ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 2, Spring 1981

STRENGTHENED MAXIMAL FUNCTIONS AND POINTWISE CONVERGENCE IN R^n

RICHARD J. BAGBY

1. **Introduction.** Questions relating to the pointwise a.e. convergence of a sequence of operators applied to a function in L^p are usually handled in terms of some maximal function which serves as a pointwise majorant for all terms in the sequence. A simple example of this is Lebesgue's differentiation theorem in R^n, where we are concerned with a sequence of averages of a function f over balls centered at x. The Hardy-Littlewood maximal function arises naturally in this problem and provides the key to this theorem as well as many other problems of pointwise convergence; see Stein [3].

If instead of taking averages of f over balls at x we take averages over more general sets, then a number of very interesting problems arise. See Guzmân [1] for a survey. Of course, for the averages to approach $f(x)$ we must require that the sets shrink to $\{x\}$ in some sense. For bounded continuous f little else in needed, but for f merely integrable the sets must shrink to $\{x\}$ regularly: the measure of each set in the sequence must be comparable to that of a ball centered at x and containing the set. For functions in L^p for $1 < p < \infty$ one expects some intermediate regularity condition to suffice; we develop such conditions here. We introduce set functions which measure the extent to which a set is concentrated near x; the appropriate regularity condition is to require that this set function be bounded by a multiple of Lebesgue measure on the sequence of sets considered. Our regularity condition is sufficiently general to allow us to deal with averages over unbounded sets.

In the process we introduce some new maximal functions which are useful for estimating convolution operators. We obtain estimates of the form

$$|K * f(x)| \leqq \|K\| F(x)$$

where F depends only on f. Such estimates are particularly useful when K depends on a parameter. In particular, when $K_\lambda(x) = \lambda^n K(\lambda x)$, the norms we introduce for K have the important property that $\|K_\lambda\| = \|K\|$. Consequently, we obtain some new sufficient conditions for pointwise

1980 AMS (MOS) subject classifications: Primary 28A15, 44A35, 46E30.
Received by the editors on July 27, 1979.

convergence of $K_\lambda * f$ which may be applicable even when K has an unbounded set of singularities.

2. **Preliminaries.** We shall denote points in R^n as x and Lebesgue measure as dx. The Euclidean length of x is $|x|$, and B_r denotes the ball $\{x: |x| \leqq r\}$.

For E a Lebesgue measurable subset of R^n, mE is its measure and χ_E is its characteristic function.

For f a measurable function on $\mathbf{R}^n$, $\|f\|_p$ denotes the usual L^p norm.

For μ a Borel measure on $\mathbf{R}^n$, $L^{pq}(d\mu)$ denotes a Lorentz space as defined in Hunt **[2]** or Stein and Weiss **[4]**. When μ is Lebesgue measure, we shall simply write L^{pq}. We shall always assume $1 < p < \infty$ and $1 \leqq q \leqq \infty$, so that $L^{pq}(d\mu)$ is a Banach space. A norm for $L^{pq}(d\mu)$ may be computed in terms of

$$f^{**}(t) = \sup_{\mu E \leqq t} t^{-1} \int_E |f| d\mu \tag{2.1}$$

from the formula

$$\|f\|^{pq} = \begin{cases} \left\{ \dfrac{q}{p} \displaystyle\int_0^\infty [t^{1/p} f^{**}(t)]^q t^{-1} dt \right\}^{1/q}, & q < \infty \\ \sup t^{1/p} f^{**}(t), & q = \infty. \end{cases} \tag{2.2}$$

When $q < \infty$ and $f \in L^{pq}(d\mu)$, the quantity $t^{1/p}f^{**}(t)$ is not only bounded but vanishes at 0 and ∞.

An alternate characterization of $L^{pq}(d\mu)$ is commonly used. Let f^* denote the non-negative non-increasing rearrangement of $|f|$ on $(0, \infty)$, i.e., for each $s > 0$ the set $\{t: f^*(t) > 0\}$ is an interval extending from 0 to $\mu\{x: |f|(x) > s\}$. Replacing $f^{**}(t)$ by $f^*(t)$ in the definition of $\|f\|_{pq}$ gives the functional denoted $\|f\|_{pq}^*$; it can be bounded from above and below in terms of $\|f\|_{pq}$.

Below we give another formula for computing $\|f\|_{pq}^*$. We suspect it is part of the folklore, although we have not found it in the literature except when $q = p$ or ∞.

PROPOSITION 2.3. *Let $M_f(s) = \mu\{x: |f(x)| > s\}$. Then*

$$\|f\|_{pq}^* = \begin{cases} \left\{ q \displaystyle\int_0^\infty M_f(s)^{q/p} s^{q-1}\, ds \right\}^{1/q}, & q < \infty \\ \sup s M_f(s)^{1/p}, & q = \infty. \end{cases}$$

PROOF. When $q = \infty$, this is well-known. When $q < \infty$, let $A = \{(s, t): f^*(t) > s > 0\}$. Then by Tonelli's theorem

$$\frac{q}{p}\int_0^\infty t^{q/p}f^*(t)^q t^{-1}\,dt = \frac{q^2}{p}\int_A\int s^{q-1}\,t^{q/p-1}\,dsdt$$
$$= q\int_0^\infty s^{q-1}M_f(s)^{q/p}\,ds.$$

The (centered) Hardy-Littlewood maximal function is defined by

$$Mf(x) = \sup_{r>0}\frac{1}{mB_r}\int_{B_r}|f(x+y)|\,dy.$$

There is a constant C, depending only on n, such that

$$m\{x\colon Mf(x) > s\} \leqq C\,\|f\|_1\,s^{-1},\quad 0 < s < \infty.$$

For $1 < p < \infty$ we also define

$$M_pf(x) = \sup_{r>0}\left\{\frac{1}{mB_r}\int_{B_r}|f(x+y)|^p\,dy\right\}^{1/p}.$$

We shall follow the usual practice of writing c for any constant whose value we do not wish to note explicitly.

