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SHAPE FIBRATIONS, II

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Introduction. The homotopy lifting property (HLP) is not a very useful notion when applied to maps $p : E \to B$ between spaces with bad local properties. For instance, if B contains no arc, then every map p has the HLP with respect to any space X and is thus a fibration. The approximate homotopy lifting property (AHLP), introduced by D. S. Coram and P. F. Duvall, Jr. [3], is useful only when E and B are ANR's. The authors therefore introduced in [12] a new class of maps $p : E \to B$ between metric compacta called *shape fibrations*. Shape fibrations reduce to approximate fibrations in the case of ANR's and preserve various nice properties of the latter even in this broader setting.

The present paper can be considered a continuation of [12]. We therefore refer to that paper for more motivation and for some definitions and results. In the present paper we first prove that for shape fibrations, whose base space has trivial shape, the inclusion of each fiber $F = p^{-1}(b) \to E$ induces a shape equivalence (Theorem 1). This yields several results of Coram and Duvall [3], including their result that the fibers of approximate fibrations are FANR's. The second main result is an exact sequence for the homotopy pro-groups (Theorem 3). The Coram and Duvall exact sequence [3] is derived as a corollary.

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2. **The homotopy lifting properties.** We recall from [12] that a level map $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ between inverse sequences $\mathbf{E} = (E_i, q_{ii'})$, $\mathbf{B} = (B_i, r_{ii'})$ of ANR's is a sequence of maps $p_i : E_i \to B_i$ such that $p_i q_{ii'} = r_{ii'} p_{i'}$ for $i \leqq i'$. The level map $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ induces a map $p : E = \lim \mathbf{E} \to B = \lim \mathbf{B}$ called the limit of $\mathbf{p}$.

Definition 1. ([12]) We say that $\mathbf{p}$ has the HLP with respect to a space X provided that each i admits a $j \geqq i$ (called lifting index) such that for any maps $h_j : X \to E_j$, $H_j : X \times I \to B_j$ with

$$(1) \qquad p_j q_j = H_{j0},$$

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there exists a homotopy $\tilde{H}_i : X \times I \to E_i$ with

$$\tilde{H}_{i0} = q_{ij}h_j \tag{2}$$

$$p_i\tilde{H}_i = r_{ij}H_j. \tag{3}$$

If in addition one can achieve that $\tilde{H}_i(x, t)$ is independent of t whenever $H_j(x, t)$ is independent of t, then we say that $\mathbf{p}$ has the *regular* HLP with respect to X.

In a similar way the AHLP is defined for a level map $\mathbf{p} : \mathbf{E} \to \mathbf{B}$. One requires that every i and $\epsilon > 0$ admit a $j \geqq i$ and a $\delta > 0$ such that whenever the distance

$$d(p_jh_j, H_{j0}) < \delta, \tag{4}$$

then there is a $\tilde{H}_i$ satisfying

$$d(\tilde{H}_{i0}, q_{ij}\, h_j) < \epsilon, \text{ and} \tag{5}$$

$$d(p_i\tilde{H}_i, r_{ij}\, H_j) < \epsilon. \tag{6}$$

For the *regular* AHLP one requires in addition that $\tilde{H}_i$ be independent of t if H_j is.

In [12] shape fibrations were defined as maps $p : E \to B$ induced by level maps $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ between ANR-sequences satisfying the AHLP. There it was shown that if one $\mathbf{p}$ has the AHLP, then so does every other level map which induces p. It was also proved that there exists a level map $\mathbf{p}' : \mathbf{E}' \to \mathbf{B}$ which induces the same p, leaves $\mathbf{B}$ unchanged and has the HLP.

In Section 3 we shall need the following simple fact:

Proposition 1. *If* $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ *is a level map which has the* HLP, *then* $\mathbf{p}$ *also has the regular* HLP.

Proof. There is no loss of generality in assuming that diam $B_j \leqq 1$. If j is the lifting index for i and we are given $h_j : X \to E_j$ and $H_j : X \times I \to B_j$ with $p_jh_j = H_{j0}$, then we define a map $\alpha : X \times I \to I$ by the formula

$$\alpha(x) = \operatorname{diam} H_j(x \times I). \tag{7}$$

Let $H_j' : X \times I \to B_j$ be given by

$$H_j'(x, t) = \begin{cases} H_j(x, t/\alpha(x)), & 0 \leqq t < \alpha(x), \\ H_j(x, 1), & \alpha(x) \leqq t \leqq 1. \end{cases} \tag{8}$$

