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DISPERSION FREE WEIGHTS, MAXIMAL REFINEMENT IDEALS, AND ATOMICITY IN CERTAIN GENERALIZED SAMPLE SPACES

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ABSTRACT. An *HD* space is a generalized sample space each of whose subspaces has, as its logic, an orthomodular poset. We study *HD* spaces which allow full sets of dispersion free weights: in a sense, these sample spaces can be said to admit "hidden variables". For such a space, we show that any proposition in the logic contains a minimal proposition, and that any maximal refinement ideal of operations is generated by a single operation. While the former ensures that the logic is atomic, the latter provides a generalized analog of the classical "grand canonical operation". In fact, we obtain a stronger result concerning atomicity, namely, that the logic of any *HD* space in which all maximal refinement ideals are principal is atomic.

1. **Introduction.** Whereas classical mechanics admits the theoretical existence of a "grand canonical operation" which in a sense refines all possible operations on a given system, the concept of a grand canonical operation as such is to a large extent meaningless in quantum mechanics, where the Heisenberg Commutation Relations preclude the possibility of determining a system with absolute accuracy. The study of operational statistics undertaken by D. J. Foulis and C. H. Randall (see [3], [7], [8], [9], [10]) seeks to provide a generalized version of classical probability and statistics, a formulation of particular value in its applicability to quantum mechanical systems. In his ground-breaking paper [6], G. W. Mackey studies the "logic" of quantum mechanics, that is, the set of "questions" (propositions), ordered by implication, together with a certain family of probability measures on this poset. The Foulis-Randall approach has the additional feature that an analog of this "logic", and probability measures thereon, are naturally induced by a generalized sample space and the accompanying generalized weight functions, thereby emphasizing the role of the physical situation.

In this paper, we will be concerned with a special kind of generalized sample space, called an *HD* space (rigorous definitions will be given presently). The logic (in the Foulis-Randall sense) of an *HD* space is a complete orthomodular lattice, (see [1] for lattice theoretical terminology) and thus a natural generalization of the classical Boolean lattice of propositions. Being the model Mackey adopts, the complete orthomodular lattice can reasonably be considered typical of quantum logics.

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A relation of refinement, closely paralleling the classical notion, can be defined on the operations in our generalized sample spaces; this relation usually orders the operations. The concept of a refinement ideal is both meaningful and useful in this connection. We shall introduce a condition which, in our generalized sample spaces, will be equivalent to the existence of "hidden variables". The aim of this paper is to demonstrate that the presence of "hidden variables" in *HD* sample spaces ensures: (i) that every maximal refinement ideal be principal, providing a generalized version of the classical "grand canonical operation" for each such ideal; (ii) that the logic of such a sample space be atomic.

2. **The Mathematical Framework.** In our approach, we wish to represent the result sets for a collection of physical operations in a single sample space, identifying indistinguishable outcomes of distinct operations. To this end, we define an *orthogonality space* to be a pair $(X, \perp)$, where X is a nonempty set and $\perp$ is a symmetric, antireflexive binary relation on X called *orthogonality.* A subset of X consisting of pairwise orthogonal elements is called an *orthogonal set,* and the collection of all orthogonal sets is denoted $\mathscr{O}(X, \perp)$. Let $\mathscr{E}(X, \perp)$ denote the collection of all orthogonal sets which are maximal as orthogonal sets with respect to set inclusion. The orthogonality space $(X, \perp)$ together with the collection $\mathscr{E}(X, \perp)$ is what we choose to regard as a *generalized sample space* (in [3], this is referred to as a *completely coherent generalized sample space*). Each element of $\mathscr{E}(X, \perp)$ represents an exhaustive set of distinct outcomes for some physical experiment. Thus maximal orthogonal sets are often called *operations*; the elements of $\mathscr{O}(X, \perp)$, being subsets of operations, are usually called *events.* Hence the intended interpretation of the relation $\perp$ is *operational rejection*: elements x and y in X are orthogonal if and only if they are distinct outcomes of some operation.

Let $(X, \perp)$ be an orthogonality space, and $W \subseteq X$. We define $W^{\perp} = \{x \in X \mid x \perp w, \text{ for all } w \text{ in } W\}$, $W^{\perp\perp} = (W^{\perp})^{\perp}$, etc., and if $x \in X$, we agree to denote $\{x\}^{\perp}$ by $x^{\perp}$. The following lemma is basic.

LEMMA 1. *Let* $A, B \subseteq X$. *Then*:
(i) $A \cap A^{\perp} = \phi$;
(ii) $A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp}$;
(iii) $A \subseteq A^{\perp\perp}$;
(iv) $A^{\perp} = A^{\perp\perp\perp}$;
(v) $\phi^{\perp} = X$, $X^{\perp} = \phi$.

