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BORN SERIES AND SCATTERING BY TIME-DEPENDENT POTENTIALS

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ABSTRACT. A general two-body Schroedinger operator without point spectrum is perturbed by a time-dependent potential. The wave operators exist and are unitary, provided that the perturbation is dominated for large times by a small time-independent potential. When there is point spectrum, the perturbation must satisfy an integrability condition with respect to time.

0. **Introduction.** This paper concerns the scattering theory of the time-dependent Schroedinger operator

$$H(t) = -\Delta + V_1(x) + q(x, t)$$

where $V_1(x)$ and $q(x, t)$ are real valued functions on $\mathbf{R}^n$, $n \geqq 3$.
Let

$$H_1 = -\Delta + V_1(x)$$

be the unperturbed Hamiltonian, and let $U(t, s)$ be the propagator for the evolution equation

$$i\frac{\partial\psi}{\partial t} = H(t)\psi.$$

We are interested in existence and unitarity of the wave operators

$$W_+(s) = \operatorname{st} - \lim_{t\to\infty} U(s, t)e^{-iH_1(t-s)}$$

in the case where $q(x, t)$ is small for large t. Previous work on this problem (we refer to [7] for a list of references) has concentrated on the case $V_1(x) \equiv 0$ in which the unperturbed system is a free particle. Some sort of integrability condition with respect to t is usually imposed, although Davies [1] has results for small coupling constant.

The characteristic result of the present paper is the following. Let $n = 3$, and take $V_1(x) \equiv 0$, so that $H_1 = -\Delta$ is the free particle Hamiltonian. If $q_0(x)$ is a potential, independent of time, with

$$\iint |x - y|^{-2}|q_0(x)q_0(y)|\, dx\, dy < 16\pi^2, \tag{1}$$

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the [6] the Born series converges and the wave operator

$$W_+(H_2, H_1) = \text{st} - \lim_{t\to\infty} e^{iH_2t}e^{-iH_1t}$$

exists and is unitary for

$$H_2 = -\Delta + q_0(x).$$

We shall show that if

$$|q(x, t)| \leqq q_0(x)$$

for all t sufficiently large, then $W_+(s)$ also exists and is unitary. This is roughly Davies' [1] result, plus the remark that only the behaviour of $q(x, t)$ for large t matters.

This result is then generalized to $V_1(x)$ of Rollnik class [9]. If $H_1 = -\Delta + V_1(x)$ has no point spectrum, one obtains the same result, except that the constant $16\pi^2$ must be replaced by a positive number $b(H_1)$, which, depending on H_1, may be quite small. The number $b(H_1)$ is related to convergence of the Born series for the wave operators $W_+(H_2, H_1)$, where

$$H_2 = -\Delta + V_1(x) + q_0(x).$$

When H_1 has an eigenfunction $\phi(x)$, it is necessary to assume in addition that

$$\int |q(x, t)| \, |\phi(x)|^2 \, dx$$

is integrable with respect to t at $+\infty$. We obtain similar results for $n \geqq 4$.

It will be assumed throughout this paper that $q(x, t)$ and $V_1(x)$ are real-valued. However, since the methods are those of [4] and [6], which treat nonselfadjoint operators, one should be able to consider the case where $q(x, t)$ and, to a lesser extent, $V_1(x)$, are complex.

1. **Unitary Evolution Groups.** The discussion will be based on the author's scattering theory of evolution groups [4; 8, Vol. II, p. 290]. The present paper is intended to be read as a sequel to [4], to which the reader is referred for many details which are omitted in the following summary.

