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ERGODIC THEOREMS OF POPULATION DYNAMICS

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ABSTRACT. The ergodic theorems of population dynamics give conditions under which closed single-sex populations converge, in some sense, to age structures (probability density functions of individuals according to age) and growth rates (changes per unit time in total number of individuals) which are independent of certain initial conditions. We review models in discrete time $n = 0, 1, 2, \cdots$ with a finite number k of age-classes. The action of age-specific birth- and death-rates, collectively called vital rates, is modeled as a linear operator (the "Leslie matrix") x_n acting on an age-structure vector y_n according to $y_{n+1} = x_{n+1}y_n$.

The strong ergodic theorem of demography (which assumes a fixed $x_n = x$), the weak ergodic theorem (which assumes x_n changing deterministically), the strong stochastic ergodic theorem (which assumes x_n determined by a homogeneous Markov chain) and the weak stochastic ergodic theorem (which assumes x_n determined by an inhomogeneous Markov chain) are each motivated by the empirical inadequacies of population projection techniques based on models assumed previously. The strong stochastic ergodic theorem establishes convergence in distribution of the normalized age structure and almost sure convergence of the limiting growth rate; it gives explicit procedures for calculating these limits.

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