Notice that

$$H_j'(x, t\alpha(x)) = H_j(x, t). \tag{9}$$

Since $H_j'(x, 0) = H_j(x, 0) = p_j h_j(x)$, there is a homotopy $\tilde{H}_i' : X \times I \to E_i$ such that

$$\tilde{H}'_{i0} = q_{ij}\, h_j, \text{ and} \tag{10}$$

$$p_i \tilde{H}_i' = r_{ij} H_j'. \tag{11}$$

We now define $\tilde{H}_i : X \times I \to E_i$ by the formula

$$\tilde{H}_i(x, t) = \tilde{H}_i'(x, t\alpha(x)). \tag{12}$$

Notice that (10) implies

$$\tilde{H}_i(x, 0) = \tilde{H}_i'(x, 0) = q_{ij}\, h_j(x). \tag{13}$$

Furthermore, by (11) and (9), we have

$$p_i \tilde{H}_i(x, t) = r_{ij}\, H_j'(x, t\alpha(x)) = r_{ij}\, H_j(x, t). \tag{14}$$

Finally, if $H_j(x, t)$ does not depend on t, then $\alpha(x) = 0$ and therefore $\tilde{H}_i(x, t) = \tilde{H}_i'(x, 0)$ does not depend on t.

REMARK 1. Similarly one can prove that a level map $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ between ANR-sequences has the regular AHLP if it has the AHLP.

3. **Shape fibrations with trivial base space.** The main result of this section is the following theorem (its consequences are discussed in Section 4):

THEOREM 1. *Let $p : E \to B$ be a shape fibration between metric compacta and let B have the shape of a point. If $e \in E$, $b = p(e) \in B$, $F = p^{-1}(b)$, then the inclusion $u : (F, e) \to (E, e)$ of the fiber F is a pointed shape equivalence.*

The proof proceeds in several steps.

(i) Since B has trivial shape, one can embed it in the Hilbert cube Q in such a way that there is a decreasing sequence of neighborhoods B_i of B each being homeomorphic with Q ([**18**], Theorem 1). Let us also embed E in Q and let $\tilde{p} : Q \to Q$ be an extension of p. Then one can find a decreasing sequence of closed ANR-neighborhoods E_i of E such that $\tilde{p}(E_i) \subseteq B_i$. The maps $p_i = \tilde{p} \mid E_i$ determine thus a level map $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ between ANR-sequences inducing p. There is no loss of generality in assuming that $\mathbf{p}$ has the HLP (one can achieve this without changing $\mathbf{B}$). Furthermore, by Proposition 1 $\mathbf{p}$ also has the regular HLP.

(ii) Let $e_i = q_i(e)$, $b_i = r_i(b)$, where $q_i : E \to E_i$, $r_i = B \to B_i$ are the natural projections. Let $F_i = p_i^{-1}(b_i)$. Then $q_{ii'}(F_{i'}) \subseteq F_i$ for $i \leqq i'$ and the inverse limit of the compact sequence $\mathbf{F} = (F_i,\ q_{ii'} \mid F_{i'})$ is $F = p^{-1}(b)$.

(iii) For each i choose a lifting index $j = j(i) \geqq i$ and let $k = k(i)$ be a lifting index for j. One can assume that $i \leqq i'$ implies $j \leqq j'$ and $k \leqq k'$. Since $B_k \approx Q$, one can find a homotopy $G_k : B_k \times I \to B_k$ such that

$$G_k(y, 0) = y,\ y \in B_k, \tag{1}$$

$$G_k(y, 1) = b_k,\ y \in B_k \tag{2}$$

$$G_k(b_k, s) = b_k,\ s \in I. \tag{3}$$

Let $h_k = 1_{E_k} : E_k \to E_k$ and let $H_k : E_k \times I \to \mathrm{B}_k$ be defined by

$$H_k(x, s) = G_k(p_k(x), s). \tag{4}$$

Notice that

$$H_{k0} = p_k = p_k h_k, \tag{5}$$

$$H_{k1} = b_k, \text{ and} \tag{6}$$

$$H_k(x, s) = b_k,\ x \in F_k,\ s \in I. \tag{7}$$

By the regular HLP there exists a homotopy $\tilde{H}_j : E_k \times I \to E_j$ such that

$$\tilde{H}_{j0} = q_{jk} \tag{8}$$

$$p_j \tilde{H}_j = r_{jk} H_k, \text{ and} \tag{9}$$

$$\tilde{H}_j(x, s) = \tilde{H}_j(x, 0) = q_{jk}(x),\ x \in F_k,\ s \in I. \tag{10}$$

We now define a map $f_i : E_k \to E_i$ by

$$f_i(x) = q_{ij} \tilde{H}_j(x, 1). \tag{11}$$

By (9) and (6) one has

$$\begin{aligned} p_i f_i(x) &= r_{ij} p_j \tilde{H}_j(x, 1) = r_{ik}\ H_k(x, 1) \\ &= r_{ik}\ (b_k) = b_i, \end{aligned} \tag{12}$$

and therefore $f_i(x) \in F_i$. Also notice that (10) implies that

$$f_i(e_k) = q_{ij} \tilde{H}_j(e_k, 1) = q_{ik}(e_k) = e_i, \tag{13}$$

so that f_i can be considered a map $f_i : (E_k, e_k) \to (F_i, e_i)$.