The main idea is to reduce the scattering theory of the equation

$$i\frac{\partial\phi}{\partial t} = H(t)\phi = H_0\phi + V(t)\phi \tag{1.1}$$

on $\mathscr{H}$ to the scattering theory of the operator

$$K = -i\frac{\partial}{\partial t} + H(t) \tag{1.2}$$

on $\mathscr{K}=L_2(\mathbf{R}; \mathscr{H})$. If the propagator $U(t, s)$ of (1.1) is strongly continuous, then K may be defined as the generator of the unitary group

$$(1.3) \qquad e^{-i\sigma K} f(t) = U(t, t-\sigma) f(t-\sigma)$$

on $\mathscr{K}$. More generally, any self-adjoint operator K on $\mathscr{K}$ satisfying

$$(1.4) \qquad KM(\phi) - M(\phi)K = iM(d\phi/dt).$$

for every scalar multiplication

$$M(\phi)f(t) = \phi(t)f(t)$$

by a function $\phi(t) \in C_c^1(\mathbf{R})$, generates a group of the form (1.3) for a certain *measurable* propagator $U(t, s)$. (In the self-adjoint case, this is a consequence of von Neumann's uniqueness theorem for representations of the CCR; see [**5**]). If we put

$$K_0 = -i\frac{\partial}{\partial t} + H_0,$$

then existence and unitarity of the wave operator

$$(1.5) \qquad W_+(K, K_0) = \operatorname{st} - \lim_{\sigma \to \pm\infty} e^{i\sigma K} e^{-i\sigma K_0}$$

is essentially equivalent to existence and unitarity of

$$(1.6) \qquad W_+(s) = \operatorname{st} - \lim_{t \to \infty} U(s, t) e^{-i(t-s)H_0}$$

for any (or every) s [**4**, Theorem 4]. For the Schroedinger equations considered below, this equivalence is exact, rather than essential. This fact, along with strong continuity of $U(t, s)$ for these equations, follows routinely from [**4**, Theorem 6] by following the argument of [**4**, §4]. (cf. [**5**]) *In the subsequent proofs, we shall therefore just deal with* $W_+(K, K_0)$.

For the record, we shall now state precisely the abstract result that is used in the following sections. We shall stick to selfadjoint problems, although the reader will see easily from [**4, 6**] the generalization to the nonselfadjoint case.

Let K_0 generate a unitary evolution group. A closed operator A is K_0-smooth if and only if $D(A) \supset D(K_0)$ and

$$\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \|A(K_0 - \lambda - i\varepsilon)^{-1} f\|^2 \, d\lambda < \infty$$

for every $f \in \mathscr{K}$. Let A and B be commuting K_0-smooth self-adjoint operators on $\mathscr{K}$ such that $AM(\phi) \supset M(\phi)A$ and $BM(\phi) \supset M(\phi)B$ for every bounded scalar function ϕ. If $Q(\zeta) = A(K_0 - \zeta)^{-1}B$ has a bounded extension, and $\|Q(\zeta)\| < 1$ for some ζ, then the method of factorization

[6] defines an operator $K \supset K_0 + AB$, which also generates a unitary evolution group. This is the precise definition of the propagator $U(t, s)$.

We now make the trivial, but useful, remark *that $W_+(s)$ depends only on $H(t)$ for $t \geqq s$.* Since $W_+(s)$ exists for all s if it exists for one, we have the privilege of replacing $H(t) = H_0 + V(t)$ by $H(t, \alpha) = H_0 + V(t)\chi_{[\alpha,\infty]}(t)$ *for arbitrarily large α.* In fact, if $P(\alpha) = M(\chi_{[\alpha,\infty]})$, then $AP(\alpha)$ and $BP(\alpha)$ satisfy the same conditions as A and B, and so we get another operator $K(\alpha) \supset K_0 + ABP(\alpha)$, which also generates a unitary evolution group. By the Localizatiom Lemma [4, p. 328], the propagators $U(t, s)$ and $U_\alpha(r, s)$ agree a.e. for $t, s \geqq \alpha$. By [4, Theorem 4], $W_+(K, K_0)$ is determined by $U(t, s)$ for large values of t and s only. We therefore obtain

THEOREM 1. *Under the assumptions above, $W_+(K, K_0)$ exists if and only if $W_+(K(\alpha), K_0)$ exists for some (and hence every) α, and*

$$W_+(K, K_0) = W_+(K(\alpha), K_0). \tag{1.7}$$

In particular, if

$$\sup_\zeta \|P(\alpha)Q(\zeta)P(\alpha)\| < 1 \tag{1.8}$$

for sufficiently large α, then $W_+(K, K_0)$ exists and is unitary.