Finally, we give a well-known result which we shall refer to occasionally.

Proposition 2.4. *Let f be a non-negative non-increasing function on* $(0,\infty)$, *such that* $\int_0^\infty t^{\alpha-1}f(t)\,dt < \infty$, where $\alpha > 0$. *Then* $\lim_{t\to\infty}t^\alpha f(t) = 0$.

Proof. $s^\alpha f(s) \leqq C\int_{s/2}^s t^{\alpha-1}f(t)dt \to 0$ as $d \to \infty$.

3. **Set functions $m_p(E)$ and $m_p^*(E)$.** First we define a pair of set functions which measure the degree to which a set is concentrated near the origin.

Definition 3.1. For $1 < p < \infty$ and $E \subset \mathbf{R}^n$ with $0 < mE < \infty$,

set

$$m_p(E) = \sup_{r>0}(mB_r)^{1/p}\,m(E \sim B_r)^{1-1/p}$$

and

$$m_p^*(E) = \frac{n}{p}\int_0^\infty (mB_r)^{1/p}\,m(E \sim B_r)^{1-1/p}\,r^{-1}dr.$$

Our first result shows the relation between mE, $m_p(E)$, and $m_p^*(E)$.

Proposition 3.2. $(1/2)\,mE \leqq m_p(E) \leqq m_p^*(E)$.

Proof. The first inequality follows from the observation that if $mB_r = (1/2)mE$, then $m(E \sim B_r) \geqq (1/2)mE$. For the second inequality, note

$$m_p^*(E) \geqq \frac{n}{p} \int_0^R (mB_r)^{1/p}\, m(E \sim B_r)^{1-1/p}\, r^{-1}\, dr$$
$$\geqq \left[\frac{n}{p} \int_0^R (mB_r)^{1/p}\, r^{-1}\, dr\right] m(E \sim B_R)^{1-1/p}$$
$$= (mB_R)^{1/p}\, m(E \sim B_R)^{1-1/p}.$$

Next we note the dependence of $m_p(E)$ and $m_p^*(E)$ on E.

PROPOSITION 3.3. *The set functions m_p and m_p^* are non-negative, monotone, and countably subadditive. The ratios $m_p(E)/mE$ and $m_p^*(E)/mE$ are invariant under dilations. If $E \subset B_R$, then we have*

$$m_p^*(E) \leqq (mB_R)^{1/p}\, (mE)^{1-1/p} \leqq mB_R.$$

PROOF. Non-negativity, monotonicity, and countable subadditivity follow from the corresponding properties for $m(E \sim B_r)^{1-1/p}$. For the dilation invariance, note

$$\lambda^n (mB_r)^{1/p} m(E \sim B_r)^{1-1/p} = (mB_{\lambda r})^{1/p}\, m(\lambda E \sim B_{\lambda r})^{1-1/p}.$$

A change of variables then give $m_p(\lambda E) = \lambda^n m_p(E)$ and $m_p^*(\lambda E) = \lambda^n m_p^*(E)$.
When $E \subset B_R$, we have

$$m_p^*(E) = \frac{n}{p} \int_0^R (mB_r)^{1/p} m(E \sim B_r)^{1-1/p}\, r^{-1} dr$$
$$\leqq \frac{n}{p} \int_0^R (mB_r)^{1/p} (mE)^{1-1/p}\, r^{-1} dr = (mB_R)^{1/p} (mE)^{1-1/p}.$$

Our next result shows that $m_p(E)$ and $m_p^*(E)$ are continuous with respect to translations of E.

PROPOSITION 3.4. For $0 < mE < \infty$ and $K > 1$, there is a $\delta > 0$ such that $|x| < \delta$ implies $m_p(x + E) \leqq Km_p(E)$ and $m_p^*(x + E) \leqq Km_p(E)$.

PROOF. For the first inequality, choose $R > 0$ such that

$$(mB_R)^{1/p} (mE)^{1-1/p} \leqq Km_p(E)$$

and then choose δ with $0 < \delta < R$ and $mB_R = K^p mB_{R-\delta}$. For $0 < r \leqq R$ we have clearly

$$(mB_r)^{1/p} m[(x + E) \sim B_r]^{1-1/p} \leqq (mB_R)^{1/p} (mE)^{1-1/p} \leqq Km_p(E).$$

For $r > R$ and $|x| < \delta$, $x + B_{r-\delta} \subset B_r$ implies

$$(mB_r)^{1/p} m[(x + E) \sim B_r]^{1-1/p} \leqq (mB_r)^{1/p} m(E \sim B_{r-\delta})^{1-1/p}$$
$$\leqq [mB_r/mB_{r-\delta}]^{1/p} m_p(E).$$

Since $mB_r/mB_{r-\delta}$ is a decreasing function of r, the desired estimate follows from our choice of δ.

The second estimate is proved similarly, except we must compare integrals instead of suprema. We choose $0 < \varepsilon < K - 1$ and then pick R so that

$$(mB_R)^{1/p}(mE)^{1-1p} \leqq \varepsilon m_p^*(E)$$

and then pick $\delta < R$ so that

$$mB_R = (K - \varepsilon)^p mB_{R-\delta}.$$

For $|x| < \delta$ we then have

$$\begin{aligned}
&\frac{n}{p}\int_0^R (mB_r)^{1/p} m[(x + E) \sim B_r]^{1-1/p}\, r^{-1}dr \\
&\qquad \leqq \frac{n}{p}\int_0^R (mB_r)^{1/p}(mE)^{1-1/p} r^{-1}dr \\
&\qquad = (mB_R)^{1/p}(mE)^{1-1/p} \leqq \varepsilon m_p^*(E)
\end{aligned}$$

