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DIFFERENTIABLE POINTS OF THE GENERALIZED CANTOR FUNCTION

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ABSTRACT. The generalized Cantor function Θ_γ has a derivative equal to $1/(1 - \gamma)$ at almost every point in the set C_γ. This was established by Darst [1] who then posed the problem of characterizing those points which are not differentiable. The differentiability of points in C_γ is determined by the spacing of the 0's and 2's in a ternary-like expansion. Points that are interval endpoints have one-sided derivatives from both sides.

1. **Introduction.** To describe a generalized Cantor set, denoted by C_γ, and the corresponding Cantor function Θ_γ, first choose a number γ satisfying $0 < \gamma < 1$. The usual Cantor set is obtained when $\gamma = 1$. The set C_γ is obtained in the same manner as the standard Cantor set by deleting a sequence $\{(a_i, b_i)\}_{i=1}^{\infty}$ of pairwise disjoint segments from the interior of the unit interval. In general, the k-th step consists of removing an open interval of length $\gamma/3^k$ from the middle of each of the 2^{k-1} closed intervals, thereby leaving 2^k closed intervals of equal length. This length is in fact equal to $(1 - \gamma_k)/2^k$, where $\gamma_k = \gamma[1 - (2/3)^k]$. The process continues, and C_γ is defined to be the set of points in $[0, 1]$ which fail to be removed. The measure of C_γ is positive and equals $1 - \gamma$. The corresponding Cantor function is defined analogously to the standard Cantor function. The function Θ_γ is a nonnegative, nondecreasing continuous function. In addition, Darst established that $\Theta_\gamma'(x) = 1/(1 - \gamma)$ for almost all x in C_γ. Characterizing the set of points in $[0, 1]$ at which Θ_γ is not differentiable is the problem this paper concerns itself with.

2. **Derivatives at Endpoints.** In establishing $\Theta_\gamma'(x) = 1/(1 - \gamma)$ for almost all x in C_γ, Darst showed that

$$\left| \frac{\Theta_\gamma(y) - \Theta_\gamma(x)}{y - x} \right| \leqq \frac{1}{1 - \gamma}$$

for all x, y in $[0, 1]$ with $x \neq y$. Our first result is that all right (left) hand interval endpoints have derivatives from the right (left) which equal $1/(1 - \gamma)$. A geometric approach will be used and a sketch of the proof given. To proceed, let x be an arbitrary right endpoint, where the length of the removed interval is $\gamma/3^k$ and k is some positive integer. For each integer $n > k$, let $J_n = (u_n, v_n)$ be the removed in-

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terval of length $\gamma/3^n$ closest to x on the right. Consequently, for each integer $n > k$, we have $x < u_{n+1} < v_{n+1} < u_n < v_n$, $v_n - u_n = \gamma/3^n$ and $u_n - x = (1 - \gamma_n)/2^n$. We will define the sequence $\{\beta_i\}$ by

$$\beta_i = 1/(1 - \gamma_{i+1} + \gamma(2/3)^{i+1}).$$

This sequence is increasing, $\beta_i < 1/(1 - \gamma_i) < \beta_{i+1}$, and converges to $1/(1 - \gamma)$. Using this fact and Darst's inequality, if we can show that for each integer $n > k$,

$$\frac{\Theta_\gamma(y) - \Theta_\gamma(x)}{y - x} \geqq \beta_{n-1}$$

for each y such that $v_{n+1} \leqq y \leqq v_n$, it will follow that Θ_γ is differentiable at x from the right. Consider the following diagram.

