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ON COPRODUCTS OF RECTANGULAR BANDS

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ABSTRACT. For rectangular bands X and Y, a characterization is given of the coproduct of X and Y in the category of rectangular bands and in the category of all bands. A characterization is also given of the coproduct of X and Y amalgamating a subband U.

1. **Introduction.** A semigroup is a *band* provided $x^2 = x$ for all x, and a band is *rectangular* provided $xz = xyz$ for all x, y, and z. A band X is rectangular if and only if X is the direct product of a left zero semigroup L and a right zero semigroup R [2, Theorem 4.1.5]. The main structure theorem for bands is that every band is a semilattice of rectangular bands [1]. More specifically, for a band B, Green's relation $\mathcal{D}$ is a congruence on B, $B/\mathcal{D}$ is a semilattice, and each $\mathcal{D}$-class is a rectangular band [2, Corollary 7.4.7].

When X_1, X_2 are bands, the coproduct (free product) of X_1 and X_2 in the category $\mathcal{B}$ of all bands is defined to be a band W and a pair of homomorphisms $i_t : X_t \to W (t = 1, 2)$ with the mapping property that for each band B and each pair of homomorphisms $\tau_t : X_t \to B (t = 1, 2)$, there exists a unique homomorphism $\theta : W \to B$ such that $\theta \circ i_t = \tau_t (t = 1, 2)$. When X_1 and X_2 are rectangular bands, the coproduct in the category of rectangular bands (rectangular band coproduct) would be defined by simply replacing band with rectangular band in the definition above.

This note will give a characterization of the band coproduct and rectangular band coproduct of two rectangular bands X and Y. The coproduct of X and Y amalgamating a subband U will also be characterized.

2. **The Rectangular Band Coproduct.**

LEMMA 2.1. *Let $X = L \times R$ and $Y = P \times Q$ be rectangular bands and let $\alpha : X \to Y$ be a homomorphism. Then $\alpha = (\alpha_1, \alpha_2)$ for some homomorphisms $\alpha_1 : L \to P$ and $\alpha_2 : Y \to Q$.*

PROOF. Let $\ell_1, \ell_2 \in L$, let $r_1, r_2 \in R$ and suppose that $\alpha(\ell_1, r_1) = (p_1, q_1)$, $\alpha(\ell_2, r_2) = (p_2, q_2)$. Now if $\ell_1 = \ell_2$, then $(p_2, q_2) =$

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$\alpha(\ell_2, r_2) = \alpha(\ell_1, r_2) = \alpha[(\ell_1, r_1)(\ell_2, r_2)] = \alpha(\ell_1, r_1)\alpha(\ell_2, r_2) = (p_1, q_1)(p_2, q_2) = (p_1, q_2)$ so that $p_1 = p_2$. Dually, if $r_1 = r_2$ then $q_1 = q_2$. Thus, there exist $\alpha_1 : L \to P$ and $\alpha_2 : R \to Q$ such that $\alpha(\ell, r) = (\alpha_1(\ell), \alpha_2(r))$ for all $(\ell, r) \in L \times R$. Also, α_1 is a homomorphism since both L, P are left zero semigroups and similarly, α_2 is a homomorphism.

PROPOSITION 2.1. *Let $X = L \times R$ and $Y = P \times Q$ be rectangular bands. Then the rectangular band $Z = (L \cup P) \times (R \cup Q)$ is a rectangular band coproduct of X and Y.*

PROOF. First, in the statement of the theorem, $L \cup P$ is to be understood to be the disjoint union of the sets L and P. Similarly, $R \cup Q$ is a disjoint union. Now let $i_1 : X \to Z$ and $i_2 : Y \to Z$ be the inclusion homomorphisns. Let $K = A \times B$ be a rectangular band and let $\alpha : X \to K$ and $\beta : Y \to K$ be homomorphisms. By Lemma 2.1, both α and β factor into product maps, say $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. Define $\theta_1 : L \cup P \to A$ by $\theta_1(x) = \alpha_1(x)$ if $x \in L$ and $\theta_1(x) = \beta_1(x)$ if $x \in P$. Dually, define $\theta_2 : R \cup Q \to B$. Now define $\theta : Z \to K$ by $\theta = (\theta_1, \theta_2)$. It follows that θ is a homomorphism, that $\theta \circ i_1 = \alpha$ and $\theta \circ i_2 = \beta$, and that θ is the only such homomorphism since $i_1(X) \cup i_2(Y)$ is a generating set for Z.

