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PERIODIC SOLUTIONS OF AN EPIDEMIC MODEL WITH A THRESHOLD

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1. **Introduction.** In a recent paper by F. Hoppensteadt and P. Waltman [1], a very general model for the spread of an infection in a population was described. A main feature of this model was the idea of a threshold which must be overcome before the infection can spread. A susceptible individual may become exposed to the disease and thereupon become infective if he/she is exposed to a sufficient amount of infectiousness. Upon recovery the individual is temporarily immune and then again susceptible. Since the susceptible class is replenished by those who have recovered and lost their immunity, it is not unexpected that solutions of the resulting mathematical equations should exhibit some recurrence properties. Indeed there is biological data that suggest recurrence, e.g. [11], as well as numerical experiments on the model equations themselves [3, 4] which indicate that periodic solutions may exist. It is the aim of this paper to demonstrate that periodic solutions of the model equations do exist in the presence of a periodic environment. More specifically, we show, under certain conditions on the prameters involved in the model, that biologically reasonable, periodic solutions of the asymptotic form of the equations derived by Hoppensteadt and Waltman do exist. The condition is essentially that the typical infective exposes a sufficiently large nuber of susceptibles over its period of infectiousness. Left open is the question of whether there are admissible initial data which give rise to solutions approaching this periodic solution. Thus we have partially answered a problem suggested in the monograph of P. Waltman [2, chapter 5].

In § 2 we will briefly describe the model of Hoppensteadt and Waltman. For a more detailed description the interested reader should consult [1] *or* [2, chapters 4 and 5]. Section 3 contains the main result, Theorem 3.3, and its proof.

2. **Brief description of the model of Hoppensteadt and Waltman.** We suppose that the population may be divided into four groups: the proportion of individuals who are susceptible to the infection but have not been exposed to it, denoted S; the proportion who have become exposed to the disease but at not yet infective, denoted E; the infectives, denoted by I; and those individuals who have recovered and are temporarily immune, denoted by R.

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Schematically, the progress of an individual is represented in the following diagram:

$$S \to E \to I \to R \to S$$

The infection is assumed to spread according to the following rules:

(i) The rate of exposure of susceptibles to infectives at time t is given by

$$r(t)\ S(t)\ I(t)$$

(ii) An individual who is first exposed at time τ becomes infective at time t if

$$\int_\tau^t [\rho_1(x) + \rho_2(x)I(x)]\, dx = m$$

where $\rho_1(x)$, $\rho_2(x)$ are given nonnegative functions and m is a nonnegative constant

(iii) An individual who becomes infective at time t recovers from the infection at time $t + \sigma$, σ a given positive constant.

(iv) An individual, upon recovering, is immune for a time interval of length $\omega \geqq 0$, whereupon he again becomes susceptible.

(v) The population size is constant, i.e.,

$$S(t) + E(t) + I(t) + R(t) \equiv 1 \qquad \text{for all } t.$$

From these assumptions Hoppensteadt and Waltman [1] derive a set of five integral equations of which three are sufficient to determine the progress of the infection. For $t > \sigma + \omega$, where we assume that at time $t = 0$, $I_0 + S_0 = 1$, these equations may be written:

$$\int_{\tau(t)}^t [\rho_1(x) + \rho_2(x)I(x)]\, dx = m, \tag{2.1}$$

$$S(t) = 1 - \int_{\tau(t-\sigma-\omega)}^t r(x)S(x)I(x)\, dx, \tag{2.2}$$

$$I(t) = \int_{\tau(t-\sigma)}^{\tau(t)} r(x)S(x)I(x)\, dx. \tag{2.3}$$

In addition we assume a periodic environment, that is, we assume $r(x + 1) = r(x)$, $\rho_1(x + 1) = \rho_1(x)$, $\rho_2(x + 1) = \rho_2(x)$ for all $x \in R$.

At this point we remark that the equations (2.1)–(2.3) reduce to the form of the equations studied in our earlier work [7] if $\rho_2 \equiv 0$ and $\rho_1(x) \equiv \rho_1 > 0$ so that $\tau(t) = t - m/\rho_1$. However, in general, as is evident from (2.1), $\tau(t)$ is coupled in a more complicated nonlinear way with I.