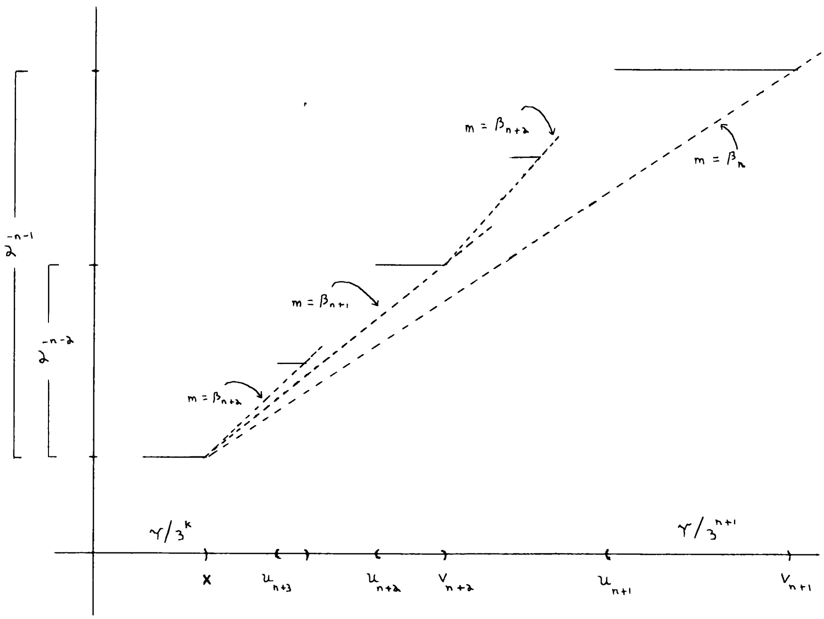


Figure 1.

In outlining a possible approach, one considers the three disjoint cases,

(i) $y \in [u_i, v_i]$ for some $i > k$

(ii) $y \in (v_{i+1}, u_i)\backslash C_\gamma$ for some $i > k$

(iii) $y \in (v_{i+1}, u_i) \cap C_\gamma$ for some $i > k$.

In parts (i) and (ii) one makes use of the nature of $\{\beta_i\}$ and the fact that the graph of Θ_γ lies above the proper piecing of the dotted-line graph. One completes part (iii) by choosing an appropriate sequence $\{r_j\}$ of right hand endpoints converging upward to y with $r_1 = x$ and such that

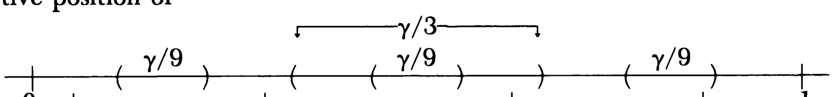
$$\frac{\Theta_\gamma(r_{j+1}) - \Theta_\gamma(r_j)}{r_{j+1} - r_j} = \beta_{k_j}.$$

It is important to note in the diagram that there will be 2^{m-1} dotted lines connecting a pair of right endpoints on the graph of Θ_γ with slope equal to β_{n+m}. This fact helps to complete part (ii).

The above result on one-sided derivatives can also be obtained without this geometric approach and employing instead an analytic method making use of Lemmas 1 and 2 from the next section.

3. **Derivatives at Nonendpoints.** Now let $x \in C_\gamma$ where x is not an interval endpoint. There exist endpoints as close as you want to x, and on either side of it. One's intuition might lead one to believe that x should therefore have right and left hand derivatives with both equalling $1/(1 - \gamma)$. This is not the case, primarily because some members of C_γ are "closer to an endpoint" than others. The idea to be used in the following is that, in computing the right hand derivative of x, the worst possible case would be to choose a sequence $\{h_n\} \searrow 0$ such that $x + h_n$ are right hand endpoints. This is "worst" in the sense that if we want the difference quotient to converge to something other than $1/(1 - \gamma)$, then this should do it.

One of the first difficulties in dealing with this problem is the inability to get a handle on the members of C_γ. For this we introduce a code system, very similar to base 3. Let $x \in [0, 1]$, and write $x = .x_1x_2 \cdots$ (code) with $x_i \in \{0, 1, 2\}$ where these three digits denote the relative position of



x at the i-th step in the construction of C_γ. It follows that $x \in C_\gamma$ if and only if $x = .x_1x_2x_3 \cdots$ (code) with $x_i \in \{0, 2\}$ for all i, and consequently $\Theta_\gamma(x) = .(x_1/2)(x_2/2) \cdots$ (base 2). The number $x = .202020 \cdots$ (code) is in C_γ but is not an endpoint for any of the intervals removed in the construction of C_γ. Note that if $x \in C_\gamma$, then x is a right (left) endpoint for some interval if, after a certain stage in the expansion (preference is given to the expansion involving 0's and 2's) of x, all the digits are 0's (2's). In general, the expansion $.00 \cdots 0x_{k_1}0 \cdots 0x_{k_2}0 \cdots 0x_{k_n}0 \cdots$ (code) with $x_{k_i} = 2$ represents the number

$$\sum_{i=1}^{\infty} \left[(1 - \gamma_{k_i})/2^{k_i} + \gamma/3^{k_i} \right].$$