3. **The Band Coproduct.** Let $X = L \times R$ and $Y = P \times Q$ be rectangular bands, and consider the coproduct W of X and Y in the category $\mathcal{B}$ of bands. Since each band is a semilattice of rectangular bands, the coproduct W will consist of X, Y, and one additional rectangular band, say $Z = G \times H$, in which each product of an element of X with an element of Y must lie. Thus, the problem of finding W is just the problem of finding G and H and then describing the multiplication between X, Y, and Z.

Suppose that $x = (\ell, r) \in L \times R = X$ and $y = (p, q) \in P \times Q = Y$. The $\mathcal{L}$-class of xy will depend on the $\mathcal{L}$-class of x, which is indexed by r, and the element $y = (p, q)$. Thus, the $\mathcal{L}$-class of xy could be denoted $G \times (r, p, q)$, from which $(r, p, q) \in H$. So, $R \times P \times Q \subset H$ and by considering yx it follows that $Q \times L \times R \subset H$. By repeating the same argument for $\mathcal{R}$-classes, it follows that $L \times R \times P$, $P \times Q \times L \subset G$. Note that this characterization of the $\mathcal{L}$ and $\mathcal{R}$ classes of xy says that

$$xy = (\ell, r)(p, q) = [(\ell, r, p), (r, p, q)]. \tag{3.1}$$

Define the sets G, H by

$$G = (L \times R \times P) \cup (P \times Q \times L),$$

and

$$H = (R \times P \times Q) \cup (Q \times L \times R).$$

An element $[(\ell, r, p), (r', p', q')] \in G \times H$ may be factored as $(\ell, r)(p, q')(\ell, r')(p', q')$. To check this, $[(\ell, r)(p, q')]\,[(\ell, r')(p', q')] = [(\ell, r, p), (r, p, q')]\,[(\ell, r', p'), (r', p', q')] = [(\ell, r, p), (r', p', q')]$ by the rectangularity of $G \times H$. A factorization table follows.

$$
(3.2)\qquad
\begin{aligned}
[(\ell, r, p), (r', p', q')] &= (\ell, r)(p, q')(\ell, r')(p', q')\\
[(\ell, r, p), (q', \ell', r')] &= (\ell, r)(p, q')(\ell', r')\\
[(p, q, \ell), (r', p', q')] &= (p, q)(\ell, r')(p', q')\\
[(p, q, \ell), (q', \ell', r')] &= (p, q)(\ell, r')(p, q')(\ell', r').
\end{aligned}
$$

When $x \in X$ and $z \in Z$, then $xz\,\mathcal{L}\,z$ since $X > Z$ in the semilattice structure. Thus, x must induce an action on the left component of z. The factorization just given (3.2) will dictate the action of $L \times R$ on z. for example, $(\ell'', r'')[(p, q, \ell), (r', p', q')] = (\ell'', r'')[(p, q)(\ell, r')(p', q')] = [(\ell'', r'')(p, q)]\,[(\ell, r')(p', q')] = [(\ell'', r'', p), (r'', p, q)]\,[(\ell, r', p'), (r', p', q')] = [(\ell'', r'', p), (r', p', q')]$ so that $(\ell'', r'')(p, q, \ell) = (\ell'', r'', p)$. A table of the actions of $L \times R$, $P \times Q$ on G, H follows.

$$
(3.3)\qquad
\begin{aligned}
&(L \times R) : (L \times R \times P) \to (L \times R \times P)\ (\ell', r')(\ell, r, p) = (\ell', r, p)\\
&(L \times R) : (P \times Q \times L) \to (L \times R \times P)\ (\ell', r')(p, q, \ell) = (\ell', r', p)\\
&(P \times Q) : (L \times R \times P) \to (P \times Q \times L)\ (p', q')(\ell, r, p) = (p', q', \ell)\\
&(P \times Q) : (P \times Q \times L) \to (P \times Q \times L)\ (p', q')(p, q, \ell) = (p', q, \ell)\\
&(R \times P \times Q) : (L \times R) \to (Q \times L \times R)\,(r, p, q)(\ell', r') = (q, \ell', r')\\
&(Q \times L \times R) : (L \times R) \to (Q \times L \times R)\,(q, \ell, r)(\ell', r') = (q, \ell, r')\\
&(R \times P \times Q) : (P \times Q) \to (R \times P \times Q)\,(r, p, q)(p', q') = (r, p, q')\\
&(Q \times L \times R) : (P \times Q) \to (R \times P \times Q)\,(q, \ell, r)(p', q') = (r, p', q').
\end{aligned}
$$