as well as

$$\begin{aligned}
&\frac{n}{p}\int_R^\infty (mB_r)^{1/p} m[(x + E) \sim B_r]^{1-1/p} r^{-1}dr \\
&\leqq \frac{n}{p}\int_R^\infty (mB_r)^{1/p} m(E \sim B_{r-\delta})^{1-1/p} r^{-1}dr \\
&\leqq (K - \varepsilon)\left(\frac{n}{p}\right)\int_R^\infty (mB_{r-\delta})^{1/p} m(E \sim B_{r-\delta})^{1-1/p} r^{-1}dr \\
&\leqq (K - \varepsilon)\left(\frac{n}{p}\right)\int_{R-\delta}^\infty (mB_r)^{1/p} m(E \sim B_r)^{1-1/p} r^{-1}dr \\
&\leqq (K - \varepsilon)\, m_p^*(E).
\end{aligned}$$

Note that δ may be estimated in terms of n, p, mE, and K.

Our last result in this section shows the dependence of m_p and m_p^* on p.

PROPOSITION 3.5. *For $1 < p < q < \infty$ there is a constant c such that $m_q^*(E) \leqq cm_p(E)$.*

PROOF. Since it clearly suffices to consider the case $0 \leqq m_p(E) < \infty$, we may choose R such that $mB_R = m_p(E)$. Clearly we have

$$\begin{aligned}
&\frac{n}{q}\int_0^R (mB_r)^{1/q} m(E \sim B_r)^{1-1/q} r^{-1}dr \\
&\qquad \leqq \frac{n}{q}\int_0^R (mB_r)^{1/p}(mE)^{1-1/q} r^{-1}dr \\
&\qquad = (mB_R)^{1/q}(mE)^{1-1/q} \leqq cm_p(E)
\end{aligned}$$

Since we also have

$$(mB_r)m(E \sim B_r)^{q-1} = [(mB_r)m(E \sim B_r)^{p-1}]^{(q-1)/(p-1)}(mB_r)^{(p-q)/(p-1)}$$
$$\leqq m_p(E)^{p(q-1)/(p-1)} mB_r^{(p-q)/(p-1)},$$

we see that

$$\frac{n}{p}\int_R^\infty (mB_r)^{1/q} m(E \sim B_r)^{1-1/q} r^{-1} dr$$
$$\leqq m_p(E)^{p(q-1)/q(p-1)} \left(\frac{n}{q}\right)\int_R^\infty (mB_r)^{(p-q)/q(p-1)} r^{-1} dr$$
$$= cm_p(E)^{p(q-1)/q(p-1)}(mB_R)^{(p-q)/q(p-1)} = cm_p(E).$$

4. Strengthened maximal functions.

DEFINITION 4.1. For $1 < p < \infty$ and f integrable over sets of finite measure, we define

$$A_p f(x) = \sup \frac{1}{m_p(E)} \int |f(x + y)| \, dy$$

and

$$A_p^* f(x) = \sup \frac{1}{m_p^*(E)} \int_E |f(x + y)| \, dy$$

where the suprema are taken over all measurable sets E with $0 < mE < \infty$.

PROPOSITION 4.2. *$A_p f$ and $A_p^* f$ are upper semi-continuous functions.*

PROOF. Let us consider only $A_p f$; the arguments for $A_p^* f$ are similar. When f vanishes a.e. the function A_p vanishes identically; otherwise $A_p f(x) > 0$ for all x. Thus, it suffices to show that for each $s > 0$, the set where $A_p f(x) > s$ is open.

If $A_p f(x) > s$, then there is a $K > 1$ and a set E with $0 < m_p(E) < \infty$ such that

$$\int_E |f(x + y)| \, dy > Ksm_p(E).$$

By (3.4), there is a $\delta > 0$ such that $M_p(z + E) \leqq Km_p(E)$ for all z with $|z| < \delta$. Thus we have

$$\int_{z+E} |f(x - z + y)| \, dy = \int_E |f(x + y)| \, dy$$
$$> Ksm_p(E) \geqq sm_p(z + E).$$

Hence $A_p f(x - z) > s$ for all z with $|z| < \delta$, and we are done.

In view of (3.2) and (3.3) we can easily establish

$$Mf(x) \leqq A_p^*f(x) \leqq A_pf(x).$$

Now we estimate A_pf and A_p^*f from above.

LEMMA 4.3 *For $f \in L^\infty$ we have*

$$A_pf(x) \leqq 2 \|f\|_\infty^{1-1/p} Mf(x)^{1/p}.$$

PROOF. Fix E with $0 < m_p(E) < \infty$. For each $r > 0$, we have

$$\begin{aligned}\int_E |f(x+y)|\,dy &\leqq \int_{B_r} |f(x+y)|\,dy + \int_{E\sim B_r} |f(x+y)|\,dy \\ &\leqq (mB_r)Mf(x) + m(E \sim B_r)\,\|f\|_\infty.\end{aligned}$$

Setting $t^{p-1} = mB_r/m_p(E)$, the definition of $m_p(E)$ shows that $m(E \sim B_r)/\, m_p(E) \leqq t^{-1}$. Hence

$$\frac{1}{m_p(E)}\int_E |f(x+y)|\,dy \leqq t^{p-1}Mf(x) + t^{-1}\,\|f\|_\infty.$$

Choose r to make $t = \|f\|_\infty^{1/p}\, Mf(x)^{-1/p}$.