This fact helps us verify that addition of two code expansions of members of C_γ can be computed as in base 3 arithmetic, provided the 2's don't overlap. For example, if $x = .202020 \cdots$ (code) and $y = .02000200000 \cdots$ (code) then $x + y = .222022202020 \cdots$ (code). In addition, if $x = .x_1x_2 \cdots x_n000 \cdots$ (code) and $y = .00 \cdots 0y_{n+1}y_{n+2} \cdots$ (code) with $x_1 \in \{0, 2\}$ and $y_i \in \{0, 1, 2\}$ then $x + y = .x_1x_2 \cdots x_ny_{n+1}y_{n+2} \cdots$ (code). The following lemma will prove useful.

LEMMA 1. *Let $x_i \in \{0, 1\}$ with $x_1 = 1$. Then*

$$\frac{\overbrace{.00 \cdots 0}^{n}x_1x_2x_3 \cdots \text{(base 2)}}{.00 \cdots 0(2x_1)(2x_2) \cdots \text{(code)}}$$

converges to $1/(1 - \gamma)$, and the convergence is uniform for all choices of (x_i).

PROOF. Let $A_k = .x_1x_2 \cdots x_k000 \cdots$ (base 2) and $B_k = .(2x_1) \cdots (2x_k)000 \cdots$ (code). Then $A_k/B_k \leqq A_{k+1}/B_{k+1}$ and in addition

$$\frac{1}{1 - \gamma[1 - (4/3)(2/3)^n]} = \frac{.00 \cdots 01000 \cdots \text{(base 2)}}{\underbrace{.00 \cdots 0}_{n}2000 \cdots \text{(code)}}$$

$$\leqq \frac{.0 \cdots 01x_2 \cdots x_k00 \cdots \text{(base 2)}}{.0 \cdots 02(2x_2) \cdots (2x_k)0 \cdots \text{(code)}}$$

and this last expression is bounded by $1/(1 - \gamma)$ because it is equal to $[\Theta_\gamma(y + h) - \Theta_\gamma(y)]/h$ where $h = .00 \cdots 02(2x_2) \cdots (2x_k)000 \cdots$ (code) and $y = .2000 \cdots$ (code). The sequence therefore converges to the desired limit.

Since every monotonic function has a derivative almost everywhere, the following will re-establish that $\Theta_\gamma'(x) = 1/(1 - \gamma)$ for almost all x in C_γ.

LEMMA 2. *Let $x \in C_\gamma$ be a nonleft endpoint. Then there exists a sequence of numbers $\{h_n\} \searrow 0$ such that $[\Theta_\gamma(x + h_n) - \Theta_\gamma(x)]/h_n$ converges to $1/(1 - \gamma)$.*

PROOF. Let $x = .22 \cdots 2x_{k_1}2 \cdots 2x_{k_n}2 \cdots$ (code) with $x_{k_i} = 0$ for all i. Define $\{h_n\}$ by $h_n = .00 \cdots 0h_{k_n}0h_{k_{n+1}}0 \cdots$ (code) with $h_{k_i} = 2$ for all $i \geqq n$. Then

$$\frac{\Theta_\gamma(x + h_n) - \Theta_\gamma(x)}{h_n} = \frac{.00 \cdots 010 \cdots 010 \cdots 010 \cdots \text{(base 2)}}{.00 \cdots 020 \cdots 020 \cdots 020 \cdots \text{(code)}}$$

