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COUNTABLE UNIONS OF 0-DIMENSIONAL DECOMPOSITIONS

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In this paper we shall investigate monotone upper semicontinuous decompositions of a locally compact connected metric space, where the projection of the nondegenerate elements is 0-dimensional.

In particular we show that if G can be described by a countable collection G_i of decompositions of M which fit together in a "semicontinuous" manner and where each G_i is shrinkable, then G is shrinkable and M/G is homeomorphic to M.

Throughout this paper M will denote a locally compact connected metric space, G a monotone 0-dimensional upper semicontinuous decomposition of M, P the projection map of M onto the decomposition space M/G, and $H(G)$ the set of all nondegenerate elements of G. By 0-dimensional we mean that $P(H(G))$ is a 0-dimensional subset of M/G. The closure of a set X will be denoted by $\mathrm{Cl}(X)$, its boundary by $\mathrm{Bd}(X)$, and the set of all points within ε of X by $N(X, \varepsilon)$.

A 0-dimensional decomposition G is shrinkable if for each open set B containing the union of the nondegenerate elements, each positive number ε, and each homeomorphism f from M onto M, there is a homeomorphism h from M onto M such that the diameter of $h(f(g))$ is less than ε for each g in G and $f(x) = h(f(x))$ for x an element of $M - f(B)$. This definition of shrinking was shown to be equivalent to the definition which uses the identity map for f (see **[1]** and **[3]**).

Since it will be used repeatedly we restate Theorem 1 of **[3]**.

THEOREM 1. *Let K be an open covering of $H(G)$ in M, then there exists a refinement K' of K such that*

1) *K' is an open covering of $H(G)$ in M,*

2) *K' is a disjoint countable collection,*

3) *If X is a compact set, then the closure of $\{k \in K' | k$ intersects $X\}$ is a compact set, and*

4) *K' is a locally null collection.*

THEOREM 2. *If G is a monotone upper semicontinuous decomposition of a locally compact connected metric space M such that $P(H(G))$ is a 0-dimensional subset of M/G and there exists a countable collection $\{G_i\}$ of shrinkable upper semicontinuous decompositions of M such that*

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1) $H(G) = \bigcup_{i=1}^{\infty} H(G_i)$,

2) *For each i if a sequence of elements of* $H(G_i)$ *converges to a set* X, *then* $X \in G$ *or* X *is a subset of some element of* $H(G_j)$ *where* $j \leqq i$, *and*

3) *Let* $L = \mathrm{Cl}(\bigcup H(G)) - \bigcup_{i=1}^{\infty}\mathrm{Cl}(\bigcup H(G_i))$, *then the upper semicontinuous decomposition* G' *of* M *such that* $H(G') = \{g \in H(G) | g$ *intersects* $\mathrm{Cl}(L)\}$ *is shrinkable, then G is shrinkable and it follows that* M/G *is homeo-then G is shrinkable and it follows that* M/G *is homeomorphic to* M.

PROOF. Let A be an open set containing $\bigcup H(G)$ and ε be a positive number.

First we shall define a sequence $\{C_i\}$ of open coverings of $H(G)$ and a sequence of homeomorphisms $\{f_i\}$ of M onto M. Let A' denote an open covering of $H(G)$ such that A' satisfies the conclusion of Theorem 1 where $\{A\}$ is the open covering K. Since $H(G')$ is a subset of $H(G)$, $\bigcup A'$ is an open set containing $\bigcup H(G')$ and hence there exists a map f_0 such that $f_0 | M - \bigcup A'$ is the identity and if g is an element of $H(G')$, then the diameter of $f_0(g)$ is less than $\varepsilon/2$. Since $(\bigcup H(G')) \bigcup L$ and the union of all elements of G whose diameter is greater than or equal to ε are disjoint closed subsets of M, there exists disjoint open sets O_1, O_2 containing them. If g, an element of $H(G)$, is a subset of O_i, set $O_g = O_i \bigcap (\bigcup A')$ otherwise pick an open subset of $\bigcup A'$ containing g which does not intersect either of the above closed sets. Let C_0 be an open refinement of $\{O_g | g \in H(G)\}$ satisfying the conclusion of Theorem 1. Assume inductively, that f_i and C_i have been chosen. Since $\bigcup C_i$ is an open set containing $H(G)$ and G_{i+1} is shrinkable, we can obtain a homeomorphism h_{i+1} which is equal to f_i when restricted to $M - \bigcup C_i$ and for each g in G_{i+1} the diameter of $h_{i+1}(g)$ is less than $\varepsilon/2$. Let J_i be the set of all elements of G whose diameter under f_i is greater than or equal to ε, and let B_i be the union of all components of $\bigcup C_i$ which intersect $\bigcup J_i$. Let