If D is any event, we can interpret $D^{\perp}$ as the set of all outcomes which *refute* the event D, and $D^{\perp\perp}$ as the set of all outcomes which *confirm* D, in the sense that an element of $D^{\perp\perp}$ refutes those outcomes which refute D. We call $D^{\perp\perp}$ the *generalized operational proposition associated with* D. It is confirmed or refuted on the basis of an execution of any operation E, where $E \subseteq D^{\perp} \cup D^{\perp\perp}$; if a result e in $D^{\perp\perp}$ is obtained, the proposition is confirmed, while if e in $D^{\perp}$ is obtained, it is refuted. When D_1 and D_2 are events with $D_1{}^{\perp\perp} \subseteq D_2{}^{\perp\perp}$, then any outcome which confirms D_1 also confirms D_2, and one says that $D_1{}^{\perp\perp}$ *implies* $D_2{}^{\perp\perp}$. The collection of all confirming sets for events in X, ordered by implication, is defined to be the *logic* of $(X, \perp)$. More formally, let

$$\Pi(X, \perp) = \{D^{\perp\perp} \mid D \in \mathscr{O}(X, \perp)\}.$$

Then $(\Pi(X, \perp), \subseteq)$ is the *generalized operational logic over* $(X, \perp)$.

Note that if a sample space $(X, \perp)$ consists of a single operation, as in the classical situation, the logic $\Pi(X, \perp)$ is simply the Boolean lattice of all subsets of X ordered by set inclusion. We refer to such sample spaces as *classical sample spaces.*

We define a *generalized weight function* on $(X, \perp)$ to be a function $\omega : X \to [0, 1]$ such that for all E in $\mathscr{E}(X, \perp)$, $\sum_{e \in E} \omega(e) = 1$. A weight function ω is said to be *dispersion free* if $\omega : X \to \{0, 1\}$. Thus a dispersion free weight function completely and precisely determines a system; each outcome either occurs or does not with absolute certainty.

In most treatments of quantum logics, probability measures simply arise as abstract measures, called "states", on the set of propositions. A feature of the Foulis-Randall approach is that probability measures (called "regular states") on the logic, are induced by generalized weight functions on the generalized sample space. If ω is a weight function, then the *regular state associated with* ω is the function $\alpha_\omega : \Pi(X, \perp) \to [0, 1]$ defined by $\alpha_\omega(D^{\perp\perp}) = \sum_{d \in D} \omega(d)$, for all events D. The mapping α_ω is well defined in the sense that if D_1, D_2 are events with $D_1{}^{\perp\perp} = D_2{}^{\perp\perp}$, then $\alpha_\omega(D_1{}^{\perp\perp}) = \alpha_\omega(D_2{}^{\perp\perp})$. It can be shown that if ω is any weight function, and D_1, D_2 are events with $D_1{}^{\perp\perp} \subseteq D_2{}^{\perp\perp}$, then $\alpha_\omega(D_1{}^{\perp\perp}) \leqq \alpha_\omega(D_2{}^{\perp\perp})$. If Ω is a set of weights such that whenever $\alpha_\omega(D_1{}^{\perp\perp}) \leqq \alpha_\omega(D_2{}^{\perp\perp})$ for all ω in Ω, then $D_1{}^{\perp\perp} \subseteq D_2{}^{\perp\perp}$, we say that Ω is a *full* set of weights. Thus Ω is full if the implication relation on $\Pi(X, \perp)$ can be completely recaptured given only a knowledge of the weights in Ω.

We can define an imbedding of the othogonality space $(X, \perp_1)$ into the orthogonality space $(Y, \perp_2)$ to be a function ϕ from the outcomes of X to the events of Y such that:

(i) $x \perp_1 y \Longleftrightarrow \phi(x) \perp_2 \phi(y)$;

(ii) $\cup\{\phi(e) \mid e \in E\} \in \mathscr{E}(Y, \perp_2)$, for any $E \in \mathscr{E}(X, \perp_1)$.

Such a function imbeds $(X, \perp_1)$ in a more detailed model, but in a way consistent with the information regarding orthogonality which is already present. In the case where $(Y, \perp_2)$ is a sample space whose structure is dictated by the possibly unknown laws governing the actual procedures which lead to the sample space $(X, \perp_1)$, then the imbedding ϕ describes how $(X, \perp_1)$ is to be "interpreted" or "explained" in terms of $(Y, \perp_2)$. (See [**10**], Section 5.)