3. **Main Results.** In this section we consider the problem of finding periodic solutions of the equations

$$\int_{\tau(t)}^{t} [\rho_1(x) + \rho_2(x)I(x)]\, dx = m, \tag{3.1}$$

$$S(t) = 1 - \int_{\tau(t-\sigma-\omega)}^{t} r(x)S(x)I(x)\, dx, \tag{3.2}$$

$$I(t) = \int_{\tau(t-\sigma)}^{\tau(t)} r(x)S(x)I(x)\, dx. \tag{3.3}$$

Throughout this section the following assumptions will be made: (A) $m > 0$, $\sigma > 0$, $\omega > 0$ and $r(x)$, $\rho_1(x)$, $\rho_2(x)$ are continuous 1-periodic functions with $r(x)$, $\rho_1(x) > 0$ and $\rho_2(x) \geqq 0$.

It is convenient to make a change of variables as follows: $y_1 = 1 - S$, $y_2 = I$. Accordingly (3.1)–(3.3) become

$$\int_{\tau(t)}^{t} [\rho_1(x) + \rho_2(x)y_2(x)]\, dx = m, \tag{3.4}$$

$$y_1(t) = \int_{\tau(t-\sigma-\omega)}^{t} r(x)y_2(x)(1 - y_1(x))\, dx, \tag{3.5}$$

$$y_2(t) = \int_{\tau(t-\sigma)}^{\tau(t)} r(x)y_2(x)(1 - y_1(x))\, dx. \tag{3.6}$$

Before proceeding to our main results we require some preliminary notation. Let X_1 be the Banach space of 1-periodic continuous functions on R equipped with the usual supremum norm. Let $K_1 \subseteq X_1$ be the cone of nonnegative functions in X_1. Let $X_2 = X_1 \times X_1$ with norm $\|(y_1, y_2)\| = \max(\|y_1\|, \|y_2\|)$ and $K_2 = ((y_1, y_2) \in K_1 \times K_1 : y_1(t) \geqq y_2(t))$. K_2 is a cone in X_2. We will convert the problem of finding 1-periodic solutions of (3.4)–(3.6) to a fixed point problem for an operator mapping K_2 into itself.

LEMMA 3.1. *Equation* (3.4) *defines a continuous mapping* $G : K_1 \to K_1$ *given by* $y \to p$ *where* $\tau(t) = t - p(t)$. *Moreover, the following hold*

(a) $$p(t) = g(t, m) - \frac{1}{\rho_1(t - g(t, m))} \int_{t-g(t,\, m)}^{t} \rho_2(x)y(x)\, dx + o(\|y\|)$$

as $\|y\| \to 0$ *where* $g = g(t, m)$ *is the unique solution of the equation.*

$$\int_{t-g}^{T} \rho_1(x)\, dx = m.$$

The function g *is* 1-*periodic in* t. *If* $\rho_1(x) \equiv \rho_1$ *then* $g(t, m) \equiv m/\rho_1$

(b) $\tau(t) = t - p(t)$ *is monotone increasing and* $p(t) \leqq m/\min \rho_1(x)$
(c) G *is Lipschitz continuous*:

$$\|p_1 - p_2\| \leqq \frac{m\|\rho_2\|}{[\min \rho_1(x)]^2} \|y_2 - y_1\|,$$

$$\textit{where } p_i = G(y_i) \quad i = 1, 2.$$

(d) *If* $t_2 > t_1$, $\tau(t) = t - p(t)$, $p = G(y)$ *then*

$$\frac{\min \rho_1(x)}{\|\rho_1\| + \|\rho_2\| \, \|y\|} \leqq \frac{\tau(t_2) - \tau(t_1)}{t_2 - t_1} \leqq \frac{\|\rho_1\| + \|\rho_2\| \, \|y\|}{\min \rho_1(x)}.$$

Proof of Lemma 3.1. Given $y \in K$, and $t \in R$, the continuity of the integral in (3.4) with respect to τ assures the existence of a unique value of τ, $\tau(t) = t - p(t)$, for which (3.4) holds. Clearly $p(t + 1) = p(t)$ for all $t \in R$ and the implicit function theorem implies that p is continuous, in fact, differentiable in t. Thus p is in K_1.

