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SUBORDINATIONS FOR TYPICALLY-REAL AND RELATED FUNCTIONS

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ABSTRACT. Let T be the typically-real functions in the open unit disk E and let C be the subclass of functions convex in the direction of the imaginary axis. For real μ, the subordination $z/2 + \mu a_2 z^2 \propto f(z)(z \in E)$ holds for all $f(z) = z + a_2 z^2 + \cdots$ in C if and only if $0 \leqq \mu \leqq 1/3$. If $\eta(z) = z/[1 + (1 - z^2)^{1/2}]$, then $2f(\eta(z))$ is in C whenever $f \in T$. From these results, we conclude for real μ that $z/4 + \mu a_2 z^2 \propto f(\eta(z)) \propto f(z)(z \in E)$ for all $f(z) = z + a_2 z^2 + \cdots$ in T if and only if $0 \leqq \mu \leqq 1/12$. If s_n is the n-th partial sum of $f(z) = z + a_2 z^2 + \cdots$, then subordinations of the form $s_n(\lambda z) \propto f(z)(z \in E)$, λ real, are obtained for $f \in T$ and $f \in C$.

1. **Introduction.** Let E denote the unit disk $|z| < 1$. The class C is all analytic functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in E such that $\{a_j\}$ is a real sequence and the intersection of $f(E)$ with each line parallel to the imaginary axis is empty or an interval. The functions $f \in C$ are univalent and the region $f(E)$ includes all points of the disk $|w| < \frac{1}{2}$ (see [**10**]). The convex univalent functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in E also map E onto a region that includes the $\frac{1}{2}$-disk [**8**, p. 45]. This property was extended to the subordinations

$$(1) \qquad \frac{1}{2} z \propto \frac{2}{3} z + \frac{1}{6} a_2 z^2 \propto f(z) = z + a_2 z^2 + \cdots \ (z \in E)$$

for all convex univalent functions f and it was shown that the constants 2/3 and 1/6 are best possible [**1**]. The subordination $2z/3 + a_2 z^2/6 \propto f(z) = z + a_2 z^2 + \cdots (z \in E)$ does not hold for all functions $f \in C$ since $f(z) = z/(1 - z^2)$ is in C and $f(E)$ omits the points ti for all real t, $|t| \geqq 1/2$. There is, nevertheless, an analog of (1) for the class C that is proved in this paper.

THEOREM 1. *Let μ be real. Then*

$$\frac{1}{2} z + \mu a_2 z^2 \propto f(z) = z + a_2 z^2 + \cdots \ (z \in E)$$

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for all $f \in C$ if and only if $0 \leqq \mu \leqq 1/3$. The factor $1/2$ cannot be replaced by a larger number.

Let $s_n(z)$ be the n-th partial sum of $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. By Theorem 1, if f is in C, then $s_1(z/2)$ and $s_2(z/2)$ are subordinate in E to f. More generally, we prove the following theorem.

THEOREM 2. *For each $f \in C$, $s_n(z/2) \propto f(z)$ for $n = 1, 2, 4, 5, \ldots$ $(z \in E)$. The factor $1/2$ is best possible.*

The validity of this result for $n = 3$ is an open question.

The typically-real functions T are the analytic functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in E such that $\operatorname{Im} z \operatorname{Im} f(z) \geqq 0$ in E [**8**, p. 54]. It is known that each $f \in T$ is univalent in

$$G = \{z : |z + i| < \sqrt{2}\} \cap \{z : |z - i| < \sqrt{2}\}$$

([**2**], [**4**], [**7**]). The function $\eta(z) = z/[1 + (1 - z^2)^{1/2}]$ maps E onto G. A relationship between T and C is given by the next result.

THEOREM 3. *If $f \in T$, then $g(z) = 2f(\eta(z)) \in C$.*

Theorem 3 verifies a conjecture of Goodman [**4**]. It proves that for each $f \in T$ the region $f(G)$ is convex in the direction of the imaginary axis. In view of Theorem 1, another consequence is the following subordination result for the class T.

