ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 2, Spring 1981

INEQUALITIES FOR THE GENERALIZED TRANSFINITE DIAMETER

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ABSTRACT. Let E be a compact subset of a metric space X and f a Lipschitzian function on X. It is shown that $d(f(E)) \leqq Md(E)$, where d is the generalized transfinite diameter of Hille [2, 3] and M is the Lipschitz constant. Also, upper and lower bounds are obtained for the transfinite diameter of the union E of two "widely separated" compact sets, E_1 and E_2, in terms of the diameters of E, E_1, and E_2, the transfinite diameters of E_1 and E_2, and the distance between E_1 and E_2.

1. **Preliminaries.** The transfinite diameter in the complex plane was first introduced by Fekete in 1923. Pólya and Szegö extended the concept to three - dimensional space and showed that the transfinite diameter coincides with the capacity. Further generalizations of this concept were made by them and by Leja. Finally Hille, in two papers [2, 3], summarized and unified the previous generalizations. His papers contain bibliographies of previous work.

The present paper extends, in Theorem 1, a result of the author [6] from the complex plane to a metric space. Also inequalities are obtained for the transfinite diameter of the union of two "widely separated" sets.

2. **Averaging processes.** Let A be a function whose domain is all finite sequences of positive numbers. A is called an *averaging process* if it satisfies the following four axioms of Kolmogoroff [4]:

(i) $A[x_1, x_2, \ldots, x_n] > 0$ for all finite sequence $\{x_k\}$ of positive numbers;

(ii) A is a continuous, symmetric function of its arguments and is a strictly increasing function of each of them;

(iii) $A[x, x, \ldots, x] = x$; and

(iv) $A[x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n] = A[y, y, \ldots, y, x_{k+1}, \ldots, x_n]$ if $y = A[x_1, x_2, \ldots, x_k]$.

In addition, we will assume a fifth axiom:

(v) $A[kx_1, \ldots, kx_n] = kA[x_1, \ldots, x_n]$ for any $k > 0$.

We shall sometimes use the notation $A_{1 \leqq i \leqq n} x_i$ for $A[x_1, \ldots, x_n]$.

It can be proved [7] that an averaging process which satisfies (v) is of

AMS (MOS) subject classification (1970), Primary 31C15.
Received by the editors on October 6, 1979.

the form $(n^{-1}\sum_{i=1}^{n} x_i^r)^{1/r}$, r a real number $\neq 0$, or is the geometric mean. These are the mean values $\mathfrak{M}_r$ of Hardy, Littlewood, and Pólya [**1**, pp. 12–32]. With one exception we shall not use these formulas but will reason directly from the axioms.

3. **Transfinite diameters.** Let X be a metric space with distance function ρ, and let E be a compact subset of X. Let $n \geqq 2$, and let A be an averaging process satisfying axioms (i)–(v). Let $x_1, \ldots, x_n \in X$. We define

$$d_n(E) = \max_{x_i, x_j \in E} \left[\underset{1 \leqq i < j \leqq n}{A} \rho(x_i, x_j) \right].$$

It is shown in [**2**] that the sequence $d_n(E)$ is non-increasing. The transfinite diameter of E (with respect to the averaging process A) is defined as $d(E) = \lim_{n\to\infty} d_n(E)$.

The most familiar examples of the transfinite diameter are the Newtonian capacity ($X = \mathbf{R}^3$, A is the harmonic mean) and the logarithmic capacity ($X = \mathbf{C}$ or $\mathbf{R}^2$, A is the geometric mean). Further examples are the elliptic capacity and the hyperbolic capacity. [**8**, pp. 89–96]

4. THEOREM 1. *Let E be a compact subset of a metric space X, and let A be an averaging process satisfying* (i)–(v). *Let f be a function from X to X satisfying the Lipschitz condition: there exists a constant M such that for every $x, y \in X$, we have $\rho(f(x), f(y)) \leqq M\rho(x, y)$. Let $E^* = f(E)$. Then $d(E^*) \leqq Md(E)$.*

PROOF. Choose $y_1, \ldots, y_n \in E^*$ such that

$$d_n(E^*) = \underset{1 \leqq i < j \leqq n}{A} \rho(y_i, y_j).$$

Then there exist $x_1, \ldots, x_n \in E$ such that

$$d_n(E^*) = A[\rho(f(x_i), f(x_j))] \leqq A[M\rho(x_i, x_j)] = MA[\rho(x_i, x_j)] \leqq Md_n(E).$$

Now, let $n \to \infty$ in the above inequality.