(iv) Now we shall prove that for $i \leqq i'$ the following diagram commutes up to pointed homotopy:

$$
\begin{tikzcd}
E_k \arrow[d, "f_i"'] & E_{k'} \arrow[l, "q_{kk'}"'] \arrow[d, "f_{i'}"] \\
F_i & F_{i'} \arrow[l, "q_{ii'}"]
\end{tikzcd}
\tag{14}
$$

Indeed, let the map $\varphi : E_{k'} \times (I \times 0 \cup I \times 1 \cup 0 \times I) \rightarrow E_j$ be given by

$$\varphi(x, s, 0) = q_{jj'} \tilde{H}_{j'}(x, s), \tag{15}$$

$$\varphi(x, s, 1) = \tilde{H}_j(q_{kk'}(x), s), \text{ and} \tag{16}$$

$$\varphi(x, 0, t) = q_{jk'}(x). \tag{17}$$

We shall define a map $\phi : E_{k'} \times I \times I \rightarrow B_j$ as follows. For $(x, s, t) \in E_{k'} \times (I \times 0 \cup I \times 1 \cup 0 \times I)$ let

$$\phi(x, s, t) = p_j \varphi(x, s, t). \tag{18}$$

Notice that

$$
\begin{aligned}
\phi(x, 1, 0) &= p_j q_{jj'} \tilde{H}_{j'}(x, 1) = r_{jj'} p_{j'} \tilde{H}_{j'}(x, 1) \\
&= r_{jk'} H_{k'}(x, 1) = r_{jk'}(b_{k'}) = b_j,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\phi(x, 1, 1) &= p_j \tilde{H}_j(q_{kk'}(x), 1) = r_{jk} H_k(q_{kk'}(x), 1) \\
&= r_{jk}(b_k) = b_j.
\end{aligned}
\tag{20}
$$

We define ϕ on $E_{k'} \times 1 \times I$ by

$$\phi(x, 1, t) = b_j, \tag{21}$$

and thus obtain $\phi : E_{k'} \times \partial(I \times I) \rightarrow B_j$. This is possible because of (19) and (20).

Now notice that

$$\phi(e_{k'}, s, t) = b_j, \ (s, t) \in \partial(I \times I), \tag{22}$$

because (10), (15), (16) and (17) imply

$$\varphi(e_{k'}, s, 0) = \varphi(e_{k'}, s, 1) = \varphi(e_{k'}, 0, t) = e_j, \tag{23}$$

and one can then apply (18) and (21). Therefore, one can extend ϕ to $e_{k'} \times I \times I$ by

$$\phi(e_{k'}, s, t) = b_j, \ (s, t) \in I \times I. \tag{24}$$

Finally, since $B_j \approx Q$, one can extend ϕ to a map $\phi : E_{k'} \times I \times I \rightarrow B_j$.

Since $(I \times I, I \times 0 \cup I \times 1 \cup 0 \times I) \approx (I \times I, I \times 0)$, one can view φ as a lifting of the initial stage of ϕ. Therefore, there exists a homotopy $\tilde{\phi} : E_{k'} \times I \times I \to E_i$ such that

$$\tilde{\phi}(x, s, t) = q_{ij}\varphi(x, s, t), \ (s, t) \in I \times 0 \cup I \times 1 \cup 0 \times I, \tag{25}$$

$$p_i\tilde{\phi} = r_{ij}\phi, \tag{26}$$

$$\tilde{\phi}(e_{k'}, s, t) = e_i, \ (s, t) \in I \times I. \tag{27}$$

The last formula is a consequence of (24), (25), (23) and of the fact that the regular HLP yields a lifted homotopy which leaves a point fixed, whenever the original homotopy leaves the point fixed.

Also notice that by (26) and (21) one has

$$p_i\tilde{\phi}(x, 1, t) = r_{ij}\phi(x, 1, t) = b_i, \tag{28}$$

so that $\tilde{\phi}(x, 1, t) \in F_i$. Therefore, the formula

$$K(x, t) = \tilde{\phi}(x, 1, t) \tag{29}$$

defines a homotopy $K : (E_{k'} \times I, e_{k'} \times I) \to (F_i, e_i)$ with the following properties:

$$\begin{aligned} K(x, 0) &= q_{ij}\varphi(x, 1, 0) = q_{ij'}\tilde{H}_{j'}(x, 1) \\ &= q_{ii'}f_{i'}(x), \end{aligned} \tag{30}$$

$$\begin{aligned} K(x, 1) &= q_{ij}\varphi(x, 1, 1) = q_{ij}\tilde{H}_j(q_{kk'}(x), 1) \\ &= f_i q_{kk'}(x). \end{aligned} \tag{31}$$

This establishes the commutativity of (14) up to pointed homotopy.

