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# ON THE SUPPORT OF THE RADON TRANSFORM

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A basic tool in the study of hyperbolic equations is the Radon transform: if $f(x)$ is a function defined in $R^n$, then one takes $Rf(s, \omega) = \int_P f(x)\, d\sigma$, where $s \in R$, $\omega \in R^n$ has norm one, $P$ is the hyperplane in $R^n$ given by $\{x \in R / x \cdot \omega = s\}$, and $d\sigma$ is the element of $(n - 1)$-dimensional measure.

The domain of the Radon transform can be taken to be $\mathscr{S}(R^n)$, the space of smooth functions which, together with all derivatives, vanish rapidly at infinity. For this case, it turns out that the Radon transform is in $\mathscr{S}(R \times S^{n-1})$.

Now, one sees directly that if $f$ were in $C_0^\infty (R^n)$ and had support in $|x| \leqq M$, then $Rf(s, \omega)$ would vanish for all $|s| \geqq M$, that is, $\int_P f(x)\, d\sigma = 0$ for all hyperplanes $P$ in $R^n$ whose distance to the origin is at least $M$.

It is an interesting result that the converse statement is true.

THEOREM. *Let $f \in \mathscr{S}(R^n)$, and suppose that $\int_P f\, d\sigma = 0$ for all hyperplanes $P$ in $R^n$ whose distance to the origin is greater than some $M > 0$. Then in fact $f \in C_0^\infty(R^n)$, with support in $|x| \leqq M$.*

This fact appears in Ludwig [1] and Lax-Phillips [2], where its proof depends on spherical harmonic expansions and the Paley-Wiener theorem. The present note will give a direct proof.

STEP ONE. *If the theorem holds in $R^{n+1}$ then it holds in $R^n$.*

To prove this, take $f \in \mathscr{S}(R^n)$. Then pass to $g \in \mathscr{S}(R^{n+1})$ where $g(x, t) = f(x)\phi(t)$, with $x \in R^n$, $t \in R$, and $\phi \in C_0^\infty(R)$. Suppose $\phi$ has support in $-\epsilon \leqq t \leqq \epsilon$.

Suppose $\int_{P_{n-1}} f\, d\sigma_{n-1} = 0$ for all hyperplanes $P_{n-1}$ in $R^n$ whose distance to the origin is at least $M \geqq 0$. Then consider $\int_{P_n} g\, d\sigma_n$, where $P_n$ is a hyperplane in $R^{n+1}$ whose distance to the origin is at least $\sqrt{M^2 + \epsilon^2}$.

If $\operatorname{supp}(g) \cap P_n = \phi$ then $\int_{P_n} g\, d\sigma_n = 0$. If not, then for each $t \in R$ let $P_{n-1}(t) \subset R^n$ denote the hyperplane $\{x \in R^n / (x, t) \in P_n \subset R^{n+1}\}$. If $|t| < \epsilon$ the geometry of the situation shows that $P_{n-1}(t)$ will have a distance at least $M$ to the origin of $R^n$.

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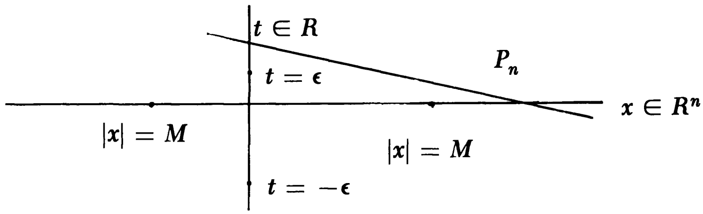


Now $\int_{P_n} g \, d\sigma_n = (\text{constant}) \times \int_{-\epsilon}^{+\epsilon} \phi(t) \int_{P_{n-1}(t)} f(x) \, d\sigma_{n-1} \, dt$. Since the integrand vanishes by the hypothesis on $f$, we see that $\int_{P_n} g \, d\sigma_n = 0$ for all hyperplanes in $R^{n+1}$ sufficiently distanct from the origin.

Thus the theorem can be applied in $R^{n+1}$, showing that $f(x)\phi(t)$ has support in $|x|^2 + t^2 \leqq M^2 + \epsilon^2$. Letting $\epsilon$ pass to zero, one gets the result for $f \in \mathscr{S}(R^n)$.

From now one can assume that $n \geqq 3$, and is odd.

STEP TWO. *The theorem holds when $f(x)$ is radial.*

For, taking $f(x) = \phi(|x|)$ where $\phi : R \to R$, one can use the particular hyperplane

$$P(s) = \{x \in R^n / x_1 = s\}, \text{ for } s > 0.$$

Then one gets

$$\int_{P(s)} f(x) \, d\sigma = \text{const.} \int_0^\infty \phi\left(\sqrt{s^2 + r^2}\right) r^{n-2} \, dr$$

$$= \text{const. } F(s),$$

where $F(s) = \int_s^\infty \phi(t)(t^2 - s^2)^{(n-3)/2} \, t \, dt$, via the change of variables $s^2 + r^2 = t^2$. Moreover, $F(s) = 0$ in $s > M$.