THEOREM 2. *Let X be a Banach space, and let E be a compact, convex subset of X. Let $f\colon E \to X$ be a C^1 mapping.* Let $E^* = f(E)$. *Then $d(E^*) \leqq Md(E)$, where $M = \sup_{v\in E}\|f'(v)\|$.*

PROOF. Since E is convex, the line segment joining two points of E lies in E. Hence, by Corollary 1 on p. 314 of [**5**], f satisfies the Lipschitz condition with $M = \sup_{v\in E}\|f'(v)\|$. Now, apply Theorem 1.

5. **Widely separated sets.**

DEFINITION. Let E_1 and E_2 be compact subsets of a metric space X. Let D_1 and D_2 be the diameters of E_1 and E_2, respectively, and let D_0 be the (minimum) distance from E_1 to E_2. The sets E_1 and E_2 are said to be *widely separated* if $D_0 > \max(D_1, D_2)$.

THEOREM 3. *Let E_1 and E_2 be widely separated infinite compact subsets of a metric space X, and let $E = E_1 \cup E_2$. Let A be an averaging process satisfying* (i)–(v). *Let D, D_1, and D_2 be the diameters of E, E_1, and E_2, respectively, and let $D_0 = \rho(E_1, E_2)$. Then*

$$A[D_0, D'] \leqq d(E) \leqq A[D, D^*],$$

where $D' = A[d(E_1), d(E_2)]$ and $D^* =$ $\max(D_1, D_2)$.

PROOF. We adopt the notation $A[(x; m), (y; n)]$ for $A[x, \ldots, x, y, \ldots, y]$, where x is repeated m times and y is repeated n times.

Choose $x_1, \ldots, x_{2n} \in E$ such that $d_{2n}(E) = A_{1 \leq i < j \leq 2n}\, \rho(x_i, x_j)$. Suppose, without loss of generality, that $x_1, \ldots, x_{n+k}$ are in E_1 and $x_{n+k+1}, \ldots, x_{2n}$ are in E_2, where $0 \leqq k < n$. Then, using Axiom (iv) repeatedly and noting that $D^* \leqq D$,

$$
\begin{aligned}
(1) \qquad d_{2n}(E) &\leqq A[(D_1; (n+k)(n+k-1)/2), \\
&\qquad (D_2; (n-k)(n-k-1)/2), (D; n^2 - k^2)] \\
&\leqq A[(D^*; n^2 - n + k^2), (D; n^2 - k^2)] \\
&\leqq A[(D^*; n^2 - n), (D; n^2)] = A[(B; 2n^2 - 2n), (D; n)],
\end{aligned}
$$

where $B = A[D, D^*]$. Letting $n \to \infty$, we obtain $d(E) \leqq B$. The limit of the right hand side of (1) is seen to be B if the specific formulas for A, $\mathfrak{M}_r$ and the geometric mean, are used.

Choose $y_1, \ldots, y_n \in E_1$ and $y_{n+1}, \ldots, y_{2n} \in E_2$ such that $d_n(E_1) = A_{1 \leqq i < j \leqq n}\, \rho(y_i, y_j)$ and $d_n(E_2) = A_{n+1 \leqq i < j \leqq 2n}\, \rho(y_i, y_j)$. Then

$$
\begin{aligned}
d_{2n}(E) &\geqq A_{1 \leqq i < j \leqq 2n}\, \rho(y_i, y_j) \\
&= A[(d_n(E_1); n(n-1)/2), (d_n(E_2); n(n-1)/2), (D_0; n^2)] \\
&\geqq A[d(E_1); n(n-1)/2), (d(E_2); n(n-1)/2), (D_0; n^2)] \\
&= A[(D'; n^2 - n), (D_0; n^2)] = A[(D_0; n), (B_1; 2n^2 - 2n)],
\end{aligned}
$$

where $B_1 = A[D_0, D']$. Taking the limit as $n \to \infty$, we get $d(E) \geqq B_1$.

5. **Examples.** 1. We will estimate the logarithmic capacity of the union of two unit circles with centers 98 units apart. Then $D = 100$, $D_1 = D_2 = 2$, $D_0 = 96$, $d(E_1) = d(E_2) = 1$, A is the geometric mean.

$$9.80 \approx \sqrt{96.1} \leqq d(E) \leqq \sqrt{100 \cdot 2} \approx 14.14.$$

2. We will estimate the Newtonian capacity of the union of two unit spheres with centers 98 units apart. In this problem all quantities appear-

ing in Theorem 3 are exactly the same as in Example 1, except that A is the harmonic mean. We obtain $1.98 \leqq d(E) \leqq 3.92$.

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