Let $(X, \perp)$ be an orthogonality space which admits a full set of dispersion free weights. Then $(X, \perp)$ can be imbedded in the classical sample space whose outcomes are the dispersion free weights on $(X, \perp)$: an outcome in X is mapped to the event defined by those dispersion free weights which assign that outcome weight one. It can also be shown that any generalized sample space which can be imbedded in a classical sample space supports a full set of dispersion free weights. We conclude that those generalized sample spaces which allow a "classical interpretation", or equivalently, which admit "hidden variables", are precisely those which support full sets of dispersion free weights.

Elements x and y in an orthogonality space $(X, \perp)$ are said to be *scattered*, denoted $x \# y$, if they are distinct and nonorthogonal. A *scattered set* is, of course, a set of pairwise scattered elements; let $\mathscr{S}(X, \perp)$ denote the collection of all scattered sets in $(X, \perp)$ which are maximal as scattered sets with respect to set inclusion. Then the following characterization of those sample spaces which admit hidden variables follows immediately from Theorem 2 in [**3**].

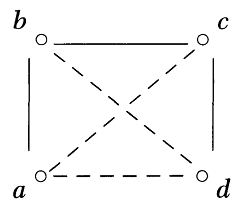
THEOREM 2. *$(X, \perp)$ admits a full set of dispersion free weights if and only if whenever $x, y \in X$ with $x \notin y^\perp$, then there is a set S in $\mathscr{S}(X, \perp)$ such that $x, y \in S$ and, for all $E \in \mathscr{E}(X, \perp)$, $E \cap S \neq \Phi$.*

In the light of this result, any maximal scattered set which intersects every operation is called a *dispersion free support set.* Let $\mathscr{S}_{df}$ represent the collection of all dispersion free support sets, that is, $\mathscr{S}_{df} = \{S \in \mathscr{S}(X, \perp) \mid S \cap E \neq \Phi, \text{ for all } E \in \mathscr{E}(X, \perp)\}$. Thus the theorem states that $(X, \perp)$ admits a full set of dispersion free weights if and only if every pair of nonorthogonal elements is contained in a dispersion free support set.

3. ***HD* spaces.** The generalized operational logic is, of course, the Foulis-Randall analog of Mackey's set of questions. In the system Mackey proposes in Axioms I through VI, this set of questions, ordered by implication, is an orthomodular poset. J. C. Dacey [**2**] has exhibited conditions on a generalized sample space which are necessary and sufficient for its logic to be an orthomodular poset. We conclude that such

sample spaces, which have come to be called *Dacey spaces*, can reasonably be considered appropriate to quantum mechanics. In addition, one might want to require that any *subspace* of a Dacey sample space, that is, any subset with the induced orthogonality, exhibit the Dacey property as well. Such a sample space, every subspace of which "inherits" the Dacey property, is called a *hereditary Dacey* (or *HD*) *space*. The logics of *HD* spaces are complete orthomodular lattices. Recall that in his formulation, Mackey adopts a model in which the logic of quantum mechanics is a complete orthomodular lattice.

The *orthogonality diagram* for $(X, \perp)$ represents elements of X by points, the points corresponding to orthogonal elements being joined by a solid line, while those corresponding to scattered elements are connected by a broken line. For example, consider the sample space



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Here $a \perp b$, $b \perp c$, $c \perp d$, $a \# c$, $a \# d$, $b \# d$.

The sample space illustrated above, usually called a *hook*, is of particular interest to us as a consequence of the following characterization of *HD* spaces due to J. C. Dacey.

THEOREM 3. *An orthogonality space $(X, \perp)$ is an HD space if and only if it contains no subspace isomorphic to a hook.*

It follows that a sample space is *HD* if and only if it is "hook-deficient".

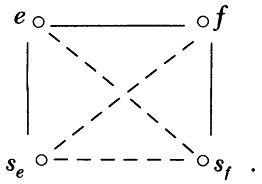
The characterization of those sample spaces which admit hidden variables given in Theorem 2 can be appreciably simplified in the case of an *HD* space:

THEOREM 4. *Let $(X, \perp)$ be an HD space. Then $(X, \perp)$ admits a full set of dispersion free weights if and only if $X = \cup \mathscr{S}_{df}$.*

PROOF. Suppose that $(X, \perp)$ admits a full set of dispersion free weights, and let x be in X. Then $x \notin x^{\perp}$ by the irreflexivity of the relation, and so, by Theorem 2, x is contained in some dispersion free support set. Thus $X = \cup \mathscr{S}_{df}$.