$$f_{i+1}(x) = \begin{cases} f_i(x) & \text{if } x \in M - B_i \\ h_{i+1}(x) & \text{if } x \in B_i. \end{cases}$$

Since $f_i(x) = h_{i+1}(x)$ on $\mathrm{Bd}(B_i)$, it follows that f_{i+1} is a homeomorphism of M onto M. We shall now choose C_{i+1}. If $g \in H(G)$ and k is the element of C_i containing g, we choose an open set O_g which satisfies the following requirements. If $g \notin J_{i+1}$, choose a subset O_g of k containing g such that O_g does not intersect any element of J_{i+1}. If $g \in J_{i+1}$, it follows from condition 2) that there exists an open set D_g whose intersection with $\mathrm{Cl}(\bigcup H(G_{i+1}))$ is empty. Also since $\bigcup J_{i+1}$ is a closed set, there exist disjoint open sets O, O' containing $\bigcup J_{i+1}$ and $\mathrm{Bd}(\bigcup C_i)$ respectively. Let $O_g = k \bigcap O \bigcap D_g$. In each of the two cases above it will also be required that O_g be a subset of $P^{-1}(N(P(g), 1/\mathrm{i}))$. Apply Theorem 1 to $\{O_g | g \in H(G)\}$ and obtain a disjoint collection C_{i+1} of open sets which cover $H(G)$.

We now define a homeomorphism which shrinks G. Let $F(x) = \lim_{i\to\infty} f_i(x)$. To show that $F(x)$ is a homeomorphism we will first show that for each x in M there exists a neighborhood D of x and an n such that for all $i \geqq n$, $f_i|D = f_n|D = F|D$. There are three cases to consider, first if $x \notin \mathrm{Cl}(\bigcup H(G))$, then there exists an n such that $P^{-1}(N(P(x), 2/n))$ does not intersect $\mathrm{Cl}(\bigcup H(G))$, thus $P^{-1}(N(P(x), 1/n))$ would not intersect any of the O_g's used to define C_n and hence $f_i(y) = f_n(y)$ whenever $i \geqq n$ and $y \in P^{-1}(N(P(x), 1/n))$. Secondly if $x \in L$, then $M - \mathrm{Cl}(B_0)$ is an open set containing x which is fixed for all $i > 0$. For the last case there exists an n such that $x \in \mathrm{Cl}(\bigcup H(G_n))$ and hence $\mathrm{Cl}(B_n) \cap \mathrm{Cl}(\bigcup H(G_n)) = \emptyset$, it then follows that $f_i|M - \mathrm{Cl}(B_n) = f_n|M - \mathrm{Cl}(B_n)$. The only other requirement we need to check out is that F is an onto mapping. Since each component of $\bigcup A'$ has a compact closure, we note that for each y in M the sequence $\{f_i^{-1}(y)\}$ lies in a compact set and has a limit point p. From the above arguments there exists an n and a neighborhood D of p such that for all $i \geqq n$, $f_i|D = f_n|D$; hence $F(p) = y$ or F is onto.

If $g \in H(G)$, then there exists an n such that $g \in H(G_n)$ and the diameter of $f_n(g)$, and hence of $F(g)$, is less than ε. Thus F is a shrinking homeomorphism for G and it follows from [3] that M/G is homeomorphic to M, which completes our proof of Theorem 2.

We note that there are several conditions which allow us to satisfy condition 3) of Theorem 2. For example, if we were to require that the nondegenerate elements form a continuous collection, then $H(G')$ is empty and 3) is trivial. Since Theorem 2 is in a form which could be applied in an iterative manner on the decomposition G', then if the elements of $H(G')$ are isolated or if for any n the n-th derived set of $H(G')$ in M/G is empty, condition 3) will be satisfied.