LEMMA 4.4. *There is a constant c, depending only on n and p, such that $A_p^*f(x) < cM_pf(x)$.*

PROOF. Given E, set $E_k = \{y \in E: 2^k < |y| \leqq 2^{k+1}\}$.

$$\begin{aligned}\int_{E_k} |f\,(x+y)|\,dy &\leqq \left(\int_{|y|\leqq 2^{k+1}} |f(x+y)|^p dy\right)^{1/p} (mE_k)^{1-1/p} \\ &\leqq M_pf(x)\,(mB_{2^{k+1}})^{1/p} m(E \sim B_{2^k})^{1-1/p}.\end{aligned}$$

Using the monotonicity of $m(E \sim B_r)$, it is routine to show

$$\sum_{k=-\infty}^{\infty} (mB_{2^{k+1}})^{1/p} m(E \sim B_{2^k})^{1-1/p} \leqq cm_p^*(E).$$

REMARK 4.5. The inequality in (4.4) can be sharpened somewhat; $M_pf(x)$ can be replaced by the quantity

$$W_pf(x) = \sup(mB_r)^{-1/p}\,\|f|_{x+B_r}\|_{p\infty}.$$

Thus $A_p^*f(x)$ is finite a.e. for $f(x) = |x|^{-n/p}$, although in this case $M_pf(x)$ is infinite everywhere.

Since for $E \subset B_r$ we have

$$\begin{aligned}\int_E |f(x+y)|\,dy &\leqq m_p(E)A_pf(x) \\ &\leqq (mB_r)^{1/p}(mE)^{1-1/p}A_pf(x),\end{aligned}$$

we can also establish the inequality $W_pf(x) \leqq A_pf(x)$.

THEOREM 4.6. *Both A_p and A_p^* are bounded in L^{rq} for $p < r < \infty$ as well as in L^∞. Moreover, there are constants depending only on n and p such that*

$$\|A_p f\|_{p\infty} \leqq c\|f\|_{p1} \text{ and } \|A_p^* f\|_{p\infty} \leqq c\|f\|_p.$$

PROOF. In view of (4.4), the estimates for $A_p^* f$ follow immediately from the corresponding estimates for $M_p f$.

For $A_p f$, we use the methods of Stein and Weiss [**4**, chap. V, §3]. Since $Mf(x) \leqq \|f\|_\infty$, (4.3) tells us that $\|A_p f\|_\infty \leqq 2\|f\|_\infty$. When $f = \chi_E$, the characteristic function of a measurable set E of finite measure, (4.3) tells us

$$A_p \chi_E(x) \leqq 2(M\chi_E(x))^{1/p}$$

and hence

$$m\{x: A_p\chi_E(x) > s\} \leqq c(mE)s^{-p}.$$

Thus, $A_p f$ is of restricted weak type (p, p). Since $A_p f$ is subadditive and positive homogenous, the desired conclusions follow at once from the Marcinkiewicz interpolation theorem and Theorem 3.13 of Stein and Weiss.

5. **Applications to differentiation.**

DEFINITION 5.1. A *differentiating sequence* is a sequence $\{E_k\}_{k=1}^\infty$ of measurable sets in $\mathbf{R}^n$, each having positive finite measure, with the property that for every $r > 0$,

$$\lim_{k\to\infty} \frac{m(E_k \sim B_r)}{mE_k} = 0.$$

Note that we do not assume that the diameters of the sets approach 0 or even that the sets are bounded. Nevertheless, the result below is trivial.

PROPOSITION 5.2. *If f is a continuous bounded function on $\mathbf{R}^n$, then for each x and each differentiating sequence $\{E_k\}$ we have*

$$\lim_{k\to\infty} \frac{1}{mE_k}\int_{E_k} f(x+y)\,dy = f(x).$$

PROOF. Suppose $|f(x+y) - f(x)| < \varepsilon$ on B_r. We then have

$$\begin{aligned}
&\left|\frac{1}{mE_k}\int_{E_k} f(x+y)dy - f(x)\right| \\
&\leqq \frac{1}{mE_k}\int_{E_k} |f(x+y) - f(x)|\,dy \\
&\leqq \frac{1}{mE_k}\int_{E_k\cap B_r} |f(x+y) - f(x)|\,dy + \frac{1}{mE_k}\int_{E_k\sim B_r} |f(x+y) - f(x)|\,dy \\
&< \varepsilon + \frac{m(E_k \sim B_r)}{mE_k}(\|f\|_\infty + |f(x)|).
\end{aligned}$$

Since f is continuous, we may take ε as small as we please by choosing r appropriately; the second term then approaches 0 for each differentiating sequence.

Let us note that in addition to the continuity of f at x, all we really need for the proof is some way to show

$$\lim_{k\to\infty} \frac{1}{mE_k} \int_{E_k \sim B_r} |f(x+y)|\, dy = 0 \text{ for each } r > 0. \tag{5.3}$$

By restricting $\{E_k\}$ appropriately, we can establish (5.3) for f in a larger class than L^∞.

DEFINITION 5.4. A differentiating sequence $\{E_k\}$ is said to be p-regular if $m_p(E_k)/mE_k$ is bounded independently of k. Similarly, if $m_p^*(E_k)/mE_k$ is bounded, we call the sequence p^*-regular.

THEOREM 5.5. *Suppose $f \in L^\infty + L^{pq}$, where $1 \leqq q < \infty$. If f is continuous at x, then*

$$\lim_{k\to\infty} \frac{1}{mE_k} \int_{E_k} f(x+y)dy = f(x)$$

for every p-regular differentiating sequence $\{E_k\}$.

PROOF. All we need to do is establish (5.3); since we have previously done this when $f \in L^\infty$, we need only consider the case $f \in L^{pq}$. For $f \in L^{pq}$, the relevant property of f is

$$\lim_{t\to 0} t^{1/p} f^{**}(t) = 0.$$

If we set $t_k = m(E_k \sim B_r)$, then $t_k \to 0$ and the definition of f^{**} shows

$$\frac{1}{mE_k} \int_{E_k \sim B_r} |f(x+y)|\, dy \leqq \frac{t_k}{mE_k} f^{**}(t_k).$$

Since $t_k \leqq (mB_r)^{-1/p} m_p(E_k) t_k^{1/p}$, we see (5.3) must hold for each p-regular sequence.