Conversely, let us suppose that the dispersion free support sets cover X, and x, y are nonorthogonal elements in X. We shall show that both x

and y are contained in some dispersion free support set. Theorem 2 will then imply that $(X, \perp)$ admits a full set of dispersion free weights. Note that if $x = y$, then both are contained in some dispersion free support set by hypothesis, so that we may assume without loss of generality that x and y are distinct. Being distinct nonorthogonal elements, x and y are, by definition, scattered. Now, by hypothesis, we can find S_x, $S_y \in \mathscr{S}_{df}$ such that $x \in S_x$, $y \in S_y$. Then the set $\{x, y\} \cup (S_x \cap S_y)$ is clearly a scattered set; since "$\#$" as well as "$\perp$" are properties of finite character, we can expand $\{x, y\} \cup (S_x \cap S_y)$ to a maximal scattered subset of $S_x \cup S_y$, call it S'. Then by another application of the Tukey lemma, $S' \subseteq S$, for some S in $\mathscr{S}(X, \perp)$. We need only show that $S \in \mathscr{S}_{df}$, so suppose to the contrary that there is some operation E with the property that $S \cap E = \Phi$. We will contradict the fact that $(X, \perp)$ is *HD* by finding a forbidden hook. Since S_x and S_y are dispersion free support sets, there exists an e in $E \cap S_x$, and an f in $E \cap S_y$. If $e = f$, then $e \in S_x \cap S_y \subseteq S$, a contradiction since $E \cap S = \Phi$. Thus e, f are distinct elements of E, an orthogonal set, whence $e \perp f$. Because $e \notin S$ and $S' \subseteq S$, we have that $e \notin S'$. But S' was a maximal scattered subset of $S_x \cup S_y$, and $e \in S_x \subseteq S_x \cup S_y$, so that there must exist s_e in S' such that $s_e \perp e$. If s_e were in S_x, we would have $s_e = e$ or $s_e \# e$; thus $s_e \notin S_x$. But $s_e \in S' \subseteq S_x \cup S_y$, so that $s_e \in S_y$. A similar argument yields the existence of some s_f in $S' \cap S_x$ which is orthogonal to f. Now $e \notin S$, while $s_f \in S' \subseteq S$, so $e \neq s_f$. However, e, $s_f \in S_x$, where S_x is a scattered set; thus $e \# s_f$. Similarly $f \# s_e$. Clearly now s_e and s_f are distinct, since $e \perp s_e$ while $e \# s_f$. We have constructed the following subspace:



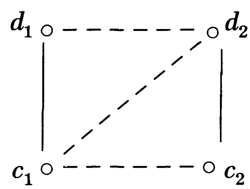
But this is a contradiction to the fact that $(X, \perp)$ is an *HD* space. Thus $S \in \mathscr{S}_{df}$, as desired.

4. **Atomicity of $\Pi(X, \perp)$.** We will presently discuss the notion of refinement mentioned earlier. But first let us define a quasi-order on the elements of an *HD* space, and study certain chains which result from this relation. Unless otherwise stated, we henceforth assume that $(X, \perp)$ is an *HD* space.

For elements x and y in X, define $x \leqq y$ if $x^{\perp\perp} \subseteq y^{\perp\perp}$, and $x < y$ if $x \leqq y$ and $x \neq y$. Then $\leqq$ is clearly reflexive and transitive. We say that an element x in X is *minimal in* X if there does not exist $y \in X$ with $y < x$. Note that comparable elements are never orthogonal. Let $C \subseteq X$ be any chain under $\leqq$, and let $D' = \{d \in X \mid d \perp c$, for some $c \in C\}$. Note that $D' \cap C = \Phi$. We define an equivalence relation on D' as follows: for d_1, d_2 in D, we write $d_1 \sim d_2$ if and only if $d_1^{\perp} \cap C = d_2^{\perp} \cap C$. Non-equivalent elements of such a set are orthogonal by the following lemma.

Lemma 5. *Let $C \subseteq X$ be a chain, and let $D' = \{d \in X \mid d \perp c$, for some $c \in C\}$. If $d_1, d_2 \in D'$, where $d_1 \not\sim d_2$, then $d_1 \perp d_2$.*

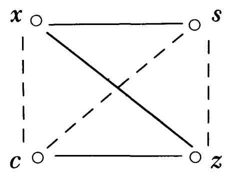
Proof. Since $d_1 \not\sim d_2$, we have $d_1^{\perp} \cap C \neq d_2^{\perp} \cap C$, so that we can assume without loss of generality that there exists some $c_1 \in (d_1^{\perp} \cap C) \setminus (d_2^{\perp} \cap C)$. The disjointness of C and D' ensures that c_1 and d_2 are distinct, whence $c_1 \# d_2$. Because $d_2 \in D'$, there is some c_2 in C with $d_2 \perp c_2$. Thus c_1 and c_2 are distinct elements of a chain, and are therefore scattered ($c_1 \# c_2$). Suppose that $d_1 \# d_2$. We then have:



If $d_1 \perp c_2$, then (c_1, d_1, c_2, d_2) is a hook (hooks are usually denoted in this fashion as ordered 4-tuples). This however contradicts the fact that $(X, \perp)$ is an HD space. Therefore $d_1 \# c_2$, and we have that $d_1 \in c_1^{\perp} \setminus c_2^{\perp}$, $d_2 \in c_2^{\perp} \setminus c_1^{\perp}$, violating the comparability of c_1 and c_2.