THEOREM 4. *Let μ be real. Then*

$$z/4 + \mu a_2 z^2 \propto f(\eta(z)) \propto f(z) = z + a_2 z^2 + \cdots \quad (z \in E)$$

for all $f \in T$ if and only if $0 \leqq \mu \leqq 1/12$. The factor $1/4$ cannot be replaced by a larger number.

Finally, we prove an analog of Theorem 2 for the class T similar to Keogh's result [**6**] for univalent functions.

THEOREM 5. *For all $f \in T$, $s_n(z/4) \propto f(\eta(z)) \propto f(z)$ $(z \in E, n = 1, 2, 3, \ldots)$ The factor $1/4$ is best possible.*

2. **Some Lemmas.** For real ϕ, let $g(z, \phi) = z(1 + z\cos\phi)/(1 - z^2)$. Since $g(e^{i\alpha}, \phi) = -(1/2)\cos\phi + (2\sin\alpha)^{-1}(1 + \cos\alpha\cos\phi)i$, the function $g(z, \phi)$ maps E onto the Riemann sphere slit along the line $\operatorname{Re} w = -(1/2)\cos\phi$ from $-e^{i\phi}/2$ through ∞ to $-e^{-i\phi}/2$. The function $g(z, \phi)$ is in C for each real ϕ.

LEMMA 1. *Let $f(z) = z + a_2 z^2 + \cdots$ be in C. If for a real ϕ and $\rho > 0$ the point $\rho e^{i\phi} \notin f(E)$, then $\rho \geqq \rho_0(\phi)$, where*

$$(2) \qquad \rho_0(\phi) = \begin{cases} (2 - \cos\phi)/(4 + 2a_2) & \text{for} \quad a_2 \leqq -\cos\phi, \\ (2 + \cos\phi)/(4 - 2a_2) & \text{for} \quad a_2 > -\cos\phi. \end{cases}$$

PROOF. The function $f \in C$ is subordinate to $2\rho g(z, \pi - \phi)$ by the definition of the class C. Thus, there is a univalent Schwarz function $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$, $|\omega(z)| < 1$, in E such that

$$f(z) = 2\rho \frac{\omega - \omega^2 \cos\phi}{1 - \omega^2} = 2\rho\omega_1 z + 2\rho(\omega_2 - \omega_1^2 \cos\phi)z^2 + \cdots$$

in E. We conclude that $2\rho\omega_1 = 1$ and $a_2 = 2\rho(\omega_2 - \omega_1^2 \cos\phi)$. Since $|\omega_2| \leqq 2|\omega_1|(1 - |\omega_1|)$ [**8**, p. 99], we have $|a_2| \leqq 1$ and

$$\left|a_2 + \frac{\cos\phi}{2\rho}\right| \leqq 2 - 1/\rho.$$

This inequality is equivalent to $\rho \geqq \rho_0(\phi)$, where $\rho_0(\phi)$ is defined by (2).

For the proof of Theorem 1, we use Keogh's modification of the Schur algorithm [5].

LEMMA 2. *Let* $P(z) = b_0 + b_1 z + b_2 z^2$, *where* b_0, b_1, b_2 *are complex numbers,* $b_0 \neq 0$. *If* $|b_2| < |b_0|$ *and*

$$(3) \qquad |b_0 \bar{b}_1 - \bar{b}_2 b_1| \leqq |b_0|^2 - |b_2|^2,$$

then $P(z)$ *has no zero in* $|z| < 1$. *If* $P(z)$ *has no zero in* $|z| < 1$, *then* $|b_2| \leqq |b_0|$ *and* (3) *holds.*

The sharpness result of Theorem 1 is a consequence of the next lemma.

LEMMA 3. *Let* μ *and* λ *be real numbers,* $\lambda > 0$. *If for all real* ϕ, *we have*

$$(4) \qquad \lambda z + \mu z^2 \cos\phi \propto g(z, \phi) = z(1 + z\cos\phi)/(1 - z^2) \quad (z \in E)$$

then $\lambda \leqq 1/2$. *If, in addition,* $\lambda = 1/2$, *then* $0 \leqq \mu \leqq 1/3$.