The second assertion is proved by noting that

$$Q_\alpha(\zeta) \equiv P(\alpha)Q(\zeta)P(\alpha) = (P(\alpha)A)\,(K_0 - \zeta)^{-1}(BP(\alpha))$$

is the method of factorization operator for the $(K(\alpha), K_0)$ scattering problem. The results of Kato [6] now apply *directly* to this problem.

If $K_0 = -i\, d/dt + H_0$, the resolvent of K_0 is given by [4, eqn. (1.8)]

$$(K_0 - \zeta)^{-1}f(t) = i\int_{-\infty}^{t} e^{i\zeta(t-s)}\, e^{-iH_0(t-s)} f(s)\, ds \tag{1.9}$$

for $\operatorname{Im} \zeta > 0$. If $\mathscr{F}$ is the Fourier transform in t, and $\hat{K}_0 \equiv \mathscr{F}K_0\mathscr{F}^*$, then

$$\hat{K}_0 f(\omega) = (\omega + H_0)f(\omega).$$

Hence

$$(\hat{K}_0 - \zeta)^{-1}f(\omega) = (H_0 + \omega - \zeta)^{-1}f(\omega) = R_0(\zeta - \omega)f(\omega). \tag{1.10}$$

We shall refer to ω as the *frequency* and to the Fourier transformed representation as the *frequency representation.*

2. **Perturbation of a Free Particle.** Let $H_0 = -\Delta$ and $H(t) = -\Delta + q(x, t)$ on $L_2(\mathbf{R}^n)$ where $q(x, t)$ is real-valued. Define

$$v_p(t) = \left\{(4\pi)^{-n/2}\int_{\mathbf{R}^n}|q(x, t)|^p\, dx\right\}^{1/p}. \tag{2.1}$$

THEOREM 2. *Let $n \geqq 3$ and $1 \leqq p < n/2 \leqq q < \infty$. There is a number $b = b(n, p, q)$ such that if*

$$\limsup_{t\to\infty} \max\{v_p(t), v_q(t)\} < b \tag{2.2}$$

then $W_+(K, K_0)$ exists and is unitary.

PROOF. Let $A(t)$ and $B(t)$ be the operators of multiplication by $|q(x,t)|^{1/2}$ and $|q(x, t)|^{1/2} \operatorname{sgn} q(x, t)$ respectively. By Kato's inequality [**6**; and 4, §4]

$$\|A(t)e^{-iH_0(t-s)} B(s)\| \leqq [v(t)v(s)]^{1/2}\beta(t - s), \tag{2.3}$$

where

$$v(t) = \max\{v_p(t), v_q(t)\}$$

and

$$\beta(t) = \min \{t^{-n/2p}, t^{-n/2q}\}\chi_{[0,\infty)}(t).$$

The operator $Q(\zeta) = A(K_0 - \zeta)^{-1}B$ is then given by

$$Q(\zeta) = i\int_{-\infty}^{t} e^{i\zeta(t-s)}A(t)e^{-iH_0(t-s)}B(s)f(s)\,ds \tag{2.4}$$

for Im $\zeta > 0$, according to (1.9). The estimate (2.3) yields immediately

$$\|P(\alpha)Q(\zeta)P(\alpha)\| \leqq \{\sup_{t\geqq\alpha} v(t)\}\int_0^{+\infty} \beta(t)\,dt \tag{2.5}$$

which gives the result with

$$b^{-1} = \int_0^{\infty} \beta(t)\,dt = \frac{2q}{2q - n} + \frac{2p}{n - 2p}. \tag{2.6}$$

Note that b tends to zero as p or q tend to $n/2$. For the physical case $n = 3$, this can be improved. Let $q(x)$ be a potential on $\mathbf{R}^3$. The *Rollnik norm* of $q(x)$ is defined by

$$\|q\|_R^2 = \iint |x - y|^{-2}|q(x)q(y)|\,dx\,dy.$$

The class of Rollnik potentials was studied by Simon [**9**] in great detail.