Now consider the equation

$$k(t, \ell, p) \equiv \int_{t-p}^{t} \rho_1(x)\, dx - \ell = 0, \quad \text{for } \ell > 0.$$

Reasoning as above and employing the implicit function theorem, we can solve this equation for p, $p = g(t, \ell)$ where g is a C^1 function of $t \in R$ and $\ell > 0$. Now writing (3.4) as

$$\int_{t-p(t)}^{t} \rho_1(x)\, dx = m - \int_{t-p(t)}^{t} \rho_2(x) y(x)\, dx,$$

allows us to conclude that

$$p(t) = g\left(t, m - \int_{t-p(t)}^{t} \rho_2(x) y(x)\, dx \right).$$

Since g is C^1 we have

$$p(t) = g(t, m) - \frac{1}{\rho_1(t - g(t, m))} \int_{t-p(t)}^{t} \rho_2(x) y(x)\, dx + 0\left(\left| \int_{t-p}^{t} \rho_2(x) y(x)\, dx \right| \right)$$

where we have used the fact that

$$\frac{\partial g}{\partial \ell}(t, m) = \frac{1}{\rho_1(t - g(t, m))} .$$

Then (a) follows by replacing the p in the integral on the right side of the above equation by the entire right hand side of this equation and expanding the resulting integral as a sum of integrals.

Part (b) is clear from equation (3.4). To verify (c), note that

$$\int_{t-p_1}^{t} \rho_1(x)\,dx - \int_{t-p_2}^{t} \rho_1(x)\,dx$$

$$+ \int_{t-p_1}^{t} \rho_2(x)y_1(x)\,dx - \int_{t-p_2}^{t} \rho_2(x)y_2(x)\,dx = 0,$$

or, combining terms

$$\int_{t-p_1}^{t-p_2} \rho_1(x)\,dx + \int_{t-p_1}^{t-p_2} \rho_2(x)y_1(x)\,dx$$

$$= \int_{t-p_2}^{t} \rho_2(x)(y_2(x) - y_1(x))\,dx.$$

Take abolute values of both sides noting that the absolute value of the sum on the left is the sum of the absolute values:

$$\left| \int_{t-p_1}^{t-p_2} \rho_1(x)\,dx \right|$$

$$\leqq \left| \int_{t-p_2}^{t} \rho_2(x)(y_2(x) - y_1(x))\,dx \right|.$$

Thus

$$\min \rho_1(x)|p_2 - p_1| \leqq \|\rho_2\| \frac{m}{\min \rho_1(x)} \|y_2 - y_1\|$$

and (c) follows.

To see that (d) holds, let $t_1 < t_2$ and notice that

$$0 = \int_{T(t_1)}^{t_1} [\rho_1(x) + \rho_2(x)y(x)]\,dx$$

$$- \int_{T(t_2)}^{t_2} [\rho_1(x) + \rho_2(x)y(x)]\,dx.$$

This yields

$$\int_{T(t_1)}^{T(t_2)} [\rho_1(x) + \rho_2(x)y(x)]\,dx = \int_{t_1}^{t_2} [\rho_1(x) + \rho_2(x)y(x)]\,dx.$$

It is easy to deduce (d) from this last equality.

We can now pose the periodic problem (3.4)–(3.6) as a fixed point problem as follows. Given $(y_1, y_2) \in K_2$ let $\tau(t) = t - p(t)$ where $p = G(y_2)$. Putting (y_1, y_2) in the right hand side of (3.5) and (3.6) with τ determined as above, we obtain from the left hand side (z_1, z_2). If $(z_1, z_2) = (y_1, y_2)$ we have found a periodic solution.

Let $F : K_2 \to K_2$ be defined by:

$$(y_1, y_2) \to (z_1, z_2) \text{ where}$$

$$z_1(t) = \int_{\tau(t-\sigma-\omega)}^{t} r(x) f(y_1(x), y_2(x))\, dx \tag{3.8}$$

$$z_2(t) = \int_{\tau(t-\sigma)}^{\tau(t)} r(x) f(y_1(x), y_1(x))\, dx \tag{3.9}$$

with

$$\tau(t) = t - p(t),\ p = G(y_2)$$

and

$$f(y_1, y_2) = \begin{cases} y_2(1 - y_1) & 0 \leqq y_1 \leqq 1 \\ 0 & y_1 > 1. \end{cases}$$

Since our interest is in solutions with values in the interval (0, 1) we will often write $y_2(1 - y_1)$ in place of f with the understanding that $y_2(1 - y_1) = 0$ if $y_1 > 1$.

It is easy to see that $(z_1, z_2) \in K_2$. Some useful properties of the mapping F are contained in the following lemma.