$$\to \frac{1}{1 - \gamma}.$$

Before proceeding, recall that the expansion of left hand interval endpoints, using only 0's and 2's, is characterized by the fact that after a certain stage all the digits are 2's. It follows that if $x \in C_\gamma$ with $x = .x_1x_2x_3 \cdots x_n0222 \cdots$ (code), then x is the left endpoint for an interval of length $\gamma/3^{n+1}$. Now suppose $x \in C_\gamma$ is not an endpoint. The code expansion for x contains infinitely many 0's and infinitely many 2's, and we write

$$x = .22 \cdots 2x_{k_1}2 \cdots 2x_{k_2}2 \cdots 2x_{k_3}2 \cdots \text{(code)}$$

with $x_{k_i} = 0$ for $i = 1, 2, \cdots$. Define a sequence $\{h_n\} \searrow 0$ by

$$h_n = \underbrace{\overbrace{.000 \cdots 0}^{k_{n+1}}20 \cdots 0}_{k_{n+2}}20 \cdots 020 \cdots \text{(code)} + \gamma/3^{k_n}.$$

A pictorial representation is given below.

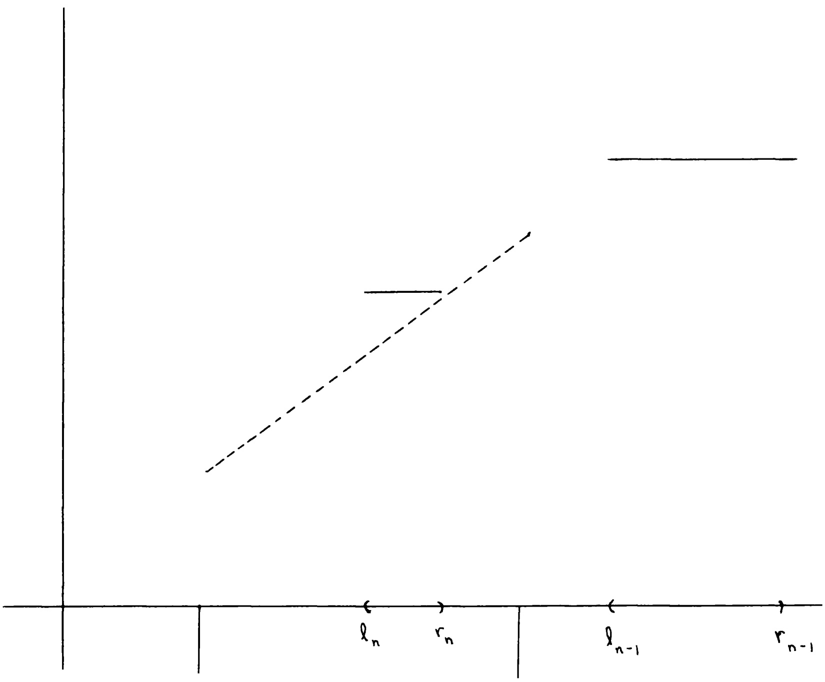


Figure 2.

Then

$$\frac{\Theta_\gamma(x+h_n)-\Theta_\gamma(x)}{h_n}$$

$$=\frac{\Theta_\gamma(.22\cdots 2x_{k_1}2\cdots 2x_{k_n}222\cdots(\text{code})+\gamma/3^{k_n})-\Theta_\gamma(x)}{h_n}$$

$$=\frac{\Theta_\gamma(.22\cdot\cdot\, 2x_{k_1}2\cdot\cdot\, 2x_{k_n}222\cdot\cdot\,(\text{code}))-\Theta_\gamma(.22\cdot\cdot\, 2x_{k_1}2\cdot\cdot\,(\text{code}))}{h_n}$$

$$=\frac{.00\cdots 010\cdots 010\cdots 010\cdots(\text{base } 2)}{\underbrace{.00\cdots 0}_{k_{n+1}}20\cdots 020\cdots 020\cdots(\text{code})+\gamma/3^{k_n}}.$$

This last expression is of the form $a_n/(b_n+c_n)$ where we know that a_n/b_n converges to $1/(1-\gamma)$. Consequently

$$\left|\frac{a_n}{b_n+c_n}-\frac{1}{1-\gamma}\right| \leqq \left|\frac{a_n}{b_n+c_n}-\frac{a_n}{b_n}\right|+\left|\frac{a_n}{b_n}-\frac{1}{1-\gamma}\right|$$

$$=\frac{a_n}{b_n}\left|\frac{1}{1+(b_n/c_n)}\right|+\left|\frac{a_n}{b_n}-\frac{1}{1-\gamma}.\right|$$

The expression b_n/c_n is the major determinant concerning the differentiability of Θ_γ at x. The following theorems result from its investigation.