(v) Notice that $\mathbf{F} = (f_i, q_{ii'} \mid F_{i'})$ is an inverse sequence of compacta and, by (14), for $i \leqq i'$ the diagram

$$\begin{tikzcd} & E \arrow[dl, "f_i q_k"'] \arrow[d, "f_{i'} q_{k'}"] \\ F_i & F_{i'} \arrow[l, "q_{ii'}"] \end{tikzcd} \tag{32}$$

commutes up to pointed homotopy. Therefore, by the continuity theorem ([9], Theorem 6.1), there is a unique shape morphism $\mathbf{f} : (E, e) \to (F, e)$ such that

$$S(q_i \mid F)\mathbf{f} = S(f_i q_k), \ i = 1, 2, \cdots, \tag{33}$$

where S denotes the shape functor.

We shall show now that $\mathbf{f}$ is the shape inverse of the shape morphism $\mathbf{u} = S(u)$ induced by the inclusion map $u : (F, e) \to (E, e)$, i.e., that the following equalities hold:

$$\mathbf{fu} = 1_F \tag{34}$$

$$\mathbf{uf} = 1_E. \tag{35}$$

(vi) Because of the uniqueness in the continuity theorem, in order to establish (34), it suffices to show

$$S(q_i \mid F)\mathbf{fu} = S(q_i \mid F)1_F, \ i = 1, 2, \cdots, \tag{36}$$

i.e.

$$S(f_i q_k u) = S(q_i \mid F), \ i = 1, 2, \cdots. \tag{37}$$

However, one can even show that

$$f_i q_k u = f_i q_k \mid F = q_i \mid F, \ i = 1, 2, \cdots. \tag{38}$$

Indeed, by (11) and (10), for $x \in F$ one has

$$f_i q_k(x) = q_{ij}\tilde{H}_j(q_k(x), 1) = q_{ik} q_k(x) = q_i(x). \tag{39}$$

(vii) Finally, in order to establish (35), it suffices to prove that

$$S(q_i)\mathbf{uf} = S(q_i)1_E, \ i = 1, 2, \cdots. \tag{40}$$

Notice that $S(q_i)\mathbf{u} = S(q_i)S(u) = S(q_i u) = S(u_i)S(q_i \mid F)$, where $u_i : F_i \to E_i$ is the inclusion map. Consequently, by (33), (40) is equivalent to

$$S(u_i f_i q_k) = S(q_i), \ i = 1, 2, \cdots. \tag{41}$$

However, $q_{ij}\tilde{H}_j$ is a pointed homotopy $(E_k \times I, e_k \times I) \to (E_i, e_i)$ because, by (10), $\tilde{H}_j(e_k, s) = q_{jk}(e_k) = e_j$. Furthermore, by (8),

$$q_{ij}\tilde{H}_j(x, 0) = q_{ik}(x), \tag{42}$$

and by (11),

$$q_{ij}\tilde{H}_j(x, 1) = f_i(x) = u_i f_i(x), \tag{43}$$

so that

$$u_i f_i \simeq q_{ik}, \tag{44}$$

and therefore

$$u_i f_i q_k \simeq q_i, \tag{45}$$

which implies (41). This completes the proof of Theorem 1.

REMARK 2. The analogous statement for $*$-fibrations ([12], Definition 5) and even for fibrations is false. Indeed, let B be any non-degenerate metric continuum which contains no arc and has trivial shape (e.g., one can take for B the pseudo-arc). Let $b_0 \in B$, $E = B \vee B$ and let $p: E \to B$ be the folding map. Then for $b \neq b_0$ the fiber $F = p^{-1}(b)$ is an 0-sphere and thus has a non-trivial shape. Nevertheless, the shape of $E = B \vee B$ is trivial (e.g., see [10], Corollary 4) and p is a fibration (every homotopy into B is fixed). It follows that p is an example of a fibration ($*$-fibration) which fails to be a shape fibration. (See Remark 5 in [12] for a direct proof that p fails to be a shape fibration.)

4. Some consequences of Theorem 1. The following corollary is an easy consequence which partially answers a question from [12]:

COROLLARY 1. *Let $p: E \to B$ be a shape fibration between metric compacta. If the points b_0, $b_1 \in B$ are contained in a subcontinuum $B' \subseteq B$ of trivial shape, then the fibers $F_0 = p^{-1}(b_0)$ and $F_1 = p^{-1}(b_1)$ have the same shape.*

PROOF. Let $E' = p^{-1}(B')$, $p' = p \mid E' : E' \to B'$. It was shown in [12, Proposition 4] that p' is also a shape fibration. Therefore, one can apply Theorem 1 and conclude that

$$Sh(F_0, e_0) = Sh(E, e_0) \text{ and}$$

$$Sh(F_1, e_1) = Sh(E, e_1)$$

for any $e_0 \in F_0$, $e_1 \in F_1$. Consequently, $Sh(F_0) = Sh(F_1)$.

REMARK 3. It follows immediately from Corollary 1 that $Sh(F_0) = Sh(F_1)$ if b_0 and b_1 belong to the same path component of B, because one can join them by an arc $A \subseteq B$ and $Sh(A) = 0$. Thus we obtain an alternate proof for Theorem 3 of [12] in the case of shape fibrations. However, one cannot obtain Theorem 3 of [12] for $*$-fibrations in this way because of Remark 2 or alternately because of the following example.