THEOREM 3.1. *Let X, Y, Z be the rectangular bands described above. Define multiplication between X and Y using* (3.1), *and define multiplication between X, Y, and Z using* (3.3).

$$
\begin{aligned}
(a, b)(c, d) &= [(a, b, c), (b, c, d)]\\
(a, b)[(c, d, e), (f, g, h)] &= [(a, b)(c, d, e), (f, g, h)]\\
[(c, d, e), (f, g, h)]\,(a, b) &= [(c, d, e), (f, g, h)(a, b)].
\end{aligned}
$$

Then $W = X \cup Y \cup Z$ is a coproduct of X and Y in the category $\mathcal{B}$ of bands.

PROOF. To see that the multiplication in W is associative, an appeal will be made to Petrich's theorem [3, Theorem 1] characterizing all bands. Since the underlying semilattice of W is to have only three points, it will be enough to check condition (ii) of that theorem. This check amounts to showing that if $(\ell', r') \in L \times R$ and $(p', q') \in P \times Q$, then the composition of the two left actions on G induced by (ℓ', r'), (p', q') reduce to a constant map, and similarly for the right compositions. Indeed,

$$(\ell', r')(p', q')(\ell, r, p) = (\ell', r')(p', q', \ell) = (\ell', r', p'),$$

and

$$(\ell', r')(p', q')(p, q, \ell) = (\ell', r')(p', q, \ell) = (\ell', r', p').$$

To check that W is a coproduct of X and Y, let B be a band and let $\theta_1 : X \to B$ and $\theta_2 : Y \to B$ be homomorphisms. Define $\theta : W \to B$ by $\theta \mid X = \theta_1$, $\theta \mid Y = \theta_2$ and then $\theta(Z)$ is dictated by the factorization schedule (3.2). For example, $\theta[(\ell, r, p), (r', p', q')] = \theta_1(\ell, r)\theta_2(p, q')$ $\theta_1(\ell, r')\theta_2(p', q')$.

To verify that θ is a homomorphism, it must be checked that $\theta(\alpha\beta) = \theta(\alpha)\theta(\beta)$ for all possible choices of α,β. Taking advantage of the symmetry in the definitions of multiplication, it is easy to see that it is sufficient to check (a) $\alpha \in X, \beta \in Y$, (b) $\alpha \in X, \beta \in Z$, and (c) $\alpha, \beta \in Z$. Since most of the arguments are similar, only one will be presented here. Free use will be made of band properties in B. In particular note that $\theta_1(X)$ is contained in some rectangular subband of B. Thus, both $\theta_1(\ell, r)\theta_2(p, q)$ and $\theta_1(\ell', r')\theta_2(p', q')$ belong to the same rectangular subband of B.

Suppose that $\alpha = (\ell'', r'') \in X$ and $\beta = [(p, q, \ell), (r', p', q')] \in Z$. Then $\theta(\alpha\beta) = \theta[(\ell'', r'')(p, q, \ell), (r', p', q')] = \theta[(\ell'', r'', p), (r', p', q')] = \theta_1(\ell'', r'')\theta_2(p, q')\theta_1(\ell'', r')\theta_2(p', q')$. On the other hand, $\theta(\alpha)\theta(\beta) = \theta(\ell'', r'')\theta[(p, q, \ell), (r', p', q')] = \theta_1(\ell'', r'')\theta_2(p, q)\theta_1(\ell, r')\theta_2(p', q) = \theta_1(\ell'', r'')\theta_2(p, q)[\theta_2(p, q')\theta_1(\ell'', r)]\theta_1(\ell, r')\theta_2(p', q') = \theta_1(\ell'', r'')\theta_2(p, q')\theta_1(\ell'', r')\theta_2(p', q')$.