THEOREM 2. *Let $x\in C\gamma$ not be an interval endpoint, and let k_n denote the position of the n-th zero (two) in the code expansion of x. If*

$$\limsup \frac{k_{n+1}}{k_n} > \frac{\ln 3}{\ln 2}$$

then Θ_γ fails to have a derivative at x from the right (left).

PROOF. There exists a number $L>\ln 3/\ln 2$ and a sequence $\{k_{n_j}\}$ such that $k_{n_j}/k_{n_j-1}\geqq L$. As before, choose $\{h_j\}\searrow 0$ by

$$h_j=\underbrace{\overbrace{.000\cdots 0}^{k_{n_j}}20\cdots 020\cdots 0}_{k_{n_j+1}}20\cdots(\text{code})+\gamma/3^{k_{n_j}-1}.$$

Then

$$\left|\frac{\Theta_\gamma(x+h_j)-\Theta_\gamma(x)}{h_j}\right|$$

$$= \frac{\overbrace{.00 \cdots 010}^{k_{n_j}} \cdots 010 \cdots \text{(base 2)}}{.00 \cdots 020 \cdots 020 \cdots \text{(code)} + \gamma/3^{k_{n_j}-1}}$$

$$\leqq \frac{2(1/2)^{k_{n_j}}}{\gamma/3^{k_{n_j}-1}}$$

$$\leqq \frac{2}{\gamma}(3/2^L)^{k_{n_j}-1}.$$

So this particular sequence of difference quotients converges to zero. From Lemma 2 there exists another sequence of difference quotients that converges to $1/(1 - \gamma)$. Hence Θ_γ is not differentiable at x from the right. The parenthetical case follows by symmetry.

One notices that for such x's as described above, the number of 2's in the code expansion far exceeds the number of 0's. Recalling the expansion for left interval endpoints, we can say that the x's from above are, in a sense, close to left endpoints. In fact, they are too close, and that is why a sequence of difference quotients converging to zero can be found. A similar result follows for members of C_γ that are close to right endpoints.

Usually it is harder to show that the derivative exists at a point than to show it doesn't. It was no exception with Θ_γ. The feeling is that if $x \in C_\gamma$ and its code expansion contains a "decent" proportion of 0's and 2's arranged in a "decent" manner then it should be a point of differentiability. A typical candidate would be $x = .02020202 \cdots$ (code). The analog to Theorem 2 follows.

THEOREM 3. *Let $x \in C_\gamma$ not be an interval endpoint, and let k_n denote the position of the n-th zero (two) in the code expansion of x. If*

$$\limsup \frac{k_{n+1}}{k_n} < \frac{\ln 3}{\ln 2}$$

then Θ_γ has a derivative at x from the right (left) equal to $1/(1 - \gamma)$.

PROOF. There exists a number $L < \ln 3/\ln 2$ such that $k_{n+1}/k_n \leqq L$ for all but a finite number of choices of n. Define a sequence of real numbers $\{h_n\} \searrow 0$ by

$$h_n = \underbrace{\overbrace{.000 \cdots 020}^{k_{n+1}} \cdots 020}_{k_{n+2}} \cdots 020 \cdots \text{(code)} + \gamma/3^{k_n}.$$