Lemma 6. *Let C be maximal as a chain in S, where $S \in \mathscr{S}(X, \perp)$. Then C is a maximal chain in X.*

Proof. Suppose that C is not a maximal chain in X. Then there exists some element x in $X \setminus C$ which is comparable to every element in C. If x were in S, x would be in C by the maximality of C in S. Thus x cannot be in S, and so, since S is a maximal scattered set, x must be orthogonal to some s in S (for otherwise $x \in S^{\#} = \Phi$, a contradiction). Since x and s are orthogonal, they are not comparable, and since x is comparable to every element in C, this means that $s \notin C$. But C is maximal as a chain in S and $s \in$ S; hence there is some c in C with $s \nleqq c$. Note that $s \# c$, since s and c are distinct elements of S, a scattered set. Thus $s^{\perp\perp} \not\subseteq c^{\perp\perp}$, which implies that $c^{\perp} \not\subseteq s^{\perp}$, or equivalently, that there is some z in $c^{\perp} \setminus s^{\perp}$. If $s = z$, we have $s \in c^{\perp}$, a contradiction since $s \# c$. So $z \notin s^{\perp}$ and $z \neq s$, yielding $z \# s$. Since x is comparable to c, $x \notin c^{\perp}$. But $x \notin C$ whereas $c \in C$, so $x \neq c$, and we have $x \# c$. Now $s \in x^{\perp} \setminus c^{\perp}$, so that $c \nleqq x$. But x and c are comparable, so it must be that $x \leqq c$, that is, $c^{\perp} \subseteq x^{\perp}$. And so $z \in c^{\perp} \subseteq x^{\perp}$. We have



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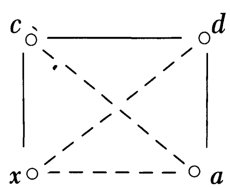
a hook and a contradiction.

THEOREM 7. *Every chain in $(X, \perp)$ which is maximal in a dispersion free support set contains an element which is minimal in X.*

PROOF. Let $C \subset X$ be a chain which is maximal in S, where S is a dispersion free support set, and let $D' = \{d \in X \mid d \perp c, \text{ some } c \in C\}$. The set D' is partitioned into equivalence classes by the relation defined earlier; choose a maximal orthogonal subset of each such equivalence class (again, by the Tukey lemma), and let D denote the union of all such subsets. Lemma 5 assures us that $D \in \mathcal{O}(X, \perp)$, so that there exists an operation E such that $D \subseteq E$. Because $S \in S_{df}$, there exists some a in $E \cap S$; we will show that a is the required minimal element.

Suppose that there exists c in C, with $a \nleqq c$. Then $c^{\perp} \not\subseteq a^{\perp}$, so that we can find some x in $c^{\perp} \setminus a^{\perp}$. If $a = x$, then $a \in c^{\perp}$, a contradiction since a and c are elements of S, a scattered set. Thus $a \# x$. But $x \in c^{\perp}$, so that $x \in D'$. Note that $x \notin E$, so that $x \notin D$. Now x cannot be orthogonal to all those elements of D which are in its equivalence class in D', since these comprise a maximal orthogonal subset of that equivalence class. Thus there exists d in D, $d \sim x$, such that $d \# x$. Since $a \# x$ and $c \perp x$, we have that a and c are distinct, whence $a \# c$; since $x \perp c$ and $d \sim x$, we have $d \perp c$, so that $a \neq d$, whence $a \perp d$. This yields



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a contradiction.

Thus $a \leqq c$, for all c in C. Since C is maximal in S, where $S \in \mathscr{S}(X, \perp)$, C is maximal in X by Lemma 6. Now the maximality of C in X ensures that $a \in C$. Note that a is minimal, since the existence of an element of X strictly less than a would contradict the maximality of C.

We now observe that an element a is minimal in X if and only if $a^{\perp\perp}$ is an atom in $\Pi(X, \perp)$. That a is minimal in X if $a^{\perp\perp}$ is an atom in the logic is clear. Suppose that a is minimal in X and that $a^{\perp\perp}$ is not an atom in the logic. Then $a^{\perp\perp}$ properly contains some nonempty proposition, say $D^{\perp\perp}$. Since $D^{\perp\perp} \neq \Phi$, part (v) of Lemma 1 ensures that $D \neq \Phi$, and so we can find some d in D. Therefore $d^{\perp\perp} \subseteq D^{\perp\perp} \subsetneqq a^{\perp\perp}$, whence $d < a$, contradicting the minimality of a.