PROOF. We have $\lambda z \propto g(z, 0)$ if and only if $\lambda \leqq 1/2$. Suppose that $\lambda = 1/2$. In this case the subordination (4) implies that $z/2 + \mu z^2 \cos\phi \neq -e^{i\phi}/2$ for $z \in E$. By Lemma 2 we conclude that $|\mu \cos\phi| \leqq 1/2$ and

$$\frac{1}{2}\left|\mu\cos\phi - \frac{1}{2}e^{i\phi}\right| \leqq \frac{1}{4} - \mu^2\cos^2\phi.$$

The last of these inequalities holds if the only if $|\mu\cos\phi| \leqq 1/2$ and $[4\mu^4 \cos^2\phi - \mu(3\mu - 1)]\cos^2\phi \geqq 0$. This is the case for all real ϕ if and only if $0 \leqq \mu \leqq 1/3$.

The proof of Theorem 5 for $n \geqq 3$ uses the following simple consequence of a result due to Remizova [9].

LEMMA 4. *If* $f \in T$, *then for* $z \in E$

$$|f(z)| \geqq \begin{cases} |z|/(1+|z|)^2 & \text{if } 0 \leqq |z| \leqq \alpha, \\ |z|(1-|z|^2)/(1+|z|^2)^2 & \text{if } \alpha < |z| < 1, \end{cases}$$

where $\alpha \approx .544$ *is the real zero of* $\alpha^3 + \alpha^2 + \alpha - 1$.

The functions $f(z) = z/(1+z)^2$ and

$$f(z) = z(1+z^2)/(1-z^2)^2 = \frac{1}{2} z/(1-z)^2 + \frac{1}{2} z/(1+z^2)$$

are in T and equality holds in Lemma 4 when $z = r > 0$.

3. **Proof of Theorem** 1. Let $f(z) = z + \alpha z^2 + \cdots$ be in C. The subordination $z/2 + \alpha t z^2/3 \propto f(z)$ holds for $t \in [0, 1]$ provided $z + 2\alpha t z^2/3 - 2\zeta$ has no zero in E whenever $\zeta \notin f(E)$. By Lemma 1, $|\zeta| \geqq \rho_0(\phi)$ where $\phi = \arg \zeta$ and $\rho_0(\phi)$ is given by (2). Thus, $z + 2\alpha t z^2/3 - 2\zeta$ has no zero in E whenever $\zeta \notin f(E)$ if $z + 2\alpha t z^2/3 - 2\rho_0(\phi)e^{i\phi}$ has no zero in E whenever ϕ is real. Indeed, in this case $z/2 + \alpha t z^2/3$ for $z \in E$ is in the region bounded by $w = \rho_0(\phi)e^{i\phi}$ and, hence, is contained in $f(E)$. By Lemma 2 the subordination $z/2 + \mu\alpha z^2 \propto f(z)$ is proved if for all real ϕ and for all $\mu = t/3$, $t \in [0, 1]$, we have

$$(5) \qquad 2\rho_0 > 2t|\alpha|/3, \quad |2t\alpha/3 + 2\rho_0 e^{i\phi}| \leqq 4\rho_0^2 - 4t^2\alpha^2/9,$$

where $\rho_0 = \rho_0(\phi)$ is given by (2). Since $-1 \leqq \alpha \leqq 1$ **[10]** and $2\rho_0 \geqq 1$, the first inequality of (5) is satisfied. To verify the second inequality, let

$$g(\phi) \equiv |2t\alpha/3 + 2\rho_0 e^{i\phi}| - 4\rho_0^2 + 4t^2\alpha^2/9.$$

Since $|2t\alpha/3 + 2\rho_0 e^{i\phi}| \geqq 2\rho_0 - 2t|\alpha|/3 > 0$, the function g is differentiable and

$$g'(\phi) = \frac{4t\alpha(\rho_0' \cos\phi - \rho_0 \sin\phi)/3 + 4\rho_0\rho_0'}{|2t\alpha/3 + 2\rho_0 e^{i\phi}|} - 8\rho_0\rho_0'.$$