Thus, θ is a homomorphism of W into B such that $\theta \mid X = \theta_1$, and $\theta \mid Y = \theta_2$ and θ is the only such since $X \cup Y$ generates W.

REMARK 3.2. The result just proved does not immediately extend to more than two rectangular bands. If X_i is rectangular ($i = 1, 2, 3$), then it is true that $\coprod_{i=1}^{3} X_i = (\coprod_{i=1}^{2} X_i) \amalg X_3$. However, $\coprod_{i=1}^{2} X_i$ is not rectangular and so the result just proved cannot be reapplied. Also, finding $(\coprod_{i=1}^{2} X_i) \amalg X_3$ does not reduce to finding the coproduct of X_3 with each of the three subrectangular bands of $\coprod_{i=1}^{2} X_i$.

4. **An Amalgamation Theorem.** Again, let $X = L \times R$ and $Y = P \times Q$ be rectangular bands. Let $U = S \times T$ be a rectangular band

and let $\alpha : U \to X$ and $\beta : U \to Y$ be injective homomorphisms. The *band coproduct of X and Y amalgamating U* is defined to be W/ρ where W is the band coproduct of X and Y and ρ is the congruence on W generated by $\rho_0 = \{(\alpha(u), \beta(u)) : u \in U\}$. The next theorem will characterize W/ρ.

THEOREM 4.1. *The band coproduct of X and Y amalgamating U is isomorphic to the rectangular band* $K = [(L \cup P) - \alpha_1(S)] \times [(R \cup Q) - \alpha_2(T)]$.

PROOF. Again, the unions $\cup$ in the statement of the theorem and all unions in the proof are disjoint. Several cases arise as $\alpha_1(S) = L$ or $\alpha_1(S) \subsetneqq L$ and similarly for R, P, and Q. Accordingly, the proof will be split into several cases. All cases may easily be seen to fall into one of the following cases.

I. $\alpha_1(S) \subsetneqq L, \alpha_2(T) \subsetneqq R$

II. $\alpha_1(S) = L, \alpha_2(T) \subsetneqq R; \beta_1(S) \subsetneqq P, \beta_2(S) = Q$

III. $\alpha_1(S) = L, \alpha_2(T) \subsetneqq R; \beta_1(S) = P, \beta_2(S) \subsetneqq Q$.

In each case the proof will consist of showing that there are inverse homomorphisms $\theta : W/\rho \to K$ and $\Psi : K \to W/\rho$.

Case I. $\alpha_1(S) \subsetneqq L, \alpha_2(T) \subsetneqq R$.

To define $\theta : W/\rho \to K$, first define $\tau : L \times R \to K$ by

$$\tau_1(\ell) = \begin{cases} \ell & \text{if } \ell \notin \alpha_1(S) \\ \beta_1\alpha_1{}^{-1}(\ell) & \text{if } \ell \in \alpha_1(S) \end{cases}, \quad \tau_2(r) = \begin{cases} r & \text{if } r \notin \alpha_2(T) \\ \beta_2\alpha_2{}^{-1}(r) & \text{if } r \in \alpha_2(T) \end{cases}$$

and also let $\sigma : P \times Q \to K$ be the inclusion homomorphism. Since W is the band coproduct of X and Y, there is a unique homomorphism $\phi : W \to K$ such that $\phi \mid X = \tau$ and $\phi \mid Y = \sigma$.

Now for $u = (s, t) \in U$, then $\phi(\alpha(u)) = \tau\alpha(u) = \tau((\alpha_1(s), \alpha_2(t))) = (\beta_1(s), \beta_2(t)) = \beta(u) = \sigma(\beta(u)) = \phi(\beta(u))$. It follows that there is a homomorphism $\theta : W/\rho \to K$ such that $\theta \circ \rho^{\natural} = \phi$.

Recall that W consists of 3 rectangular subbands; X, Y, and Z where $XY \subset Z$ (Theorem 3.1). However, for $u \in U$, then $\alpha(u) \in X, \beta(u) \in Y$, but $\alpha(u)\rho = \beta(u)\rho$. It follows that $X\rho$, $Y\rho$, and $Z\rho$ all lie in the same rectangular subband of W/ρ and hence that W/ρ is itself rectangular.