EXAMPLE 1. In Remark 2 of this paper and in Remark 5 of [12] we mentioned a certain fibration ($*$-fibration) which fails to be a shape fibration. The following similar example is perhaps more instructive. It was mentioned to us by Coram and Duvall as an example of a fibration with fibers of different homotopy type. Let C be the $\sin 1/x$ curve (domain $(0, 1]$) and let A be the limit arc. Let $B = C \cup A$ and let $E = C \cup (A \times S^1)$, where $A \times x_0$ is identified with A for some fixed $x_0 \in S^1$. Finally, let $p: E \to B$ be defined by $p \mid C =$ identity and $p \mid A \times S_1 = \pi$ where $\pi: A \times S^1 \to A$ is the first projection. It follows immediately from Corollary 1 that $p : E \to B$ is not a shape fibration, however, it is instructive to construct the "obvious" level map of ANR-sequences which induces p and see that it fails to have the AHLP for the space S^1.

Another consequence of Theorem 1 is the pointed version of a result of Coram and Duvall ([3], Corollary 2.5).

COROLLARY 2. *Let $p : E \to B$ be an approximate fibration between compact* ANR*'s. For every point $e \in E$ the fiber (F, e), $F = p^{-1}(b)$, $b = p(e)$, is a pointed* FANR.

PROOF. Consider the map $p \times 1 : E \times Q \to B \times Q$. It is readily seen that $p \times 1$ is an approximate fibration and therefore a shape fibration ([12], Corollary 1). By a recent result of R. D. Edwards (see [2]), $B \times Q$ is a Q-manifold. Therefore, the point $(b, 0) \in B \times Q$ admits a closed neighborhood V homeomorphic with Q. The restriction of $p \times 1$ to $V' = (p \times 1)^{-1}(V)$ is a shape fibration ([12], Proposition 4) whose base space has trivial shape. Therefore, by Theorem 1, the inclusion $(F', e') \to (V', e')$, where $F' = (p \times 1)^{-1}(b, 0)$, $e' = (e, 0)$, is a pointed shape equivalence. Since $E \times Q$ is also a Q-manifold, there is a closed ANR-neighborhood U' of F' contained in V'. Clearly, the inclusion $(F', e') \to (U', e')$ is a pointed shape domination and therefore (f', e') is a pointed FANR (the unpointed argument like the one in [7], Theorem 6, applies to the pointed case as well). Finally, notice that $(F', e') = (F, e) \times 0$.

REMARK 4. We have shown in ([12], Example 6) that the well-known Taylor map $p : E \to Q$ of a certain continuum E with non-trivial shape onto Q fails to be a shape fibration in spite of the fact that p is cell-like (i.e., all fibers have trivial shape). This is an immediate consequence of Theorem 1 because $Sh(F) = 0$ and therefore different from $Sh(E) \neq 0$.

REMARK 5. The following question, suggested by the example of Remark 4 and by Theorem 1, was put to the authors: If $p : \mathrm{E} \to B$ is a shape fibration and a cell-like map, is it a shape equivalence? The answer is negative because of an example due to D. A. Edwards and H. M. Hastings ([6], Example (5.5.10)).

Using the same Adams map [1]. which is at the basis of the Taylor example, Edwards and Hastings have defined an inverse sequence $\mathbf{E}$ with terms E_i being the direct products of a certain finite polyhedron Y with i copies of the $2r$-sphere S^{2r}, where r is a certain integer. They also consider a sequence $\mathbf{B}$ with B_i the direct product of i copies of S^{2r} and a level map $\mathbf{p} = (p_i) : \mathbf{E} \to \mathbf{B}$, where $p_i : Y \times (S^{2r})^i \to (S^{2r})^i$ is the projection. Since each p_i is a fibration, $\mathbf{p}$ has the HLP and therefore induces a shape fibration $p : E \to B$. They show that p is cell-like but not a shape equivalence.

5. **The homotopy pro-groups of (*E*, *F*).** The main result of this section is the following theorem.

THEOREM 2. *Let $p : E \to B$ be a shape fibration between metric compacta and let $e \in E$, $b = p(e)$, $F = p^{-1}(b)$. Then p induces an isomorphism of the homotopy pro-groups (e.g., see* [11])

$$\mathbf{p}_* : \text{pro-}\pi_n(E, F, e) \to \text{pro-}\pi_n(B, b).$$

The proof proceeds in several steps.