If $(X, \perp)$ admits a full set of dispersion free weights, the atomicity of $\Pi(X, \perp)$ now follows quite easily. For if $D^{\perp\perp}$ is any nonempty proposition in the logic, as we have seen, we can choose d in D. Since $X = \cup \mathscr{S}_{dp}$ d is contained in some dispersion free support set, indeed, d is contained in some chain which is maximal in that dispersion free support set. But this chain is bounded below by a minimal element. Thus there exists a minimal element, say a, in X below d. Noting that $a^{\perp\perp} \subseteq d^{\perp\perp} \subseteq D^{\perp\perp}$, we conclude that $a^{\perp\perp}$ is an atom below the proposition $D^{\perp\perp}$. We have shown:

THEOREM 8. *Let $(X, \perp)$ be an HD space. If $(X, \perp)$ admits a full set of dispersion free weights, then $\Pi(X, \perp)$ is atomic.*

In a very concrete way, minimal elements in $(X, \perp)$ represent "limiting" outcomes. Our theorem assures that whenever $(X, \perp)$ admits a full set of dispersion free weights, there are sufficiently many limiting outcomes present in $(X, \perp)$ to provide atoms in $\Pi(X, \perp)$. Thus our generalized situation is analogous to that encountered in classical mechanics, where the atoms in the Boolean lattice of propositions are indicative of the presence of limiting outcomes in the sample space. We shall, in fact, prove a stronger result concerning atomicity in what follows.

5. **Refinement Ideals in $\mathscr{E}(X, \perp)$.** If $(X, \perp)$ is any orthogonality space, we can define a relation of "refinement" or "coarsening" on operations in X. If $E, F \in \mathscr{E}(X, \perp)$, we say that F *refines* E, or that E *coarsens* F, denoted $E \sqsubset F$, if for every e in E, there exists $D \subseteq F$ such that $e^{\perp\perp} = D^{\perp\perp}$. That is to say, $E \sqsubset F$ whenever every element of E is equivalent in $\Pi(X \perp)$ to some event contained in F. It can be shown [11] that if the orthogonality space is point determining ($e^{\perp\perp} = f^{\perp\perp}$ implies $e = f$), then $(\mathscr{E}(X, \perp) \sqsubset)$ is a partially ordered set.

A nonempty collection of operations which is upward directed by refinement, and to which all available coarsenings have been adjoined, is called a *refinement ideal.* Specifically, $\mathscr{I}$ is a refinement ideal if and only if the following conditions hold:

(i) if $E, F \in \mathscr{I}$, there is some $G \in \mathscr{I}$ such that $E \sqsubset G$, $F \sqsubset G$;

(ii) if $F \in \mathscr{I}$ and $E \in \mathscr{E}(X, \perp)$ such that $E \sqsubset F$, then $E \in \mathscr{I}$.

A refinement ideal $\mathscr{I}$ is *maximal* if no refinement ideal properly contains it; $\mathscr{I}$ is *principal* if there is an operation F in $\mathscr{I}$ such that for all E in $\mathscr{I}$, $E \sqsubset F$.

In his treatment of classical probability, Kolmogorov [5] assumes, in effect, the existence of a "grand canonical operation", that is, an operation which refines all possible experiments in a given system. We will presently show that maximal refinement ideals in an *HD* space which admits hidden variables share this property with classical probability, offering a generalized version of the classical "grand canonical operation". First we need the following result, whose proof can be found in [11].

THEOREM 9. *Let $(X, \perp)$ be an HD space, and let E, F be operations in $\mathscr{E}(X, \perp)$. Then $E \sqsubset F$ if and only if for every e in E, there is an f in F such that $f \leqq e$.*

If x and y are elements of an *HD* space $(X, \perp)$, it can be shown that the propositions $x^{\perp\perp}$ and $y^{\perp\perp}$ commute in $\Pi(X, \perp)$ if and only if x and y are either comparable or orthogonal in X. In the next theorem, we show that if $\mathscr{I}$ is a refinement ideal, then $\cup\mathscr{I}$ is a set of elements corresponding to mutually commuting propositions in $\Pi(X, \perp)$. Boolean lattices are characterized by the property that any pair of elements commute: thus refinement ideals, as one would hope, reflect classical behavior in $\Pi(X, \perp)$.