Suppose that $\alpha > -\cos\phi$. Since $\rho_0 = (1/2)(2+\cos\phi)/(2-\alpha)$ in this case, we obtain

$$(6) \quad g'(\phi) = -\frac{\sin\phi}{(2-\alpha)^2}\left\{\frac{4t\alpha(2-\alpha)(1+\cos\phi)/3 + 2 + \cos\phi}{|2t\alpha/3 + 2\rho_0 e^{i\phi}|} - 4 - 2\cos\phi\right\}.$$

At the critical point $\phi = 0$, we have $2\rho_0 = 3/(2-\alpha)$. Since $3/(2-\alpha) + 2t\alpha/3 \geqq 1 - 2|\alpha|/3 \geqq 1/3 > 0$, we obtain

$$g(0) = \frac{3}{2-\alpha} + \frac{2}{3}t\alpha - \frac{9}{(2-\alpha)^2} + \frac{4}{9}t^2\alpha^2$$
$$= \frac{-27(1+\alpha) + 2t\alpha(2-\alpha)^2(3+2t\alpha)}{9(2-\alpha)^2}.$$

If $-1 \leqq \alpha \leqq 0$, then $g(0) \leqq 0$. If $0 < \alpha \leqq 1$, then

$$-27(1+\alpha) + 2t\alpha(2-\alpha)^2(3+2t\alpha) \leqq -27(1+\alpha)$$
$$+ 2\alpha(2-\alpha)^2(3+2\alpha) = -27 - 3\alpha - 10\alpha^3 - 4\alpha^2 - 4\alpha^2(1-\alpha^2) < 0.$$

Thus, $g(0) \leqq 0$ for all $\alpha \in [-1, 1]$ and all $t \in [0, 1]$.

If there is a critical point $\phi \neq 0$ of g, then at this point

$$|2t\alpha/3 + 2\rho_0 e^{i\phi}| = \frac{4t\alpha(2-\alpha)(1+\cos\phi)/3 + 2 + \cos\phi}{2(2+\cos\phi)}$$

by (6). By substitution of this equality into the definition of g, we obtain

$$g(\phi) = \frac{1}{2} + \frac{4}{9}t^2\alpha^2 + \frac{2}{3}t\alpha(2-\alpha) - \frac{2t\alpha(2-\alpha)}{3u} - \frac{u^2}{(2-\alpha)^2},$$

where $u = 2 + \cos\phi \geqq 2 - \alpha \geqq 1$. Since $u^3 \geqq (2-\alpha)^3 > t\alpha(2-\alpha)^3/3$ for all $t \in [0, 1]$, $\alpha \in [-1, 1]$, we have

$$\frac{\partial}{\partial u}\left[\frac{2t\alpha(2-\alpha)}{3u} + \frac{u^2}{(2-\alpha)^2}\right] = \frac{2u}{(2-\alpha)^2} - \frac{2t\alpha(2-\alpha)}{3u^2} > 0.$$

Therefore by setting $u = 2 - \alpha$ we conclude that

$$g(\phi) \leqq -1/2 + 2t\alpha/3 - 2(3-2t)t\alpha^2/9.$$

Thus $g(\phi) \leqq 0$ for $-1 \leqq \alpha \leq 0$. For $0 < \alpha \leqq 1$, we have

$$g(\phi) \leqq -\frac{1}{2} + \frac{2}{3}t\alpha(1-\alpha) + \frac{4}{9}t^2\alpha^2$$
$$\leqq -\frac{1}{2} + \frac{2}{3}\alpha(1-\alpha) + \frac{4}{9}\alpha^2 = -\frac{1}{2}\left(1 - \frac{2}{3}\alpha\right)^2.$$

We conclude that $g(\phi) \leqq 0$ at each critical point of g when $\alpha \geqq -\cos\phi$ since $\max g(\phi) \leqq 0$.

