ of partitions of $[0, 1]$ with norm (= length of longest subinterval) tending to zero, such that $\mathscr{P}_{n+1}$ is a refinement of $\mathscr{P}_n$, for each n, and such that $\mathscr{P}_1$ contains the points of $S_\varepsilon(K)$. Since $K \in \mathscr{K}$, $S_\varepsilon(K)$ is finite by Theorem 2.6. Corresponding to $\mathscr{P}_n$, let s_n and t_n be points in $[t_{i_n - 1}, t_{i_n})$ such that for the function $x_n \in K$ we have $|x_n(s_n) - x_n(t_n)| > \varepsilon$. Without loss of generality, assume that $s_n < t_n$, for each n.

Since K is conditionally compact, the sequence (x_n) contains a subsequence which converges to some element x_0 in D. By compactness of $[0, 1]$, the sequence (t_n) contains a subsequence converging to some t' in $[0, 1]$. We take the intersection of these two subsequences and for notational convenience denote it again by (x_n), (t_n), (s_n). We thus have $\lim x_n = x_0$, $\lim s_n = t'$, and, because $|s_n - t_n| \to 0$, $\lim t_n = t'$. Also, $|x_n(s_n) - x_n(t_n)| > \varepsilon$, and $s_n < t_n$ are in the interval $[t_{i_n-1}, t_{i_n})$ of $\mathscr{P}_n$.

We now note that by Lemma 3.3 and 3.4, $t' \in S_\varepsilon(K)$ and since $t' \in \mathscr{P}_1$, $t' \in \mathscr{P}_n$ for every n.

We next note that since $\lim d(x_n, x_0) = 0$, and $K \in \mathscr{K}$, that $\lim \|x_n - x_0\|_\infty = 0$, by Theorem 3.1. Thus, we can find n_0 such that $\|x_n - x_0\|_\infty < \varepsilon/3$ for $n \geqq n_0$.

Next, observe that since $\lim s_n = t' = \lim t_n$, and $s_n < t_n$, we have that either (i) $t' \leqq s_n$, infinitely often, or (ii) $t_n < t'$, infinitely often, or both occur, ($s_n < t' \leqq t_n$ is impossible, since $t' \in \mathscr{P}_n$). Both (i) and (ii) lead to a contradiction.

Suppose (i) holds. Find $\delta > 0$ such that $w_{x_0}([t', t' + \delta)) < \varepsilon/3$. Find n so large that $t_n - t' < \delta$ and $\|x_n - x_0\|_\infty < \varepsilon/3$ and (i) holds. Then

$$\begin{aligned}
&|x_n(t_n) - x_n(s_n)| \\
&\quad \leqq |x_n(t_n) - x_0(t_n)| + |x_0(t_n) - x_0(s_n)| + |x_0(s_n) - x_n(s_n)| \\
&\quad < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

But $|x_n(t_n) - x_n(s_n)| > \varepsilon$, a contradiction.

If (ii) holds, find $\delta > 0$ such that $w_{x_0}((t' - \delta, t')) < \varepsilon/3$, and n such that $t' - s_n < \delta$ and (ii) holds. The same inequalities result.

Finally, sufficiency is immediate from Theorem 2.6, since the condition of the theorem implies that, given $\varepsilon > 0$, $S_\varepsilon(K) \subset \{t', t_n, \ldots, t_N\}$.

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