THEOREM 3. *For $n = 3$, suppose that there exists an $\alpha > 0$ and a Rollnik potential $q_0(x)$ with*

$$|q(x, t)| \leqq q_0(x) \tag{2.7}$$

for $t \geqq \alpha$, and

$$\|q_0\|_R < 4\pi \tag{2.8}$$

Then $W_+(K, K_0)$ exists and is unitary.

PROOF. Let Λ be multiplication by $q(x)^{1/2}$, and $C(t)$ multiplication by the function

$$C(x, t) = \begin{cases} q(x, t)\,[q_0(x)]^{-1} & \text{if } q_0(x) > 0 \\ 0 & \text{if } q_0(x) = 0 \end{cases}$$

For $t \geqq \alpha$, one has $|C(x, t)| \leqq 1$. Let $A(t) = C(t)\Lambda$ and $B(t) = \Lambda$. Then

$$\|P(\alpha)Q(\zeta)P(\alpha)\| = \|P(\alpha)C[\Lambda(K_0 - \zeta)^{-1}\Lambda]P(\alpha)\| \leqq \|\Lambda(K_0 - \zeta)^{-1}\Lambda\|.$$

In frequency space, $\Lambda(K_0 - \zeta)^{-1}\Lambda$ is multiplication by $\Lambda(H_0 + \omega - \zeta)^{-1}\Lambda$, and so

$$\|\Lambda(K_0 - \zeta)^{-1}\Lambda\|^2 = \sup_\omega \|\Lambda(H_0 + \omega - \zeta)^{-1}\Lambda\|^2 \leqq (4\pi)^2\|q_0\|_R^2.$$

Combining these formulas gives the result.

REMARK 1. Theorem 3 admits a simple physical interpretation. The condition $\|V\|_R < 4\pi$ for a potential $V(x)$ on $\mathbf{R}^3$ is a familiar condition for convergence of the Born series [**9**; §I. 3]. It guarantees that the system with Hamiltonian $H = -\Delta + V(x)$ cannot form bound states [**9**, §§III.3 and III.4], and implies unitarity of the wave operators $W_+(H, H_0)$ [**6**]. Physically, this happens because the potential $V(x)$ is too weak to hold in the particle and form a bound state. In Theorem 3, $q(x, t)$ is dominated uniformly by such a potential for large times, and is, therefore, also unable to bind the particle. So there is again unitarity.

REMARK 2. For $n = 3$, the condition of [**4**, §4] requires that for some $p > 3/2$, the norm $\|q(\cdot, t)\|_p$ be in $L_{r\pm\varepsilon}$ for $r = 2p/(2p - 3)$. Theorem 3 is therefore the limiting case $p = 3/2$ and $r = \infty$. For $p > 3/2$, *long range* potentials $q(x, t)$ are permitted, which may bind even for small coupling. If, for example, $q(x, t) = \kappa(t)q_0(x)$, the condition of [**4**] says that the long range behaviour of $q_0(x)$ can be compensated for by letting the coupling $\kappa(t)$ go to zero sufficiently fast.

REMARK 3. There are potentials not satisfying (2.7) for which there is unitarity. For example, on $\mathbf{R}^3$, let

$$K = -i\frac{\partial}{\partial t} - \Delta + q(x_1 - t, x_2, x_3).$$

Define

$$\mathscr{U}f(x, t) = e^{-i(x/2-t/4)}f(x_1 - t, x_2x_3; t).$$

By Galilean invariance, or just computation,

$$\mathscr{U}^*K\mathscr{U} = -i\frac{\partial}{\partial t} - \Delta + q(x_1, x_2, x_3).$$

Hence, $W_+(K, K_0)$ is unitary if $\|q\|_R < 4\pi$.