(i) Choose a level map $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ between Q-manifold-sequences which induces p. There is no loss of generality in assuming that $\mathbf{p}$ has the HLP ([12], Theorem 2). Let $e_i = q_i(e)$, $b_i = r_i(b)$. By induction on i one can define for every i a lifting index $j = j(i) > i$ and a closed neighborhood Q_i of b_i, homeomorphic with Q and such that

$$r_{ii'}(Q_{i'}) \subseteq \operatorname{Int} Q_i,\ i < i', \tag{1}$$

$$\lim(Q_i, r_{ii'} \mid Q_{i'}) = \{b\}. \tag{2}$$

Furthermore one can choose closed ANR-neighborhoods C_i of Q_i so small that

$$r_{ii'}(C_{i'}) \subseteq \operatorname{Int} Q_i,\ i < i', \tag{3}$$

and therefore

$$\lim(C_i, r_{ii'} \mid C_{i'}) = \{b\}. \tag{4}$$

Next, one chooses closed ANR-neighborhoods F_i of $p_i^{-1}(Q_i)$ so small that

$$F_i \subseteq p_i^{-1}(C_i). \tag{5}$$

Notice that (3) implies

$$r_{ii'}p_{i'}(F_{i'}) \subseteq \operatorname{Int} Q_i \subseteq Q_i \text{ for } i < i',$$

and therefore

$$q_{ii'}(F_{i'}) \subseteq p_i^{-1}(Q_i) \subseteq F_i.$$

Furthermore, (4) implies

$$\lim(F_i, q_{ii'} \mid F_{i'}) = F. \tag{6}$$

Also notice that $p_i : (E_i, F_i, e_i) \to (B_i, C_i, b_i)$ induces $p_{i*} : \pi_n(E_i, F_i, e_i) \to \pi_n(B_i, C_i, b_i)$ and the homomorphisms p_{i*} define the morphism of pro-groups

$$\mathbf{p}*: \text{pro-}\pi_n(E, F, e) \to \text{pro } \pi_n(B, C, b).$$

(ii) In order to show that this is an isomorphism of pro-groups, it suffices to produce for each i some $k > i$ and a homomorphism

$$g: \pi_n(B_k, C_k, b_k) \to \pi_n(E_i, F_i, e_i)$$

which makes the following diagram commutative:

(7)
$$\begin{tikzcd}
\pi_n(E_i, F_i, e_i) \arrow[d, "p_{i*}"'] & & \pi_n(E_k, F_k, e_k) \arrow[ll, "q_{ik*}"'] \arrow[d, "p_{k*}"] \\
\pi_n(B_i, C_i, b_i) & & \pi_n(B_k, C_k, b_k) \arrow[ll, "r_{ik*}"] \arrow[llu, "g"']
\end{tikzcd} .$$

We choose for k a lifting index with respect to $j = j(i)$. Every element $\alpha \in \pi_n(B_k, C_k, b_k)$ is given by a map

$$\phi: (I^n, \partial I^n, J^{n-1}) \to (B_k, C_k, b_k),$$

where $J^{n-1} = (\partial I^{n-1}) \times I \cup (I^{n-1} \times 1)$.

Let $\varphi: J^{n-1} \to E_k$ be the constant map e_k. Notice that

(8) $$p_k\varphi = b_k = \phi \mid J^{n-1}.$$

Since $(I^n, J^{n-1}) \approx (I^n, I^{n-1} \times 0)$, one can view φ as a map $I^{n-1} \times 0 \to E_k$ and ϕ as a homotopy $I^{n-1} \times I \to B_k$ with the initial stage equal to $p_k\varphi$. Therefore, there is a map $\tilde{\phi}: I^n \to E_j$ such that

(9) $$\tilde{\phi} \mid J^{n-1} = e_j, \text{ and}$$

(10) $$p_j\tilde{\phi} = r_{jk}\phi.$$

By (3), $r_{jk}\phi(\partial I^n) \subseteq Q_j$ and therefore

(11) $$\tilde{\phi}(\partial I^n) \subseteq p_j^{-1}(Q_j) \subseteq F_j.$$

This means that $\tilde{\phi}$ is a mapping $(I^n, \partial I^n, J^{n-1}) \to (E_j, F_j, e_j)$ and thus determines an element $[\tilde{\phi}] \in \pi_n(E_j, F_j, e_j)$.

We now define g by

(12) $$g[\phi] = q_{ij*}[\tilde{\phi}] = [q_{ij}\, \tilde{\phi}].$$

(iii) We first need to show that g is well-defined, i.e., independent of the choice of $\tilde{\phi}$ and ϕ. Assume that

$$\phi': (I^n, \partial I^n, J^{n-1}) \to (B_k, C_k, b_k)$$

is another representative of α and that $\tilde{\phi}'$ satisfies (9) and (10) (with ϕ replaced by ϕ'). Then there exists a homotopy

$$H:(I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (B_k, C_k, b_k)$$

such that

$$H_0 = \phi, H_1 = \phi'.$$

Let us consider the map

$$h:(I^n \times 0) \cup (I^n \times 1) \cup (J^{n-1} \times I) \to E_j$$

given by

(13) $$h \mid I^n \times 0 = \tilde{\phi}$$

(14) $$h \mid I^n \times 1 = \tilde{\phi}', \text{ and}$$

(15) $$h \mid J^{n-1} \times I = e_j.$$

Notice that (10) implies

(16) $$p_j h = r_{jk} H \mid (I^n \times 0) \cup (I^n \times 1) \cup (J^{n-1} \times I).$$