Now the exponent $(n - 3)/2 = m$ is a nonnegative integer; for the case $n = 3$ and $m = 0$ we have $F'(s) = -s\phi(s)$, proving that $\phi(s) = 0$ in $s > M$ as desired.

For $m \geqq 1$ we have

$$-\frac{1}{2ms} \frac{d}{ds} F(s) = \int_s^\infty \phi(t)(t^2 - s^2)^{m-1} \, t \, dt.$$

Thus $\phi(s)$ can always be obtained from $F(s)$ by successively dividing by $s$ and differentiating. Hence $F(s) = 0$ in $s > M$ implies that $\phi(s) = 0$ in $s > M$.

STEP THREE. *Given $f \in \mathscr{S}(R^n)$ with $\int_P f \, d\sigma = 0$ for all hyperplanes $P \subset R^n$ whose distance to the origin is at least $M$. Then $\int_{|x|=r} f(x) \, d\sigma = 0$ for all $r \geqq M$.*

To prove this, consider the functions $f(Vx)$ where $V$ ranges over the orthogonal linear transformations of $R^n$ to itself.

Take $g(x) = \int_{V\in 0(n)} f(Vx)$, with integration by Haar measure in the compact group $0(n)$. Evidently $g(x) = g(V_0x)$, showing that $g$ is constant over the spheres $|x| = r$. To identify $g$, note that $\int_{|x|=r} f(Vx)\,d\sigma = \int_{|x|=r} f(x)\,d\sigma$ for all $V \in 0(n)$; then, changing the order of integration in

$$\int_{|x|=r} d\sigma \int_{V\in 0(n)} f(Vx),$$

one sees that $\int_{|x|=r} g(x)\,d\sigma$ is equal to $C_0 \int_{|x|=r} f(x)\,d\sigma$, where $C_0$ is the total measure of $0(n)$. Since $g$ is radial, we have that $g(x)$ is a nonzero multiple of the integral of $f$ over the sphere of radius $|x|$.

But the transformations $V$ preserve hyperplanes and their distances. Thus each function $x \mapsto f(Vx)$ satisfies the hypothesis, as does their average $g(x)$. Now, $g(x) \in \mathscr{S}$: since $g$ is radial we must have $g \equiv 0$ for $|x| > M$, according to Step Two. This gives the result.

STEP FOUR. *Given $f \in \mathscr{S}(R^n)$, with $\int_P f\,d\sigma = 0$ for hyperplanes $P \subset R^n$ whose distance to the origin is greater than $M > 0$. Then $f(x) = 0$ for $|x| > M$.*

Take $\epsilon > 0$. Then, for $a \in R^n$ with $|a| < \epsilon$, the displaced function $f(x + a)$ will continue to satisfy the hypothesis, but with the slightly larger constant $M + \epsilon$. From the preceding step we have

$$\int_{|x-a|=r} f(x)\,d\sigma = 0,$$

valid when $|a| < \epsilon$ and $r > M + \epsilon$.

Put $a = a_0 + tE_j$ where $E_j$ is the unit coordinate vector in the $j$th direction. We differentiate the equation with respect to $t$ at $t = 0$, recalling that the finite difference quotients of $f$ converge uniformly to $\partial f/\partial x_j$. The result is that

$$\int_{|x-a|=r} \frac{\partial f}{\partial x_1}(x)\,d\sigma = 0$$

in the same range of $r$ and $a$.

Now take the vector field $V = f(x)E_j$, whose divergence is $\partial f/\partial x_j$. From the preceding formula we have

$$\int_{r<|x-a|<R} \operatorname{div} V\,dR^n = 0,$$

valid for $|a| < \epsilon$, $M + \epsilon < r < R$. Gauss' theorem gives

$$\int_{|x-a|=r} V.N\, d\sigma = \int_{|x-a|=R} V.N\, d\sigma.$$

The second integral tends to zero with large $R$ because $f$ and its derivatives decrease rapidly and uniformly. Hence

$$\int_{|x-a|=r} V.N\, d\sigma = \int_{|x-a|=r} f(x) \frac{x_j - a_j}{|x - a|}\, d\sigma = 0,$$

when $r > M + \epsilon$.

The foregoing has established that if $f$ satisfies $\int_{|x-a|=r} f(x)\, d\sigma = 0$ for all $r > M + \epsilon$ and $|a| < \epsilon$, then $x_j f(x)$ will have the same property. By repeated application, one sees that every polynomial multiple $P(x) f(x)$ will have a zero integral over the relevant spheres.

But then $f(x) \equiv 0$ in $|x| > M + \epsilon$ by Weierstrass approximation.

This completes the proof of the theorem.

## References

1. Donald Ludwig, *The Radon Transform on Euclidean Space,* Comm. Pure Appl. Math., XIX, (1966), 49–81.

2. P. D. Lax and R. S. Phillips, *The Paley-Wiener Theorem for the Radon Transform,* Comm. Pure Appl. Math. XXIII (1970), 409–424.

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