Lemma 3.2. *F maps K_2 into itself. Moreover we have*

(a) $F(0, 0) = (0,0)$

(b) *F is completely continuous*

(c) *F is Frechet differentiable at* (0, 0) *in the direction of the cone with Frechet derivative given by*

$$[F'(0, 0)\ (h_1, h_2)]\,(t) = \left(\int_{t-\sigma-\omega-g(t-\sigma-\omega,m)}^{t} r(x)h_2(x)\, dx,\ \int_{t-\sigma-g(t-\sigma,m)}^{t-g(t,m)} r(x)h_2(x)\, dx \right). \tag{3.10}$$

Proof. It is immediate that $F(0, 0 = (0, 0)$. Let us show F is continuous. Let $(y_1, y_2), (y_1^0, y_2^0) \in K_2$ with $\|(y_1, y_2)\|, \|(y_1^0, y_2^0)\| \leqq M$, $\tau = t - p$, $\tau^0 = t - p^0$ with $p = G(y_2)$ and $p^0 = G(y_2^0)$. From Lemma 3.1 we have $|\tau(t) - \tau^0(t)| = |p(t) - p^0(t)| \leqq m\|\rho_2\|/(\min \rho_1(x))^2 \|y_2 - y_2^0\|$. Let $F(y_1^0, y_2^0) = (z_1^0, z_2^0)$, $F(y_1, y_2) = (z_1, z_2)$. Then

$$z_1(t) - z_1^0(t) = \int_{\tau(t-\sigma-\omega)}^{t} r(x)y_2(x)\,(1 - y_1(x))\,dx$$

$$- \int_{\tau^0(t-\sigma-\omega)}^{t} r(x)y_2^0(x)\,(1 - y_1^0(x))\,dx$$

$$= \int_{\tau(t-\sigma-\omega)}^{\tau^0(t-\sigma-\omega)} r(x)y_2(x)\,(1 - y_1(x))\,dx$$

$$+ \int_{\tau^0(t-\sigma-\omega)}^{t} r(x)\,[y_2(x)\,(1 - y_1(x))$$

$$- y_2^0(x)\,(1 - y_1^0(x))].$$

Thus (see (3.11)–(3.13) below)

$$|z_1(t) - z_1^0(t)| \leqq \frac{\|r\|}{4}\,|\tau^0(t-\sigma-\omega) - \tau(t-s-\omega)|$$

$$+ 3\,\|r\|\,|t - \tau^0(t-\sigma-\omega)|$$

$$\cdot \max(\|y_1 - y_1^0\|, \|y_2 - y_2^0\|)$$

$$\leqq \frac{m\|r\|\,\|\rho_2\|}{4(\min \rho_1(x))^2}\,\|y_2 - y_2^0\|$$

$$+ 3\,\|r\|\left(\sigma + \omega + \frac{m}{\min \rho_1}\right)$$

$$\cdot \max(\|y_1 - y_1^0\|, \|y_2\,y_2^0\|)$$

$$\leqq \left(\frac{m\|r\|\,\|\rho_2\|}{4(\min \rho_1)^2}, 3\|r\|\left(\sigma + \omega + \frac{m}{\min \rho_1}\right)\right)$$

$$\cdot \|(y_1, y_2) - (y_1^0, y_2^0)\|.$$

In the above calculation we have used the following easily verified facts:

(3.11) $$\max_{0 \leqq y_2 \leqq y_1} |f(y_1, y_2)| = 1/4$$

(3.12) $$|f(y_1, y_2) - f(x_1, x_2)| \leqq 3 \max(|y_1 - x_1|, |y_2 - x_2|).$$

A similar calculation yields

$$|z_2(t) - z_2^0(t)| \leqq \max\left(\frac{m\|r\|\,\|\rho_2\|}{4(\min \rho_1)^2}, 3\|r\|\left(\frac{\|\rho_1\| + M\|\rho_2\|}{\min \rho_1}\right)\sigma\right)$$

$$\cdot \|(y_1, y_2) - (y_1^0, y_2^0)\|.$$

Hence

$$\|(z_1, z_2) - (z_1{}^0, x_2{}^0)\| \leqq C(M)\|(y_1, y_2) - (y_1{}^0, y_2{}^0)\|$$

$$C(M) = \max\left(\frac{m\|r\| \ \|\rho_2\|}{4(\min \rho_1)^2}, 3\|r\|(\sigma + \omega + m/\min \rho_1), \right.$$

$$\left. 3\|r\|\sigma \left(\frac{\|\rho_1\| + M\|\rho_2\|}{\min \rho_1} \right)\right)$$

demonstrating the continuity of F.