We may note that (5.5) gives us a principle of localization for p-regular differentiating sequences: if $f \in L^\infty + L^{pq}$, then the limiting behavior of the averages of f over $x + E_k$ depends only on the behavior of f near x, even though each E_k may be unbounded.

EXAMPLE 5.6. We show that for $f \in L^{p\infty}$ the conclusion of (5.5) may fail even for p^*-regular differentiating sequences. First we choose

$$f(x) = \begin{cases} x^{-1/p}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

For $x > 1$ and $0 < t < 1$ set $E_t = (-x, \ -x + t^q) \cup (0, t)$ where q satisfies $1/p + 1/q = 1$. Since $mE_t \leqq 2t$, we have

$$\frac{1}{mE_t} \int_{E_t} f(x + y)dy \geqq \frac{1}{2t} \int_0^{t^q} y^{-1/p}dy = q/2.$$

It is simple to check that $m_p^*(E_t)/mE_t$ is bounded independently of t, so any sequence $t_k \to 0$ gives a p^*-regular differentiating sequence such that (5.5) fails at x, $1 < x < \infty$.

It turns out that differentiating sequences obtained by shrinking a fixed set can produce less pathology. For example, if $E_k = \lambda_k E$ where $\lambda_k \to 0$ and $m_p^*(E) < \infty$, then $\{E_k\}$ is a p^*-regular differentiating sequence such that the conclusion of (5.5) holds for all f in $L^{P\infty}$. This is a special case of one of our results in §6 below. However, merely assuming $m_p(E) < \infty$ is not sufficient. For a counterexample, use the same function f above,

$$E = \bigcup_{k=0}^{\infty} (2^k, 2^k + 2^{-kq/p})$$

and $\lambda_k = 2^{-k}\,|x|$ where $-2^k < x < 0$.
One can then compute

$$\frac{1}{mE_k} \int_{E_k} f(x + y)dy > c\,|x|^{-1/p},$$

even though f vanishes on a neighborhood of x.

THEOREM 5.7. *Suppose $f \in L^{p1} + L^\infty$. Then for all x outside an exceptional set of measure* 0, *we have*

$$\lim_{k\to\infty} \frac{1}{mE_k} \int_{E_k} f(x + y)dy = f(x)$$

for all p-regular differentiating sequences $\{E_k\}$. If we consider only p^*-regular sequences, we need only require $f \in L^p + L^\infty$.

PROOF. The argument is fairly standard. In view of the principle of localization given by (5.5), we may as well assume $f \in L^{p1}$.

Let $E_{\varepsilon,K}$ be the set of all x such that for some differentiating sequence $\{E_k\}$ with $m_p(E_k) \leqq KmE_k$, we have

$$\limsup_{k\to\infty} \frac{1}{mE_k} \int_{E_k} |\, f(x + y) - f(x)\,|\, dy \geqq \varepsilon.$$

Then it suffices to prove $mE_{\varepsilon,K} = 0$.

For any bounded continuous function g we have

$$\frac{1}{mE_k}\int_{E_k}|f(x+y)-f(x)|\,dy$$
$$\leqq KA_p(f-g)(x)+|f(x)-g(x)|+\frac{1}{mE_k}\int_{E_k}|g(x+y)-g(x)|\,dy.$$

By (5.5), the last term approaches 0 as $k\to\infty$; hence for each $x\in E_{\varepsilon,k}$ we must have

$$KA_p(f-g)(x)+|f(x)-g(x)|\geqq\varepsilon.$$

Hence by (4.6), $mE_{\varepsilon,k}\leqq c[(K+1)\,\|f-g\|_{p1}\,\varepsilon^{-1}]^p$, and since bounded continuous functions are dense in L^{p1} we must have $mE_{\varepsilon,k}=0$.

In the case of p^*-regular differentiating sequences, we estimate analogously using A_p^* instead of A_p, so that we may obtain

$$mE_{\varepsilon,k}\leqq C[(K+1)\,\|f-g\|_p\,\varepsilon^{-1}]^p.$$

EXAMPLE 5.8. Here we construct a function $f\in L^p(\mathbf{R}^1)$, $2<p<\infty$, such that for each x in a set of positive measure there is a p-regular differentiating sequence $\{E_k\}$ for which

$$\lim_{k\to\infty}\frac{1}{mE_k}\int_{E_k}f(x+y)dy=\infty.$$

First we construct a Cantor-like set $P\subset[0,1]$ having positive measure. Let $P_1=[0,1]$. Assuming P_k consists of 2^{k-1} closed subintervals of $[0,1]$ with midpoints a_{jk}, we define

$$P_{k+1}=P_k\sim\bigcup_{j=1}^{2^{k-1}}(a_{jk}-\varepsilon_k,a_{jk}+\varepsilon_k)$$

where $\varepsilon_k=2^{-k-kq}$ and q satisfies $1/p+1/q=1$. We then set $P=\bigcap_{k=1}^{\infty}P_k$. Clearly

$$m([0,1]\sim P)=\sum_{k=1}^{\infty}2^{k-1}\cdot 2\varepsilon_k$$
$$=\sum_{k=1}^{\infty}2^{-kq}=\frac{1}{2^q-1}<1,$$

so that $mP>0$. Moreover if $x\in P$ and k is given, then for some j we have $|a_{jk}-x|\leqq 2^{-k}$.