Suppose $\alpha < -\cos\phi$. Since $-f(-z) = z - \alpha z^2 + \cdots$ is in C and $-\alpha > \cos\phi = -\cos(\pi + \phi)$, we have by what has already been proved that $2\rho_0 > 2t|\alpha|/3$ and for all ϕ

$$4\rho_0^2 - 4t\alpha^2/9 \geqq |2t(-\alpha)/3 + 2\rho_0 e^{i(\pi+\phi)}| = |2t\alpha/3 + 2\rho_0 e^{i\phi}|,$$

where $2\rho_0 = (2 + \cos(\phi + \pi))/(2 + \alpha) = (2 - \cos\phi)/(2 + \alpha)$. Therefore $z/2 + t\alpha z^2/3 - \rho_0 e^{i\phi}$ has no zero in E for each real ϕ and each $t \in [0, 1]$. Since the bounds on λ and $\mu = t/3$ are obtained from Lemma 3, this completes the proof.

4. **Proof of Theorem** 3. A function f is in T if and only if there is a probability measure β on $I = [-1, 1]$ such that

$$f(w) = \int_I \frac{w}{1 - 2sw + w^2} d\beta \qquad (w \in E)$$

[8, p. 54]. If we set $z = 2w/(1 + w^2)$, then $w = \eta(z) = z/[1 + (1 - z^2)^{1/2}]$ and

$$g(z) \equiv 2f(\eta(z)) = \int_I \frac{z}{1 - sz} d\beta \qquad (z \in E).$$

Thus,

$$zg'(z) = \int_I \frac{z}{(1 - sz)^2} d\beta \qquad (z \in E)$$

and the integrand is typically-real for each $s \in [-1, 1]$. It follows that

$$\operatorname{Im} z \operatorname{Im}\{zg'(z)\} = \int_I \operatorname{Im} z \operatorname{Im} \frac{z}{(1 - sz)^2} d\beta > 0$$

for $\operatorname{Im} z \neq 0$, that is, $zg'(z)$ is typically-real in E. Hence, $g(z) = 2f(\eta(z))$ is in C [3].

5. **Proof of Theorem** 4. If $f(w) = w + a_2w^2 + \cdots$ is in T, then $2f(\eta(z)) = z + a_2z^2/2 + \cdots$ is in C by Theorem 3. From Theorem 1, we conclude that $z/2 + \nu a_2 z^2/2 \propto 2f(\eta(z))$ for $z \in E$, where $0 \leqq \nu \leqq 1/3$. Since $\eta(z)$ is a Schwarz function, it follows that $f(\eta(z)) \propto f(z)$ for $z \in E$. Thus

$$(7) \qquad z/4 + \mu a_2 z^2 \propto f(\eta(z)) \propto f(z) \qquad (z \in E)$$

for $0 \leqq \mu = \nu/4 \leqq 1/12$.

It remains to show that for a real μ the subordination (7) holds for all $f \in T$ only if $0 \leqq \mu \leqq 1/12$. Toward this end, consider for each real ϕ the function

$$h(w, \phi) = \frac{w(1 + 2\cos\phi w + w^2)}{(1 - w^2)^2} \qquad (w \in E).$$

This function is a convex combination of the functions $w/(1 - w)^2$ and $w/(1 + w)^2$ in T and, hence, itself is in T. Since $w = \eta(z)$ implies $z = 2w/(1 + w^2)$, substitution shows $2h(\eta(z), \phi) = g(z, \phi)$, where g is the function in Lemma 3. The bounds on μ now follow from that lemma. We also conclude from this function that the factor 1/4 in the subordination (7) cannot be enlarged.

6. **Proof of Theorem** 5. We have $s_1(z/4)$ and $s_2(z/4)$ are subordinate to $f(\eta(z))$ and 1/4 is best possible by Theorem 4. The remainder of the argument is similar to that of Keogh [6]. Suppose $n \geqq 3$. For $f \in T$, let $\Delta = \{w: w = f(z) \text{ and } |z| < 1/4\}$, $G = \{z: |z + i| < \sqrt{2}\} \cap \{z: |z - i| < \sqrt{2}\}$ and let d be the positive distance from $\partial f(G)$ to $\partial\Delta$. If $r_n(z) = f(z) - s_n(z)$ and $|r_n(z/4)| \leqq d$, then $f(z/4) - r_n(z/4) \propto f(\eta(z))$ for $z \in E$. To estimate d, let P, Q respectively be points on the boundaries of $f(G)$ and Δ such that the open line segment γ from P to Q is of length d and lies in $f(G)/\Delta$. Let σ be the preimage in E of γ under the mapping $w = f(\eta(z))$. Then