Let us now show F maps bounded sets into compact sets. Let $M > 0$ and $B(0, M)$ denote the ball of radius M centered at $0 \in X$. Then $F(K_2 \cap B(0, M)) \subseteq F_1(K_2 \cap B(0, M)) \times \overline{F_2(K_2 \cap B(0, M))}$ where $F = (F_1, F_2)$ so that it suffices to show that $\overline{F_i(K \cap B(0, M)}$ $\subseteq X_1$ is compact. We do this in case $i = 1$, the other case being analogous. Clearly $F_1(K_2 \cap B(0, M))$ is bounded, in fact, $F_1(K_2) \subseteq B(0, 1/4\|r\|(\sigma + \omega + m/\min \rho_1))$. Let $z_1 = F_1(y_1, y_2)$ where $(y_1, y_2) \in K_2 \cap B(0, M)$ and $t_1 > t_2$.

Then

$$\begin{aligned} z_1(t_1) - z_1(t_2) &= \int_{\tau(t_1-\sigma-\omega)}^{t_1} r(x)y_2(x)(1 - y_1(x))\,dx \\ &\quad - \int_{\tau(t_2-\sigma-\omega)}^{t_2} r(x)y_2(x)(1 - y_1(x))\,dx \\ &= \int_{t_2}^{t_1} r(x)y_2(x)(1 - y_1(x))\,dx \\ &\quad - \int_{\tau(t_2-\sigma-\omega)}^{\tau(t_1-\sigma-\omega)} r(x)y_2(x)(1 - y_1(x))\,dx. \end{aligned}$$

Thus

$$\begin{aligned} &|z_1(t_1) - z_1(t_2| \\ &\quad \leqq \frac{\|r\|}{4}|t_1 - t_2| + \frac{\|r\|}{4}|\tau(t_1 - \sigma - \omega) - \tau(t_2 - \sigma - \omega)| \\ &\quad \leqq \frac{\|r\|}{4}|t_1 - t_2| + \frac{\|r\|}{4}\left(\frac{\|\rho_1\| + M\|\rho_2\|}{\min \rho_1} \right)|t_1 - t_2| \\ &\quad = \frac{\|r\|}{4}\left(1 + \frac{\|\rho_1\| + M\|\rho_2\|}{\min \rho_1} \right)|t_1 - t_2| \end{aligned}$$

where we have used (3.11) and (d) of lemma 3.1. This shows that

$F_1(K_2 \cap B(0, M))$ is equicontinuous and the Ascoli-Arzela theorem completes the proof.

To prove (3.10) it suffices to consider F_1 and F_2 separately. We linearize F_2 as follows.

$$z_2(t) = \int_{\tau(t-\sigma)}^{\tau(t)} r(x)y_2(x)(1 - y_1(x)\,dx$$

$$= \int_{t-\sigma-g(t-\sigma,m)}^{t-g(t,m)} r(x)y_2(x)\,dx + \int_{t-g(t,m)}^{\tau(t)} r(x)y_2(x)\,dx$$

$$- \int_{t-\sigma-g(t-\sigma,m)}^{\tau(t-\sigma)} r(x)y_2(x)\,dx - \int_{\tau(t-\sigma)}^{\tau(t)} r(x)y_2(x)y_1(x)\,dx.$$

We must show that each of the lst three terms is $0(\|(y_1, y_2)\|)$ as $\|(y_1, y_2)\| \to 0$. It is apparent that the last term is $0(\|(y_1, y_2)\|)$. Consider the second term:

$$\int_{t-g(t,\ m)}^{\tau(t)} r(x)y_2(x)\,dx \,| \leqq \|r\|\ \|y_2\|\ |\tau(t) - t + g(t, m)|$$

$$= \|r\|\ \|y_2\| \left| \frac{1}{\rho_1(t - g(t, m))} \int_{t-g(t,m)}^{t} \rho_2(x)y_2(x)\,dx + 0(\|y_2\|) \right|$$

$$= 0(\|(y_1, y_2)\|)\,.$$

The third term is handled in a similar fashion. Also, the linearization of F, employs essentially the same ideas. This completes the proof of Lemma 3.2.

We are now in position to state our main result.

THEOREM 3.3. *If* (a) *holds and*

$$(\min r)\,\sigma \left(\frac{\min \rho_1}{\max \rho_1} \right) > 1. \tag{3.13}$$

Then (3.4)–(3.6) *has at least one* 1*-periodic solution,* (y_1, y_2)*, with* $0 < y_2 \leqq y_1 < 1$ *for all* $t \in R$.