Then

$$\frac{1}{1-\gamma} \geqq \frac{\Theta_\gamma(x+h_n)-\Theta_\gamma(x)}{h_n}$$

$$= \frac{\Theta_\gamma(x+h_n-\gamma/3^{k_n})-\Theta_\gamma(x)}{h_n}$$

$$= \frac{.\overbrace{000\cdots 010}^{k_{n+1}}\cdots 010\cdots 010\cdots \text{(base 2)}}{.000\cdots 020\cdots 020\cdots 020\cdots \text{(code)} + \gamma/3^{k_n}}$$

and this last expression tends to $1/(1-\gamma)$ because, for all except finitely many n,

$$\frac{b_n}{c_n} = \frac{.00\cdots 020\cdots 020\cdots \text{(code)}}{\gamma/3^{k_n}}$$

$$\geqq \frac{.00\cdots 0100000\cdots \text{(code)}}{\gamma/3^{k_n}}$$

$$\geqq \frac{1-\gamma_{k_{n+1}}}{\gamma}(3/2^L)^{k_n},$$

which tends to infinity. So the sequence of slopes $\{(\Theta_\gamma(x+h_n)-\Theta_\gamma(x))/h_n\}$ of secant lines drawn from x to the right hand endpoints of certain "plateaus" of Θ_γ, located to the right of x, converges to $1/(1-\gamma)$. This is enough to guarantee that every sequence of difference quotients from the right of x also converges to $1/(1-\gamma)$. To see this, we proceed as in [**2**].

The only difficulty that could occur would be when $x+h$ is exterior to all intervals of length $\gamma/3^{k_n}$. So we first let

$$l_n = x + h_n - \gamma/3^{k_n} \text{ and } r_n = x + h_n$$

and notice that any $\hat{h}_n$ where $0 < \hat{h}_n < l_{n-1} - r_n$ it follows that $\Theta_\gamma(r_n + \hat{h}_n) = \Theta_\gamma(r_n) + \Theta_\gamma(\hat{h}_n)$. Then, for $r_n < x + h < l_{n-1}$, we have

$$\frac{\Theta_\gamma(x+h)-\Theta_\gamma(x)}{h}$$

$$= \frac{\Theta_\gamma(x+r_n-x+\hat{h}_n)-\Theta_\gamma(x)}{r_n-x+\hat{h}_n} \quad (\hat{h}_n = x+h-r_n)$$

$$= \frac{\Theta_\gamma(r_n)-\Theta_\gamma(x)+\Theta_\gamma(\hat{h}_n)}{r_n-x+\hat{h}_n}.$$

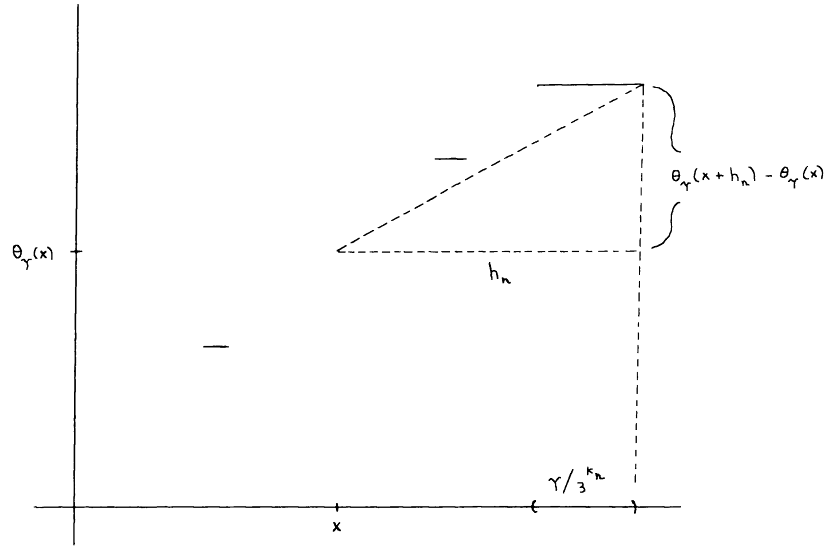


Figure 3.