Next we construct a function f which vanishes on P and whose restriction to any component of $[0,1]\sim P$ is not in L^{p1}, although $f\in L^p$. Fix θ with $1/p<\theta<1/2$ and choose c large enough so that

$$F(t)=t^{1-1/p}\left[\log\frac{c}{t}\right]^{-\theta}$$

defines a concave function on [0, 1]. Then for each k there is a non-negative, non-increasing function f_k supported on $(0, \varepsilon_k)$ such that

$$\int_0^t f_k(s) = \varepsilon_k^{1-1/p} F(t/\varepsilon_k), \quad 0 < t \leqq \varepsilon_k.$$

By monotonicity, $f_k(t) \leqq 1/t\int_0^t f_k(s)ds$ and hence

$$\|f_k\|_p^p \leqq \int_0^{\varepsilon_k} t^{-1}[\log(c\varepsilon_k/t)]^{-\theta p} dt = c.$$

Now set

$$f(x) = \sum_{k=1}^{\infty} 2^{-k/p} k^{-\theta} \sum_{j=1}^{2^k-1} f_k(|x - a_{jk}|).$$

Since the intervals $(a_{jk} - \varepsilon_k, a_{jk} + \varepsilon_k)$ are pairwise disjoint, we have

$$\|f\|_p^p = \sum_{k=1}^{\infty} 2^{-k} k^{-\theta p} \sum_{j=1}^{2^k-1} 2\|f_k\|_p^p$$
$$= \sum_{k=1}^{\infty} ck^{-\theta p} < \infty.$$

Now we carefully select some sets over which we will average f. If we set

$$t_k = 2^{(k-K)/(p-1)} \varepsilon_K$$

for $k = 1, \ldots, K$, we may then compute

$$\int_0^{t_k} f_k(t)dt = 2^{(k-K)/p} \varepsilon_K^{1-1/p} [\log c + \log 2^{(K-k)(2q)}]^{-\theta}$$
$$= c2^{(k-K)/p} \varepsilon_K^{1-1/p} [1 + c(K - k)]^{-\theta},$$

Consequently, for each $x \in P$ there is a set E_K consisting of pairwise disjoint intervals of lengths $t_1, \ldots, t_K$ such that

$$\int_{E_k} f(x + y)dy = \sum_{k=1}^{K} 2^{-k/p} k^{-\theta} \int_0^{t_k} f_k(t)dt$$
$$= 2^{-K/p} \varepsilon_K^{1-1/p} \sum_{k=1}^{K} k^{-\theta}[1 + c(K - k)]^{-\theta}$$
$$\geqq c2^{-K/p} \varepsilon_K^{1-1/p} K^{1-2\theta}.$$

Moreover, due to the location of the points a_{jk}, E_K can be selected so that the interval of length t_k lies inside $(-2^{-k}, 2^{-k})$.

Since the lengths $t_1, \ldots, t_K$ form an increasing geometric series, we see mE_K is on the order of $t_K = \varepsilon_K$. Also, when $r = 2^{-k-1}$, $m(E_K \sim B_r)$ is bounded by a fixed multiple of t_k, so that

$$(mB_r)^{1/p} m(E_K \sim B_r)^{1-1/p} \leqq c 2^{-k/p} t_k^{1-1/p} = c 2^{-K/p} \varepsilon_K^{1-1/p}.$$

Since we may clearly estimate $m_p(E_K)$ by considering only $r = 2^{-k-1}$, $k = 1, \ldots, K$, we thus have

$$m_p(E_K) \leqq c 2^{-K/p} \varepsilon_K^{1-1/p}.$$

While $m_p(E_K) \to 0$ as $K \to \infty$, the best estimate we have for $m_p(E_K)/mE_K$ is

$$m_p(E_K)/mE_K \leqq c 2^{-K/p} \varepsilon_K^{-1/p} = 2^{Kq/p}$$

so that the sequence $\{E_K\}$ is not p-regular. If we set

$$E_K^* = E_K \cup (-m_p(E_K), m_p(E_K)),$$

then clearly $\{E_K^*\}$ is a p-regular differentiating sequence. Moreover, we have

$$\frac{1}{mE_K^*} \int_{E_K^*} f(x + y) dy \geqq c K^{1-2\theta} \to \infty$$

since $\theta < 1/2$.

The example above also shows that A_p is not of weak-type (p, p) and hence is not dominated by M_p.

6. **Applications to convolution operators.** First we obtain bounds for convolution operators in terms of the maximal functions A_p and A_p^*.

THEOREM 6.1. *Let K be a non-negative measurable function on $\mathbf{R}^n$, and let $E_t = \{x : K(x) > t\}$. If f is non-negative and integrable over sets of finite measure, then*

$$K * f(x) \leqq \left(\int_0^\infty m_p(E_t)\, dt \right) A_p f(x)$$

and

$$K * f(x) \leqq \left(\int_0^\infty m_p^*(E_t)\, dt \right) A_p^* f(x).$$

PROOF. Let $G = \{(y, t) \colon K(y) > t > 0\}$. Then since

$$K(y) = \int_0^\infty \chi_G(y, t) dt,$$

$$K * f(x) = \int_G \int f(x - y)\, dy dt = \int_0^\infty \left(\int_{E_t} f(x - y) dy \right) dt.$$

We may then estimate the inner integral by either $m_p(E_t) A_p f(x)$ or $m_p^*(E_t) A_p^* f(x)$.

COROLLARY 6.2. *If* $K_\lambda(x) = \lambda^n K(\lambda x)$, *then* $\sup_\lambda K_\lambda * f(x)$ *satisfies the estimates for* $K * f(x)$ *given above.*

PROOF. For a fixed $\lambda > 0$, note that

$$\{y: K_\lambda(y) > t\} = \lambda^{-1}E_{\lambda^{-n}t}$$

so that by (3.3),

$$\int_0^\infty m_p\{y: K_\lambda(y) > t\}dt = \int_0^\infty m_p(\lambda^{-1}E_{\lambda^{-n}t})dt$$
$$= \int_0^\infty \lambda^{-n} m_p(E_{\lambda^{-n}t})\,dt = \int_0^\infty m_p(E_t)dt.$$

Exactly the same arguments apply to m_p^*. Thus, when K is replaced by K_λ, the estimates in (6.1) hold uniformly in λ.