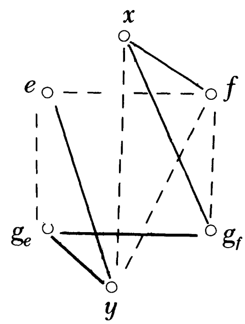
THEOREM 10. *Let $(X, \perp)$ be an HD space, $\mathscr{I}$ a refinement ideal in $\mathscr{E}(X, \perp)$. Then any two elements of $\cup\mathscr{I}$ are either comparable or orthogonal.*

PROOF. Suppose that e and f are elements of $\cup\mathscr{I}$ which are neither orthogonal nor comparable. Since $e, f \in \cup\mathscr{I}$, there are operations E and F in $\mathscr{I}$ with e in E and f in F; there is also an operation G in $\mathscr{I}$ which refines both E and F. By Theorem 9, then, there exist elements g_e and g_f in G such that $g_e \leqq e$ and $g_f \leqq f$. Now since e and f are not comparable, there exist x and y where $x \in f^{\perp} \setminus e^{\perp}$ and $y \in e^{\perp} \setminus f^{\perp}$. Further, since $e \# f$, we have $x \neq e$ and $y \neq f$, so that $x \# e$ and $y \# f$. Also note that $x \in f^{\perp} \subseteq g_f^{\perp}$, $y \in e^{\perp} \subseteq g_e^{\perp}$. If $x \perp y$, then (f, x, y, e) is a hook, a contradiction. So $x \# y$.

Either $e \neq g_e$ or $f \neq g_f$, for otherwise $g_e \# g_f$, contradicting the fact that g_e and g_f belong to the operation G.

Suppose that $e \neq g_e$ and $f = g_f$. Then $e \# g_e$. Also, since $y \# f$ while $y \perp g_e$, g_e and g_f are distinct and therefore orthogonal. But then (e, y, g_e, g_f) is a hook, a contradiction.

Therefore $e \neq g_e$ and $f \neq g_f$. If $g_e = g_f$, (x, g_e, y, e) is a hook. Thus $g_e \neq g_f$, and so $g_e \perp g_f$. We have



If g_f and y were orthogonal, (y, g_f, x, f) would be a hook; hence $g_f \# y$. Similarly, if $e \perp g_f$, (e, g_f, x, f) is a hook, whence $e \# g_f$. But now (e, y, g_e, g_f) is a hook, a final contradiction.

THEOREM 11. *Let* $(X, \perp)$ *be an HD space. Consider*:

(i) $(X, \perp)$ *admits a full set of dispersion free weights*;

(ii) *maximal refinement ideals in* $\mathscr{E}(X, \perp)$ *are principal*;

(iii) $\Pi(X, \perp)$ *is atomic.*

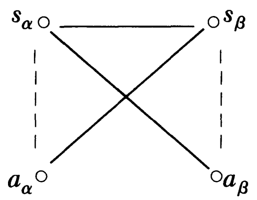
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROOF. (i) $\Rightarrow$ (ii).

Suppose that $(X, \perp)$ admits a full set of dispersion free weights, and let $\mathscr{I}$ be a maximal refinement ideal in $\mathscr{E}(X, \perp)$. Let $S \in \mathscr{S}_{df}$. Then $S \cap (\cup \mathscr{I})$ is easily seen to be a chain, since $S \cap (\cup \mathscr{I}) \subseteq S$, where no two elements of S are orthogonal, and nonorthogonal elements contained in the union of a refinement ideal are comparable by Theorem 10. Note that any chain of the form $S \cap (\cup \mathscr{I})$ can be expanded to a chain which is maximal in S, a dispersion free support set, and so by Theorem 7, such a chain is bounded below by some minimal element.

We define an equivalence relation on the dispersion free support sets: for $S, T \in \mathscr{S}_{df}$, say $S \approx T$ if and only if $S \cap (\cup \mathscr{I}) = T \cap (\cup \mathscr{I})$. Choose a representative, say S_α, from each equivalence class, and choose a minimal element a_α below $S_\alpha \cap (\cup \mathscr{I})$. Let A' denote the set of a_α so chosen.