Since $[\Theta_\gamma(r_n) - \Theta_\gamma(x)]/(r_n - x)$ converges to $1/(1 - \gamma)$ and $\Theta_\gamma(\hat{h}_n)/\hat{h}_n$ also converges to $1/(1 - \gamma)$ it follows that the above combination also converges to $1/(1 - \gamma)$. Thus

$$\frac{1}{1-\gamma} \geqq \lim_{\substack{h \to 0 \\ r_n < x+h < l_{n-1}}} \frac{\Theta_\gamma(x + h) - \Theta_\gamma(x)}{h}$$

$$= \lim_{n \to \infty} \frac{[\Theta_\gamma(r_n) - \Theta_\gamma(x)] + \Theta_\gamma(\hat{h}_n)}{[r_n - x] + \hat{h}_n}$$

$$= \frac{1}{1-\gamma}$$

and we already know that

$$\frac{1}{1-\gamma} \geqq \lim_{\substack{h \to 0 \\ l_n \leqq x+h \leqq r_n}} \frac{\Theta_\gamma(x + h) - \Theta_\gamma(x)}{h} = \frac{1}{1-\gamma},$$

consequently the derivative from the right exists and equals $1/(1 - \gamma)$. The parenthetical case again follows by symmetry.

For example, if $x = .20220222022220222220 2 \cdots$ (code) then $\Theta_\gamma'(x) = 1/(1 - \gamma)$. The remaining situation is when

$$\limsup \frac{k_{n+1}}{k_n} = \frac{\ln 3}{\ln 2}$$

and one should suspect that differentiability may or may not occur in this case. The following result, done in collaboration with Professor Darst, establishes this.

THEOREM 4. *Let $x \in C_\gamma$ not be an interval endpoint, and let k_n denote the position of the n-th zero (two) in the code expansion of x. If*

$$\limsup \frac{k_{n+1}}{k_n} = \frac{\ln 3}{\ln 2}$$

then Θ_γ may or may not have a derivative at x from the right (left).

PROOF. We first shall exhibit an $x \in C_\gamma$ satisfying the above conditions and which is differentiable from the right. From the proof of Theorem 3, it suffices to construct an increasing sequence $\{k_n\}$ of positive integers satisfying $k_n \to \infty$, $k_{n+1}/k_n \to \ln 3/\ln 2$, $k_{n+1}/k_n < \ln 3/\ln 2$ and $3^{k_n}/2^{k_{n+1}} \to \infty$. Initiate the sequence with $k_1 = 100$ and recursively define

$$k_{n+1} = \left[\frac{(\ln 3 - 1/k_n^{1/2})}{\ln 2} \right] k_n$$

where the brackets denote the greatest integer function. Then

$$\frac{(\ln 3 - 1/k_n^{1/2})}{\ln 2} k_n - 1 \leqq k_{n+1} \leqq \frac{(\ln 3 - 1/k_n^{1/2})}{\ln 2} k_n$$

so $k_n \to \infty$, $k_{n+1}/k_n \to \ln 3/\ln 2$ and $k_{n+1}/k_n < \ln 3/\ln 2$. Consequently $k_{n+1}/k_n \leqq (\ln 3 - 1/k_n{}^{1/2})/\ln 2$, so $\exp(k_n^{1/2}) \leqq 3^{k_n}2^{-k_{n+1}}$ and thus $3^{k_n}/2^{k_{n+1}} \to \infty$.

To exhibit an $x \in C_\gamma$ satisfying the main condition and which is not differentiable from the right, we examine the proof of Theorem 2. In this case it suffices to construct an increasing sequence $\{k_n\}$ of positive integers satisfying $k_n \to \infty$, $k_{n+1}/k_n \to \ln 3/\ln 2$, $k_{n+1}/k_n > \ln 3/\ln 2$ and $3^{k_n}/2^{k_{n+1}} \to 0$. Again let $k_1 = 100$ and define

$$k_{n+1} = \left[\frac{(\ln 3 + 1/k_n^{1/2})}{\ln 2} \right] k_n + 1;$$

then a similar argument to the one above completes the proof.

These results are "measure-theoretically adequate" in the sense that the measure of the set $\{x \in C_\gamma\colon \limsup k_{n+1}/k_n = \ln 3/\ln 2\}$ is zero. Letting S denote this set, we apply [3, p. 107, no. 13b] with $g = \Theta_\gamma$ and $E = \{\Theta_\gamma(x)\colon x \in S\}$ and conclude that the measure of S is zero.

References

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