REMARK 6.3. The estimate for convolutions given in Stein [3] is

$$\sup_\lambda K_\lambda * f(x) \leqq \left(\int \psi(y)dy\right) Mf(x)$$

where ψ is a decreasing radial function with $0 \leqq K \leqq \psi$. When $K = \psi$, then $E_t = B_r$ for some r and hence

$$(1/2)\, mE_t \leqq m_p(E_t) \leqq m_p^*(E_t) \leqq mE_t$$

by (3.2) and (3.3). Since $\int\psi = \int_0^\infty mE_t dt$, in this case we have

$$\frac{1}{2}\int \psi \leqq \int_0^\infty m_p(E_t)dt \leqq \int_0^\infty m_p^*(E_t)dt \leqq \int \psi.$$

While Stein's estimate is applicable when f is merely integrable, it is not applicable if K has a singularity away from the origin. Indeed, Zo [5] showed that for such a kernel, there is always an integrable f such that $\sup_\lambda K_\lambda * f(x)$ is completely unmanageable, although there is some hope when $f \in L^p$, $p > 1$. He gives an example of such an estimate when K has compact support. Below we give a slight extension of his estimate.

THEOREM 6.4. *Let f and K be non-negative measurable functions on* $\mathbf{R}^n$, *and let* $K^{(r)}$ *be K truncated to vanish on* B_r. *Then*

$$\sup_\lambda K_\lambda * f(x) \leqq c\left(\int_0^\infty (mB_r)^{1/p} \left\|K^{(r)}\right\|_q r^{-1}dr\right) M_p f(x),$$

where $1/p + 1/q = 1$.

PROOF. Since $(K_\lambda)^{(r)}(x) = \lambda^n K^{(\lambda r)}(\lambda x)$, we see that

$$\|(K_\lambda)^{(r)}\|_q = \lambda^{n/p}\|K^{(\lambda r)}\|_q$$

and

$$(mB_r)^{1/p}\|(K_\lambda)^{(r)}\|_q = (mB_{\lambda r})^{1/p}\|K^{(\lambda r)}\|_q,$$

so that $\int_0^\infty (mB_r)^{1/p}\|K^{(r)}\|_q\, r^{-1}\, dr$ is unchanged when K is replaced by K_λ. Thus it suffices to estimate $K * f(x)$. But we have

$$\int_{2^k \leq |y| \leq 2^{k+1}} f(x-y)K(y)dy \leq \left(\int_{|y|\leq 2^{k+1}} f(x-y)^p dy\right)^{1/p}\|K^{(2^k)}\|_q$$
$$\leq (mB_{2^{k+1}})^{1/p} M_p f(x)\, \|K^{(2^k)}\|_q.$$

When we sum over k, the right-hand side is bounded by the desired integral.

We may note that the integrals appearing in (6.1) and (6.4) may be used to estimate $\|K\|_1$; we simply take $f \equiv 1$.

Our next result compares the integrals in (6.1) to some more familiar-looking functionals. This allows us to gauge the relative strengths of the estimates in (6.1) and (6.4).

THEOREM 6.5. *For K a non-negative measurable function on $\mathbf{R}^n$ and $1/p + 1/q = 1$ we have*

$$\int_0^\infty m_p(E_t)dt \leq (mB_1)^{1/p}\, \|K\|^*_{L^{q1}(|x|^{nq/p}dx)} \leq q^{1/q}\int_0^\infty m_p^*(E_t)dt$$

and

$$\int_0^\infty m_p^*(E_t)\, dt = \int_0^\infty (mB_r)^{1/p}\, \|K^{(r)}\|^*_{q1}\, r^{-1}\, dr.$$

PROOF. The last equation is the simplest; we have

$$\int_0^\infty m_p^*(E_t)dt = \int_0^\infty\left(\int_0^\infty (mB_r)^{1/p}\, m(E_t \sim B_r)^{1/q}\, r^{-1}dr\right)dt$$
$$= \int_0^\infty (mB_r)^{1/p}\left(\int_0^\infty m(E_t \sim B_r)^{1/q}dt\right)r^{-1}dr$$

and since $m(E_t \sim B_r)$ is the distribution function for $K^{(r)}$ with respect to Lebesgue measure,

$$\int_0^\infty m(E_t \sim B_r)^{1/q}dt = \|K^{(r)}\|^*_{q1}$$

by (2.3).

For the inequalities we need to estimate the distribution function of K with respect to $d\mu = |x|^{nq/p}\, dx$. By definition,

$$M_K(t) = \int_{E_t} |x|^{nq/p} dx.$$

Since $|x|^{nq/p} \geqq r^{nq/p}$ on $E_t \sim B_r$, we have $M_K(t) \geqq r^{nq/p} m(E_t \sim B_r)$, which gives

$$(mB_1)^{1/p} M_K(t)^{1/q} \geqq (mB_r)^{1/p} m(E_t \sim B_r)^{1/q}$$

and hence $m_p(E_t) \leqq (mB_1)^{1/p} M_K(t)^{1/q}$. Again using (2.3), integration gives the first inequality.

For the second inequality, let us note that

$$|x|^{nq/p} \chi_{E_t}(x) = \frac{nq}{p} (mB_1)^{-1} \int_0^\infty (mB_r)^{q/p} \chi_{E_t \sim B_r}(x)\, r^{-1} dr.$$

Hence

$$\begin{aligned} M_K(t) &= \int |x|^{nq/p} \chi_{E_t}(x) dx \\ &= \frac{nq}{p} (mB_1)^{-1} \int_0^\infty (mB_r)^{q/p} m(E_t \sim B_r) r^{-1} dr \\ &= \frac{nq}{p} (mB_1)^{-1} \int_0^\infty [(mB_r)^{1/p} m(E_t \sim B_r)^{1-1/p}]^q r^{-1} dr \\ &\leqq \frac{nq}{p} (mB_1)^{-1} m_p(E_t)^{q-1} \int_0^\infty (mB_r)^{1/p} m(E_t \sim B_r)^{1-1/p} r^{-1} dr \\ &= q (mB_1)^{-1} m_p(E_t)^{q-1} m_p^*(E_t) \\ &\leqq q (mB_1)^{-1} m_p^*(E_t)^q. \end{aligned}$$

Taking a q-th root and integrating gives the last inequality.