Since j is a lifting index for i, one obtains a homotopy $\tilde{H}: I^n \times I \to E_i$ with the following properties:

(17) $$\tilde{H}_0 = q_{ij}\tilde{\phi}$$

(18) $$\tilde{H}_1 = q_{ij}\tilde{\phi}'$$

(19) $$\tilde{H} \mid J^{n-1} \times I = e_i, \text{ and}$$

(20) $$p_i\tilde{H} = r_{ik}H.$$

Notice that $H(\partial I^n \times I) \subseteq C_k$ and therefore, by (3), $r_{ik}H(\partial I^n \times I) \subseteq Q_i$. Consequently, by (20), $p_i\tilde{H}(\partial I^n \times I) \subseteq Q_i$, which implies

(21) $$\tilde{H}(\partial I^n \times I) \subseteq p_i^{-1}(Q_i) \subseteq F_i.$$

In other words, $\tilde{H}$ is a map $(I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (E_i, F_i, e_i)$. By (17) and (18) we conclude that indeed $[q_{ij}\tilde{\phi}] = [q_{ij}\tilde{\phi}']$.

(iv) We now prove that g is a homomorphism of groups. Let $\alpha = \alpha'\alpha''$ and let $\alpha' = [\phi']$, $\alpha'' = [\phi'']$. Then α is represented by the map

$$\phi:(I^n, \partial I^n, J^{n-1}) \to (B_k, C_k, b_k)$$

given by

$$(22)\qquad \phi(x, s, t) = \begin{cases} \phi'(x, 2s, t), & 0 \leqq s \leqq 1/2, \\ \phi''(x, 2s-1, t), & 1/2 \leqq s \leqq 1, \end{cases}$$

where $x \in I^{n-2}$, $t \in I$.

Notice that ϕ' and ϕ'' induce $\tilde{\phi}', \tilde{\phi}'' : (I^n, I^{n-1}, J^{n-1}) \to (E_j, F_j, e_j)$ such that the analogues of (9) and (10) hold. We now define

$$\tilde{\phi} : (I^n, \partial I^n, J^{n-1}) \to (E_j, F_j, e_j)$$

by

$$(23)\qquad \tilde{\phi}(x, s, t) = \begin{cases} \tilde{\phi}'(x, 2s, t), & 0 \leqq s \leqq 1/2, \\ \tilde{\phi}''(x, 2s-1, t), & 1/2 \leqq s \leqq 1, \end{cases}$$

where $x \in I^{n-2}$, $t \in I$. From (22) and (23), and from (9) and (10) applied to $\tilde{\phi}'$ and $\tilde{\phi}''$, one obtains (9) and (10) for $\tilde{\phi}$, which proves that

$$(24)\qquad g[\phi] = q_{ij*}[\tilde{\phi}].$$

However, by (23), $[\tilde{\phi}] = [\tilde{\phi}'][\tilde{\phi}'']$ and therefore

$$g(\alpha'\alpha'') = g(\alpha) = g[\phi] = q_{ij*}[\tilde{\phi}']q_{ij*}[\tilde{\phi}'']$$
$$= g[\phi']\, g[\phi''] = g(\alpha')g(\alpha'').$$

(v) Now we shall establish that

$$(25)\qquad p_{i*}g = r_{ik*}.$$

If $\alpha = [\phi]$, then

$$p_{i*}g(\alpha) = [p_i q_{ij}\tilde{\phi}] = [r_{ij}\, p_j\tilde{\phi}] \text{ and } r_{ik*}(\alpha) = [r_{ik}\, \phi].$$

However, by (10), $r_{ij}\, p_j\tilde{\phi} = r_{ik}\, \phi$, and one obtains (25).

(vi) Finally, let us establish that

$$(26)\qquad gp_{k*} = q_{ik*}.$$

Let $\beta \in \pi_n(E_k, F_k, e_k)$ be given by a map

$$\varphi : (I^n, \partial I^n, J^{n-1}) \to (E_k, F_k, e_k),\ \beta = [\varphi].$$

Then $p_{k*}(\beta) = [\phi]$, where $\phi = p_k\varphi$.

Put $\tilde{\phi} = q_{jk}\varphi$ and notice that $\tilde{\phi}$ satisfies (9) and (10) because

$$p_j\tilde{\phi} = p_j q_{jk}\varphi = r_{jk}\, p_k\varphi = r_{jk}\, \phi.$$

Therefore,

$$gp_{k*}(\beta) = q_{ij*}[\tilde{\phi}] = [q_{ik}\varphi]$$
$$= q_{ik*}[\varphi] = q_{ik*}(\beta).$$

This concludes the proof of Theorem 2.