We claim that $A' \in \mathcal{O}(X, \perp)$. For let a_α, a_β be distinct elements of A'. Then a_α and a_β are minimal elements below chains of the form $S_\alpha \cap (\cup \mathscr{I})$, $S_\beta \cap (\cup \mathscr{I})$, respectively, where $S_\alpha \not\approx S_\beta$. Thus we can assume without loss of generality that there exists some s_α in $[S_\alpha \cap (\cup \mathscr{I})] \setminus [S_\beta \cap (\cup \mathscr{I})]$. Since $s_\alpha \in \cup \mathscr{I}$, there is an operation F in $\mathscr{I}$ with s_α in F. But $S_\beta \in \mathscr{S}_{df}$, so we can find some s_β in $S_\beta \cap F$. Now $s_\beta \in S_\beta \cap (\cup \mathscr{I})$, whereas $s_\alpha \notin S_\beta \cap (\cup \mathscr{I})$, and so s_α, s_β are distinct elements of an operation F, whence $s_\alpha \perp s_\beta$. Also $a_\alpha \leqq s_\alpha$, so that $s_\alpha{}^\perp \subseteq a_\alpha{}^\perp$, yielding $s_\beta \perp a_\alpha$. Similarly, $s_\alpha \perp a_\beta$. Thus if $a_\alpha = s_\alpha$ or $a_\beta = s_\beta$, we have $a_\alpha \perp a_\beta$ as desired. So we may suppose that $a_\alpha \neq s_\alpha$ and $a_\beta \neq s_\beta$; but $a_\alpha \leqq s_\alpha$ and $a_\beta \leqq s_\beta$, so $a_\alpha \# s_\alpha$ and $a_\beta \# s_\beta$ must hold. We now have this subspace:



If a_α and a_β were scattered we would have a hook, a contradiction. Thus $a_\alpha \perp a_\beta$.

Since $A' \in \mathcal{O}(X, \perp)$, we can expand A' to a maximal orthogonal set, call it A. We shall show that A generates $\mathscr{I}$. Let $E \in \mathscr{I}$. To show that $E \sqsubset A$, it suffices by Theorem 9 to show that for every e in E, there is some a in A, with $a \leqq e$. So let e be in E. By hypothesis, $(X, \perp)$ admits a full set of dispersion free weights, equivalently, $X = \cup \mathscr{S}_{df}$, so there is a dispersion free support set S with $e \in S$. We have chosen a minimal element a_α below $S \cap (\cup \mathscr{I})$, and $a_\alpha \in A' \subseteq A$. Thus $a_\alpha \leqq e$, a_α in A, whence $E \sqsubset A$. Thus $\mathscr{I} \subseteq \{E \in \mathscr{E}(X, \perp) \mid E \sqsubset A\}$, and the maximality of $\mathscr{I}$ forces equality.

(ii) $\Rightarrow$ (iii)

Assume that maximal refinement ideals in $\mathscr{E}(X, \perp)$ are principal. We will show that every element in X lies above a minimal element; this is equivalent to showing that every non-empty proposition in $\Pi(X, \perp)$ contains a minimal non-empty proposition. Let x be arbitrarily chosen. Then $x \in \cup \mathscr{I}$, for some maximal refinement ideal $\mathscr{I}$. But $\mathscr{I} = \{E \in \mathscr{E}(X, \perp) \mid E \sqsubset A\}$, for some operation A. Thus there exists a in A with the property that $a \leqq x$. If a is not minimal, there exists f in X, $f < a$. Then $A \setminus \{a\} \subseteq a^\perp \subseteq f^\perp$, so that $A \setminus \{a\} \cup \{f\} \in \mathcal{O}(X, \perp)$, whence $A \setminus \{ \cup \{f\} \subseteq F$, for some operation F. Now it is easily seen that $\mathscr{I} \subsetneq \{E \in \mathscr{E}(X, \perp) \mid E \sqsubset F\}$, contradicting the maximality of $\mathscr{I}$. Hence a is a minimal element below x.

At this point, we remark that these implications are strict, that is, in an *HD* space (iii) $\not\Rightarrow$ (ii) and (ii) $\not\Rightarrow$ (i).

6. **Conclusion.** One can demonstrate that, in a finite *HD* space, every maximal scattered set intersects every maximal orthogonal set. An *HD* space which, though infinite, exhibits this property, is called a *finitary HD space*; thus $(X, \perp)$ is finitary if and only if for every S in $\mathscr{S}(X, \perp)$ and every E in $\mathscr{E}(X, \perp)$, $S \cap E \neq \Phi$. Consequently, every finitary *HD* space $(X, \perp)$ admits a full set of dispersion free weights, since $\mathscr{S}(X, \perp) = \mathscr{S}_{df}$. Thus Theorem 11 guarantees that maximal refinement ideals in $\mathscr{E}(X, \perp)$ are principal, and implies that $\Pi(X, \perp)$ is a complete atomic orthomodular lattice. In addition, it has been shown [4] that any *HD* space can be completed in a natural and canonical way to form a finitary *HD* space. Thus, by merely appending a certain set of outcomes and extending orthogonality in a natural way, any *HD* space is contained in one which admits hidden variables. And not only will the completed *HD* space admit a full set of dispersion free weights, but its maximal refinement ideals will be principal, and its logic will be atomic.

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