REMARK 4. More generally, the potential

$$q(x, t) = q(x - a(t)) \tag{2.9}$$

on $\mathbf{R}^3$ cannot satisfy (2.7) if $a(t)$ is an unbounded motion. For, let E be a bounded set of positive measure on which $|q(x)| \geqq \delta \geqq 0$. If $a(t)$ is unbounded, one can choose $a(t_n)$ so that the sets $E_n = E - a(t_n)$ are all disjoint. If (2.7) were true, one would have $q_0(x) \geqq \delta$ on the union of the E_n, which has infinite measure.

However, if $q(x) \in L_p \cap L_q$, $1 \leqq p < 3/2 < q \leqq \infty$, and has small enough norm, Theorem 2 gives unitarity for the potential (2.9). As the referee has noted, this shows that Theorems 2 and 3 are not strictly comparable.

The interaction (2.9) represents the field of a fixed system moving along the path $a(t)$, and is therefore of considerable interest.

REMARK 5. For comparison with our results, we note that Davies [**1**, p. 160–1] requires that $|q(x, t)| \leqq q_0(x)$, where $q_0 \in L_p \cap L_q$, $1 \leqq p < n/2 < q \leqq \infty$.

3. **Perturbation of a Two-body Problem.** We wish to study the Hamiltonian

$$H(t) = -\Delta + V_1(x) + q(x, t)$$

where again $q(x, t)$ is small at $t = \pm\infty$. For this, we need some standard doctrine of the two-body operator $H_1 = -\Delta + V_1(x)$. Let $H_0 = -\Delta$, $A_1(x) = |V_1(x)|^{1/2}$ and $B_1(x) = |V_1(x)|^{1/2}$ sgn $V_1(x)$. For $n \geqq 4$, *we shall assume that* $V_1(x) \in L_p(\mathbf{R}^n) \cap L_q(\mathbf{R}^n)$ *for some p and q with* $1 \leqq p < n/2 < q \leqq \infty$, and define

$$|||V_1||| = \max\{\|V_1\|_p, \|V_1\|_q\}.$$

For $n = 3$, *we assume that* V_1 *is Rollnik*, and put

$$|||V_1||| \equiv \|V_1\|_R.$$

Under these conditions, the operator

$$X(\zeta) = A_1(H_0 - \zeta)^{-1}B_1$$

is compact, the limits $X(\lambda \pm i0)$ exist in operator norm, $X(\lambda \pm i0) \to 0$ in norm as $|\lambda| \to \infty$, and one has the bound

$$\|X(\zeta)\| \leqq C(n, p, q)\ |||V||| \tag{3.1}$$

where $C(n, p, q) = (4\pi)^{-1}$ for $n = 3$. We shall also assume that $\mathrm{I} + X(\lambda \pm i0)$ is invertible for $\lambda \geqq 0$, so that there is no nonnegative eigenvalue, or zero energy resonance. There are only a finite number $\lambda_1, \ldots, \lambda_m$ of negative eigenvalues, counting multiplicity. If $\phi_1, \ldots, \phi_m$ are the corresponding normalized eigenvectors, then

$$[I + X(\zeta)]^{-1} = \sum_{k=1}^{m} (\zeta - \lambda_k)^{-1} \langle \cdot, B_1\phi_k \rangle A_1\phi_k + Y(\zeta), \tag{3.2}$$

where $Y(\zeta)$ is uniformly bounded and norm continuous on the closed cut plane. Convenient references for this theory are Reed-Simon [8, Volume IV], Ginibre-Moulin [3] for $n \geqq 4$, Simon [9] for $n = 3$. The Zemach-Klein theorem that $X(\lambda \pm i0)$ tends to zero at infinity is proved by Faris [2], and, for $n = 3$, in [9, p. 23].