In [1] it was shown that if G is a pointlike upper semicontinuous decomposition of E^3 such that $H(G)$ is countable and $\bigcup H(G)$ is a G_δ set, then $E^3/G \cong E^3$. If $H(G)$ is countable, it could always be considered as the union of a countable number of decompositions each one with only one non-degenerate element. Conditions 1) and 2) of Theorem 2 would be satisfied. In Theorem 3 we replace condition 3) of Theorem 2 by a G_δ type of condition which would, in the case that each G_i has only one non-degenerate element, be equivalent to $\bigcup H(G)$ is a G_δ set.

THEOREM 3. *If G is a monotone 0-dimensional upper semicontinuous decomposition of M such that there exists a countable collection of shrinkable upper semicontinuous decompositions of M such that*

1) $H(G) = \bigcup_{i=1}^{\infty} H(G_i)$,

2) *If a sequence of elements of $H(G_i)$ converges to a set X, then $X \in G$ or X is a subset of some element of $H(G_j)$ where $j \leqq i$, and*

3) *There exists a sequence $\{D_i\}$ of open sets each containing $\bigcup H(G)$ such*

that $L = \mathrm{Cl}(\bigcup H(G)) - \bigcup_{i=1}^{\infty} \mathrm{Cl}(\bigcup H(G_i))$ *does not intersect* $\bigcap_{i=1}^{\infty} D_i$, *then then* G *is shrinkable and* M/G *is homeomorphic to* M.

PROOF. Since the proofs of Theorem 2 and Theorem 3 are similar we will outline the changes that need to be made rather than stating another proof. Let f_0 be the identity and $A' = C_0$. In the choice of C_i require that each O_g be a subset of D_i. The only change in establishing that F is a shrinking homeomorphism would involve points of L, but the new condition 3) would guarantee that such a point x would be left out of C_n at some stage and the choice of C_{n+1} would require that a neighborhood of x is fixed for all f_i where $i \geqq n + 1$. The rest of the proof would be as in Theorem 2.

In the proofs of Theorems 2 and 3 we note that condition 1) in each of the theorems could be relaxed. For example, if each element of G is an element of some G_i and each element of G_i is a subset of some element of G and if $i > k$ and $g \in H(G_i)$ and $g' \in H(G_k)$ and g intersects g' implies g is a subset of g', then the proofs given would be valid without any changes.

In [2] Bing gave an example of a point-like decomposition of E^3 which was countable but E^3/G was not homeomorphic to E^3. In [3] we indicated that this example was the union of two shrinkable decompositions; hence, even when there are only two decompositions, if condition 2) of Theorems 2 and 3 were left out, the theorems would be false. If the example in [2] were considered as the union of a countable number of decompositions each with only one non-degenerate element, then every condition except condition 3) would be satisfied, but the conclusion is false.

The following example is rather simple, but it has some interesting properties. Let M be the compact subset of E^2 given by taking a sequence of circles the n-th one centered at $(1/2^n, 0)$ of radius $1/2^{n+2}$ where $n > 1$, together with the points on the x axis which are exterior to each of the circles and lie between $(0, 0)$ and $(1, 0)$. Let G_i be the decomposition of M which has one non-degenerate element namely the i-th circle.

For each i, G_i is a pointlike 0-dimensional upper semicontinuous decomposition of M such that M/G_i is homeomorphic to M, but G_i is not shrinkable. This is in sharp contrast to a 3-manifold where a pointlike 0-dimensional upper semicontinuous decomposition of a 3-manifold is homeomorphic to the 3-manifold if and only if it is shrinkable.

Let G be the decomposition of M such that $H(G) = \bigcup H(G_i)$, then G satisfies all of the hypotheses of Theorems 2 and 3 except that M/G_i is homeomorphic to M, but G_i is not shrinkable. M/G is not homeomorphic to M, but to a line segment.

In the above example each non-degenerate element is pointlike, but unlike a manifold there was no homogeneous condition between arbitrary points. There are three different types of points. It is not known if there exists an example of a space with a local homeomorphism between neighborhoods of points where M/G is homeomorphic to M, but G is not shrinkable.

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