If we pass to the shape groups

$$\check{\pi}_n(E, F, e) = \lim(\pi_n(E_i, F_i, e_i), q_{ii'*})$$
$$\check{\pi}_n(B, b) = \lim(\pi_n(B_i, b_i), r_{ii'*}),$$

then Theorem 2 yields the following corollary.

COROLLARY 3. *Let $p : E \to B$ be a shape fibration between metric compacta and let $e \in E$, $b = p(e)$, $F = p^{-1}(b)$. Then p induces an isomorphism of the shape groups.*

$$p* : \check{\pi}_n(E, F, e) \to \check{\pi}_n(B, b).$$

This result generalizes ([3], Theorem 3.4).

6. **The exact homotopy sequence of a shape fibration.** For any compact pair (E, F, e) the homotopy pro-groups form an exact sequence

$$\begin{aligned} \cdots \to \text{pro-}\pi_n(F, e) \to \text{pro-}\pi_n(E, e) \to \\ \text{pro-}\pi_n(E, F, e) \to \text{pro-}\pi_{n-1}(F, e) \to \cdots \end{aligned} \tag{1}$$

of pro-groups (see [11], 5.2). If we combine this fact with Theorem 2, we obtain

THEOREM 3. *Let $p : E \to B$ be a shape fibration between metric compacta, $e \in E$, $b = p(e)$, $F = p^{-1}(b)$. Then the following sequence of homotopy pro-groups is exact*

$$\begin{aligned} \cdots \to \text{pro-}\pi_n(F, e) \xrightarrow{\mathrm{i}_*} \text{pro-}\pi_n(E, e) \xrightarrow{\mathrm{p}_*} \\ \text{pro-}\pi_n(B, b) \xrightarrow{\delta} \text{pro-}\pi_{n-1}(F, e) \to \cdots. \end{aligned} \tag{2}$$

Here $\mathrm{i}*$ and $\mathrm{p}*$ are morphisms of pro-groups induced by the inclusion map $i : F \to E$ and by the map $p : \mathrm{E} \to B$ respectively, δ is the composition of the inverse of the morphism of pro-groups induced by $p : (E, F, e) \to (B, b, b)$ (Theorem 2) and of the boundary morphism pro-$\pi_n(E, F, e) \to$ pro-$\pi_{n-1}(F, e)$ *induced by the boundary homomorphisms $\pi_n(E_i, F_i, e_i) \to \pi_{n-1}(F_i, e_i)$.*

REMARK 6. It is an immediate consequence of Theorem 3 that a cell-like shape fibration p induces isomorphisms of homotopy pro-groups. In fact this is true for any cell-like map ([4], Theorem 1.1). Bearing in mind the various Whitehead theorems for shape, this also suggests the question settled in Remark 5.

We now show that Corollary 3.5 of [3] is a consequence of Theorem 3.

COROLLARY 4. (CORAM AND DUVALL). *Let* $p : E \to B$ *be an approximate fibration between compact* ANR*'s, and let* $e \in E$, $b = p(e)$, $F = p^{-1}(b)$. *Then the following sequence of groups is exact*:

$$\cdots \check{\pi}_n(F, e) \xrightarrow{i_*} \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \xrightarrow{\delta_*} \check{\pi}_{n-1}(F, e) \to \cdots. \tag{3}$$

For the proof we need the following lemma.

LEMMA 1. *Let* (X, x) *be a pointed* FANR. *Then the natural morphism of the shape group* $\check{\pi}_n(X, x)$ *into the homotopy pro-group* pro-$\pi_n(X, x)$ *is an isomorphism of pro-groups.*

PROOF OF LEMMA 1. A pointed FANR (X, x) is a pointed shape retract of an ANR (Y, x) (the argument given in ([7], Theorem 6) applies to the pointed case as well). Since the homotopy pro-groups of (Y, x) are isomorphic to the corresponding homotopy groups, we conclude that the homotopy pro-groups of (X, x) are dominated by groups. However, D. A. Edwards and R. Geoghegan have shown that for a pro-group **G**, which is dominated by a group, the natural projection from the inverse limit $G = \lim \mathbf{G}$ to **G** is an isomorphism of pro-groups ([5], Proposition 3.3). Therefore, $\check{\pi}_n(X, x) \to$ pro-$\pi_n(X, x)$ is an isomorphism.

PROOF OF COROLLARY 4. The pro-groups pro-$\pi_n(E, e)$ and pro-$\pi_n(B, b)$ are naturally isomorphic with the homotopy groups $\pi_n(E, e)$ and $\pi_n(B, b)$ respectively because E and B are ANR's. By Corollary 2, (F, e) is a pointed FANR. Therefore, by Lemma 1 the homotopy pro-groups pro-$\pi_n(F, e)$ are naturally isomorphic with the shape groups $\check{\pi}_n(F, e)$. Consequently, under the assumptions of Corollary 4, the exact sequence (2) assumes the form (3).

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