$$d = \int_\sigma |f'(\eta(z))\eta'(z)|\,|dz|.$$

Since $\eta(z) = z/[1 + (1 + z^2)^{1/2}]$, the points z such that $|\eta(z)| < \lambda\,(<1)$ are contained in the disk $|z| < \mu = 2\lambda/(1 - \lambda^2)$. Indeed,

$$|z| = 2|\eta|/|1 + \eta^2| \leqq 2\lambda/(1 - \lambda^2) = \mu.$$

In particular, if $\lambda = 1/4$, then $\mu = 8/15 < \alpha$, where α is given in Lemma 4. Since $2f(\eta(z)) \in C$ by Theorem 3, we have $2zf'(\eta(z))\eta'(z)$ is in T. From Lemma 4, we conclude that

$$|f'(\eta(z))\eta'(z)| \geqq \frac{1 - |z|^2}{2(1 + |z|^2)^2} \qquad (\alpha \leqq |z| < 1).$$

Hence

$$d \geqq \frac{1}{2}\int_\alpha^1 \frac{1 - r^2}{(1 + r^2)^2}\,dr = \frac{1}{2}\int_\alpha^1 \frac{d}{dr}\left(\frac{r}{1 + r^2}\right)dr$$
$$= \frac{1}{4} - \frac{1}{2}\,\frac{\alpha}{1 + \alpha^2} \approx .04018.$$

Since $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in T implies $|a_j| \leqq j$ $(j = 2, 3, \ldots)$ [8, p. 55], we also have for $n \geqq 3$ that

$$|r_n(z/4)| \leqq \sum_{j=n+1}^{\infty} j4^{-j} = (3n + 4)/(9\cdot 4^n) \leqq 13/576 \approx .023.$$

This proves $|r_n(z/4)| < d$ for $n \geqq 3$ and, therefore, $s_n(z/4) \propto f(\eta(z))$ for all positive integers n.

7. **Proof of Theorem** 2. We have $s_1(z/2)$ and $s_2(z/2)$ are subordinate to $f \in C$ and that $1/2$ is best possible by Theorem 1. Define $r_n(z) = f(z) - s_n(z)$. For $0 < \lambda < 1$, $s_n(\lambda z) = f(\lambda z) - r_n(\lambda z) \propto f(z)$ if $|r_n(\lambda z)| \leqq d$, where d is the distance of $\partial f(E)$ to the boundary of $\{w\colon w = f(z) \text{ and } |z| < \lambda\}$. Then, by an argument like that in §6,

$$d = \int_\sigma |f'(z)|\,|dz|,$$

where σ is a certain contour in E. By Lemma 4, we have for $\lambda \leqq \alpha$

$$d \geqq \int_\lambda^\alpha dr/(1 + r)^2 + \int_\alpha^1 (1 - r^2)/(1 + r^2)^2 dr$$
$$= 1/2 - \alpha/(1 + \alpha^2) + \alpha/(1 + \alpha) - \lambda/(1 + \lambda).$$

Since $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in C implies $|a_j| \leqq 1$ $(j = 2, 3, \ldots)$ [10], it follows that

$$|r_n(\lambda z)| \leqq \sum_{j=n+1}^{\infty} \lambda^j = \lambda^{n+1}/(1 - \lambda).$$

In particular, if $\lambda = 1/2$ and $n \geqq 4$, we conclude that $|r_n(z/2)| \leqq 1/16$ and $d > 1/6 - \alpha^2(1 - \alpha)/2 \approx .0992$. This proves $s_n(z/2) \propto f(z)$ for $n \geqq 4$.

The above procedure proves $s_3(\lambda z) \propto f(z)$ for $\lambda \leqq .484$. We conjecture the subordination is true for all $f \in C$ when $\lambda = 1/2$.

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