Let Λ be multiplication by a function $\Lambda(x)$ with $|||\Lambda^2|||$ finite. Using the formula [6]

$$R_1(\zeta) = R_0(\zeta) - R_0(\zeta)B_1[I + X(\zeta)]^{-1}A_1R_0(\zeta), \tag{3.3}$$

for $R_1(\zeta) = (H_1 - \zeta^{-1})$, we find that

$$\Lambda R_1(\zeta)\Lambda = \sum_{k=1}^{m} (\zeta - \lambda_k)^{-1} \langle \cdot, \Lambda\phi_k \rangle \Lambda\phi_k + Y_1(\zeta), \tag{3.4}$$

where $Y_1(\zeta)$ has the properties claimed for $Y(\zeta)$ above.
In fact,

$$Y_1(\zeta) = \Lambda R_0(\zeta)\Lambda - [\Lambda R_0(\zeta)B_1]Y(\zeta)[A_1R_0(\zeta)\Lambda], \tag{3.5}$$

which yields the estimate

$$\| Y_1(\zeta) \| \leqq M |||\Lambda^2|||, \tag{3.6}$$

where

$$M = C(n, p, q) + C^2(n, p, q) |||V_1||| \sup_{\zeta} \| Y(\zeta) \|. \tag{3.7}$$

We can now state a result.

THEOREM 4. *Under the conditions above, there is a positive number* $b(H_1)$ *such that if*

$$|q(x, t)| \leqq q_0(x)$$

for $t \geqq \alpha$, *where* $q_0(x)$ *is a potential with*

$$|||q_0||| < b(H_1)$$

and if

$$\int_0^\infty \int_{\mathbf{R}^n} |q(x, t)| \, |\phi_k(x)|^2 \, dx \, dt < \infty \tag{3.8}$$

for $k = 1, \ldots, m$, *then* $W_+(K, K_1)$ *exists and is unitary.*

PROOF. As in the proof of Theorem 3,

$$|q(x, t)|^{1/2} = C(x, t)q(x)^{1/2},$$

where $|C(x, t)| \leqq 1$ 'for $t \geqq \alpha$. Let Λ, A, C and S be the operators of multiplication by $q(x)^{1/2}$, $|q(x, t)|^{1/2}$, $C(x, t)$ and sgn $q(x, t)$ respectively. One has $A = C\Lambda$ and $\|P(\alpha)C\| \leqq 1$. Let $B = SA = SC\Lambda$ and $K_1 = -i\partial/t\partial + H_1$. We want to estimate the norm of $P(\alpha)Q(\zeta)P(\alpha)$ where $Q(\zeta) = A(K_1 - \zeta)^{-1}B$.

Let $T(\zeta) = \Lambda(K_1 - \zeta)^{-1}\Lambda$, so that $Q(\zeta) = CT(\zeta)CS$. In ω-space, we have by (3.4)

$$\begin{aligned} T(\zeta)f(\omega) &= \Lambda R_1(\zeta - \omega)\Lambda f(\omega) \\ &= \sum_{k=1}^{m}(\zeta - \omega - \lambda_k)^{-1} \langle f(\omega), \Lambda\phi_k\rangle \Lambda\phi_k + Y_1(\zeta - \omega)f(\omega) \\ &= T_d(\zeta) + T_c(\zeta). \end{aligned}$$

Put $Q_d(\zeta) = CT_d(\zeta)CS$ and $Q_c(\zeta) = CT_c(\zeta)CS$, so that $Q(\zeta) = Q_d(\zeta) + Q_c(\zeta)$. By (3.6), we have

$$\|T_c(\zeta)\| \leqq M \,|||\Lambda^2||| = M|||q_0|||,$$

and hence also

$$\cdot\|P(\alpha)Q_c(\zeta)P(\alpha)\| = \|[P(\alpha)C]T_c(\zeta)[CP(\alpha)S]\| \leqq M|||q_0|||.$$