REMARK 6.6. A comparison of

$$\|K\|^*_{L^{q1}(|x|^{nq/p}dx)}, \int_0^\infty (mB_r)^{1/p} \|K^{(r)}\|_q\, r^{-1} dr,$$

and

$$\int_0^\infty (mB_r)^{1/p} \|K^{(r)}\|^*_{q1}\, r^{-1} dr$$

shows that the last dominates the second, while the first two do not seem to be directly comparable. This is what one would expect from the relative strengths of $A_p f$, $M_p f$, $A_p^* f$.

Now we turn our attention to some problems of pointwise convergence.

THEOREM 6.7. *Let K be an integrable function on $\mathbf{R}^n$, and let $I = \int K(x) dx$. Assume one of the following holds:*

(a) $\int_0^\infty (mB_r)^{1/p} \|K^{(r)}\|_q r^{-1}dr < \infty$,

(b) $\int_0^\infty m_p(E_t)dt < \infty$,

or

(c) $\int_0^\infty m_p^*(E_t)dt < \infty$,

where $1/p + 1/q = 1$, $1 < p < \infty$, $E_t = \{x: |K(x)| > t\}$, *and* $K^{(r)}$ *is the restriction of* K *to the complement of* B_r. *Then we have*

(i) $\lim_{\lambda\to\infty} K_\lambda * f(x) = \text{If}(x)$ *at each point where* f *is continuous provided* $f \in L^p + L^\infty$ *and* (a) *holds*, $f \in L^{ps} + L^\infty$ *with* $s < \infty$ *and* (b) *holds, or* $f \in L^{p\infty} + L^\infty$ *and* (c) *holds, and*

(ii) $\lim_{\lambda\to\infty} K_\lambda * f(x) = If(x)$ *almost everywhere provided* $f \in L^p + L^\infty$ *and* (a) *holds or* $f \in L^{p1} + L^\infty$ *and* (b) *holds.*

PROOF. As in (5.5), (i) follows whenever we can prove

$$\lim_{\lambda\to\infty} \int_{|y|\geqq r} |f(x-y)|\,|K_\lambda(y)|dy = 0,$$

and since K is integrable this is true when $f \in L^\infty$.

If case (a) we have

$$\int_{|y|\geqq r} |f(x-y)|\,|K_\lambda(y)|dy \leqq \|f\|_p\Big(\int_{|y|\geqq r} |K_\lambda(y)|^q\,dy\Big)^{1/q}$$
$$= \|f\|_p \lambda^{n/p} \|K^{(\lambda r)}\|_q \to 0$$

by (2.4) and (a). The argument for case (c) is similar; we use the Lorentz space version of Hölder's inequality.

The argument in case (b) is more complicated. Since

$$|K_\lambda(y)| = \lambda^n|K(\lambda y)| = \lambda^n \int_0^\infty \chi_{E_t}(\lambda y)dt$$
$$= \lambda^n \int^\infty \chi_{\lambda^{-1}E_t}(y)\,dt,$$

we have

$$\int_{|y|\geqq r} |f(x-y)|\,|K_\lambda(y)|dy = \lambda^n \int_0^\infty \Big(\int_{\lambda^{-1}E_t\sim B_r} |f(x-y)|\,dy\Big)dt$$
$$\leqq \lambda^n \int_0^\infty \rho_{t,\lambda} f^{**}(\rho_{t,\lambda})\,dt$$

where we have set $\rho_{t,\lambda} = m(\lambda^{-1}E_t \sim B_r)$. Since

$$\rho_{t,\lambda}^{1-1/p} \leqq m_p(\lambda^{-1}E_t)\,(mB_r)^{-1/p},$$

we have

$$\lambda^n \rho_{t,\lambda}^{1-1/p} \leqq m_p(E_t)(mB_r)^{-1/p}.$$

Since $\rho_{\lambda,t}^{1/p} f^{**}(\rho_{t,\lambda})$ is uniformly bounded and tends to 0 as $\lambda \to \infty$ for $f \in L^{p,s}$, we then have

$$\lim_{\lambda\to\infty} \lambda^n \int_0^\infty \rho_{t,\lambda} f^{**}(\rho_{t,\lambda})\,dt = 0$$

by dominated convergence.

The proof of (ii) is fairly routine. If g is a bounded continuous function, then we have

$$\limsup_{\lambda\to\infty} |K_\lambda * f(x) - If(x)| \leqq |I| \, |f(x) - g(x)| + cM_p(f-g)(x)$$

in case (a) and

$$\limsup_{\lambda\to\infty} |K_\lambda * f(x) - If(x)| \leqq |I| \, |f(x) - g(x)| + cA_p(f-g)(x)$$

in case (b); as in (5.7) the limit must be 0 almost everywhere. Note that (i) allows us to localize so that once again it suffices to consider $f \in L^p$ in case (a) and $f \in L^{p1}$ in case (b).

References

1. M. de Guzmán, *Differentiation of integrals in* $\mathbf{R}^n$, Lecture notes in Mathematics **481**, Springer-Verlag, 1975.

2. R. A. Hunt, *On L(p, q) spaces*, L'Ens. Math. **12**(1966), 249–275.

3. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

4. ——— and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

5. F. Zo, *A note on approximation of the identity*, Studia Math. **40** (1976), 111–122.

New Mexico State University, Las Cruces, NM 88001