To make a biological interpretation of this result we must first take account of the fact that the population has been normalized to one in our work. This was not done in [1]. The effect of this is that our $r(x)$ is actually N times the $r(x)$ appearing in [1], where N is the constant population size. Thus our $r(x)$ = (population size) $\times$ (contact rate at time x). The duration of infectiousness, σ, times the contact rate is a measure of the number of susceptibles exposed by the typical infective over his period of infectiousness. Thus (3.13) says that the minimum number of susceptibles expected to be exposed by the typical infective over his period of infectiousness is larger than $\max \rho_1 / \min \rho_1$ $(1/N)$.

We remark that a slightly more precise result obtains if $\rho_1(x) \equiv \rho_1$. In this case $g(t, m) \equiv m/\rho_1$ by Lemma 3.1 (a), and the linearization (3.10) of F contains only constant delays. Thus the results of our earlier work [7] involving constant delays become available. In particular Theorems 3.1 and 3.3 of that paper hold.

The proof of Theorem 3.3 is based on the following fixed point theorem, a proof of which appears in the author's Ph.D. Thesis [6] and in a paper of the author and J. A. Gatica [8] as well as in the work of Roger Nussbaum [9] and H. Amann [10].

FIXED POINT THEOREM. *Let $A : K \to K$ be a completely continuous operator defined on the cone K contained in a Banach space. Suppose Av 0, A is differentiable in the direction of the cone at $x = 0$, and the following hold*:

(a) *$A'(0)$ has an eigenvector $k \in K$ corresponding to an eigenvalue $\lambda > 1$, and 1 is not an eigenvalue corresponding to an eigenvector in K.*

(b) *There exists a positive number R such that if $x \in K$, $\|x\| = R$, and $Ax = \mu x$ then $\mu \leqq 1$.*

Then A has a nonzero fixed point $x \in K$ with $\|x\| \leqq R$.

PROOF OF THEOREM 3.3. In view of Lemma 3.2, it suffices to check that (a) and (b) of the fixed point theorem hold. Since the Frechet derivative of F, (3.10), does not involve h_1, it is not hard to show [7, Theorem 3.1] that the spectrum of $F'(0, 0)$ is identical to the spectrum of the operator $T : X_1 \to X_1$ given by

$$(Th)(t) = \int_{t-\sigma-g(t-\sigma,m)}^{t-g(t,m)} r(x)h(x)\, dx.$$

The operator T is compact and positive ($TK_1 \subset K_1$). Moreover by an extension of the argument in [7, Lemma 2], T can be shown to be strongly positive, that is, if $x \in K_1 - \{0\}$ then some iterate $T^n x \in$ interior of K_1 (n depending on x). Hence T has a unique norm-one eigenvector which lies in K_1 (see Theorem 2.11 in [5]).

Now

$$(T1)(t) = \int_{t-\sigma-g(t-\sigma,m)}^{t-g(t,m)} r(x)\, dx \geqq (\min r)(\sigma + g(t - \sigma, m) - g(t, m))$$

and

$$(\sigma + g(t - \sigma, m) - g(t, m)) \geqq \frac{\sigma \min \rho_1}{\max \rho_1}$$

by arguments similar to those used in Lemma 3.1.

Thus

$$T1 \geqq \sigma(min\ r)\left(\frac{\min \rho_1}{\max \rho_1}\right)1.$$

This proves, by Theorem 2.5 in [5], that the eigenvalue λ corresponding to the eigenvector in K_1 satisfies $\lambda \geqq \sigma$ (min r) (min $\rho_1/\max \rho_1$). Hence, under the conditions of Theorem 3.3, (a) of the fixed point theorem is satisfied.

Now suppose that $F(x_1, x_2) = \mu(x_1, x_2)$ where $(x_1, x_2) \in K_2$ and $\|(x_1, x_1)\| = \|x_1\| = R$. Then for some t^*,

$$\mu R = \mu x_1(t^*) = \int_{\tau(t^*-\sigma-\omega)}^{t^*} r(x) y_2(x)(1 - y_1(x))\,dx$$

$$\leqq \frac{\|r\|}{4}(t^* - \tau(^* - \sigma - \omega))$$

$$\leqq \frac{\|r\|}{4}(\sigma + \omega + p(t^* - \sigma - \omega))$$

$$\leqq \frac{\|r\|}{4}\left(\sigma + \omega + \frac{m}{\min \rho_1}\right).$$

If we take $R = \|r\|/4\ (\sigma + \omega + m/\min \rho_1)$ then (b) holds. The fixed point theorem gives the existence of the periodic solution (y_1, y_2). The estimates of Theorem 3.3 are proved exactly as in Theorem 3.1 in [7].

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