The discrete part looks better in t-space:

$$Q_d(\zeta)f(t) = i\sum_{k=1}^{m}\int_{-\infty}^{t} e^{i(\zeta-\lambda_k)(t-s)} A(t)\phi_k \langle f(s), B(s)\phi_k\rangle\, ds. \tag{3.9}$$

A simple estimate gives

$$\|P(\alpha)Q_d(\zeta)P(\alpha)\| \leqq \sum_{k=1}^{m}\int_{\alpha}^{\infty} \|A(t)\phi_k\|^2\, dt, \tag{3.10}$$

where the integrals are convergent by (3.8).

The theorem now follows with

$$b(H_1) = M^{-1}. \tag{3.11}$$

For if $|||q_0||| < M^{-1}$, then $\|P(\alpha)Q_c(\zeta)P(\alpha)\| < 1 - \varepsilon$ for some $\varepsilon > 0$. If α is taken sufficiently large to make the right side of (3.10) less than ε, we obtain $\|P(\alpha)Q(\zeta)P(\alpha)\| < 1$ as desired.

When there is no point spectrum, we obtain a direct analogue of Theorem 3.

COROLLARY 1. *If H_1 has no point spectrum, or zero energy resonance, there is a positive number $b = b(H_1, p, q)$ such that if*

$$|q(x,t)| \leqq q_0(x)$$

for $t \geqq \alpha$, *and* $|||q_0||| < b$, *then* $W_+(K, K_1)$ *exists and is unitary.*

This even works if $q(x, t)$ is independent of t.

COROLLARY 2. *If* H_1 *has no point spectrum, or zero energy resonance and* $|||q_0||| < b$, *then* $H \equiv H_1 + q_0(x)$ *is unitarily equivalent to* H_1, *and* $W_\pm(H, H_1)$ *exist and are unitary.*

REMARK 1. *How small is* b? For $n = 3$, with $\sigma_p(H_1)$ empty, equations (3.7) and (3.11) give the estimate

$$b^{-1} \geqq 4\pi + 16\pi^2 \|V_1\|_R \sup_\zeta \|[I + X(\zeta)]^{-1}\|.$$

If there is a resonance-intuitively, a second sheet pole of $[I + X(\zeta)]^{-1}$ — near the positive real axis, the second term will be quite large, and hence b quite small. Physically, this means that V_1 is close to holding a bound state, so that even a small perturbation may cause unitarity to fail. If the resonance is close to zero, this can certainly happen, but for a positive energy resonance, the situation is not clear; at least, not to the author.

REMARK 2. The condition (3.8) is *really* an L_1 condition. For, since the ground state $\phi_1(x)$ is generally positive, taking $k = 1$ shows that $q(x, t)$ is L_1 in t for a.e. x. However, the x-space norm can be very weak; in fact, because of the exponential bounds on the $\phi_k(x)$ [**9**, Chapter III], it suffices that

$$\int_{\mathbf{R}^3} |q(x,t)|\, e^{-a|x|}\, dx$$

be in $L_1[\alpha, \infty; dt]$ for $0 < a$ and $a^2 < 4 \min\{|\lambda_1|, \ldots, |\lambda_m|\}$.

REMARK 3. The results of [**4**, §4] for $H_1 = -\Delta$ include *long range* $q(x, t)$, as long as an appropriate norm vanished rapidly enough. I have not been able to extend this result to $H_1 = -\Delta + V_1$.

REMARK 4. In all cases discussed, the wave operators $W_+(\alpha)$ can be computed by a convergent Dyson series; cf. [**4**, §4] and Davies [**1**].

REMARK 5. Theorem 4 does not apply to potentials of the form (2.9). One might conjecture an extension of Theorem 3 to pertubations of two-body problems which would cover this case, but I have no such result at present.

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