By Proposition 2.2, K is the rectangular band coproduct of $[L - \alpha_1(S)] \times [R - \alpha_2(T)]$ and $P \times Q$. Define f: $[L - \alpha_1(S)] \times [R - \alpha_2(T)] \to W/\rho$ by $f(\ell, r) = (\ell, r)\rho$ and $g : P \times Q \to W/\rho$ by $g(p, q) = (p, q)\rho$. Then there is a homomorphism $\Psi : K \to W/\rho$ such that

$$\Psi \mid [L - \alpha_1(S)] \times [R - \alpha_2(T)] = f \text{ and } \Psi \mid P \times Q = g.$$

To check that $\theta\Psi$ and $\Psi\theta$ are identity maps, it will of course be sufficient to verify that $\theta\Psi$ and $\Psi\theta$ are identity maps on a set of generators for K and W/ρ respectively, i.e., for

$$([L - \alpha_1(S)] \times [R - \alpha_2(T)]) \cup (P \times Q) \subset K$$

$$\text{and for } (X\rho) \cup (Y\rho) \subset W/\rho.$$

That $\theta\Psi$ is the identity is easy to see and will not be checked here.

To check that $\Psi\theta$ is the identity on $(X\rho) \cup (Y\rho)$, first let $(\ell, r) \in X$ with $\ell \in L - \alpha_1(S)$, $r = \alpha_2(t)$. Then $\Psi\theta[(\ell, r)\rho] = \Psi\phi(\ell, r) = \Psi\tau(\ell, r) = \Psi(\ell, \beta_2(t)) = \Psi[(\ell, r')(\beta_1(s), \beta_2(t))]$ (where $r' \in R - \alpha_2(T)$ and $s \in S) = \Psi(\ell, r')\Psi(\beta_1(s), \beta_2(t)) = \Psi(\ell, r')\Psi\beta(s, t) = [(\ell, r')\rho][\beta(s, t)\rho] = [(\ell, r')\rho][(\alpha(s, t))\rho] = (\ell, \alpha_2(t))\rho = (\ell, r)\rho$. The case that $\ell = \alpha_1(s)$ and $r \in R - \alpha_2(T)$ is similar. The other two cases, (i) that $\ell \in L - \alpha_1(S)$, $r \in R - \alpha_2(T)$ and (ii) that $(\ell, r) \in \alpha(U)$ are shorter and will not be checked here.

Case II. $\alpha_1(S) = L, \alpha_2(T) \subsetneqq R; \beta_1(S) \subsetneqq P, \beta_2(T) = Q$.

It will be shown that W/ρ is isomorphic to $P \times R$, which is sufficient since $P \times R \cong K$.

Let $\tau : L \times R \to P \times R$ and $\sigma : P \times Q \to P \times R$ be defined by

$$\tau_1 = \beta_1\alpha_1^{-1}, \tau_2 = 1_R; \sigma_1 = 1_P, \sigma_2 = \alpha_2\beta_2^{-1}.$$

Let $\phi : W \to P \times R$ be the homomorphism such that $\phi/L \times R = \tau$ and $\phi \mid P \times Q = \sigma$. Further, for $u = (s, t) \in U$, then $\phi\alpha(u) = \tau\alpha(u) = \tau(\alpha_1(s), \alpha_2(t)) = (\beta_1(s), \alpha_2(t)) = \sigma(\beta_1(s), \beta_2(t)) = \sigma\beta(u) = \phi\beta(u)$. Thus, there is a homomorphism $\theta : W/\rho \to P \times R$ such that $\theta \circ \rho^\natural = \phi$.

Choose (and fix) $\ell_0 \in L$; $q_0 \in Q$, say $q_0 = \beta_2(t_0)$. Define $\Psi : P \times R \to W/\rho$ by $\Psi(p, r) = (p, q_0)\rho(\ell_0, r)\rho$. That Ψ is a homomorphism follows since W/ρ is rectangular.

Now for $(p, r) \in P \times R$, then $\theta\Psi(p, r) = \theta[(p, q_0)\rho(\ell_0, r)\rho] = \phi[(p, q_0)(\ell_0, r)] = \phi(p, q_0)\phi(\ell_0, r) = \sigma(p, q_0)\tau(\ell_0, r) = (p, \alpha_2\beta_2^{-1}(q_0)) \cdot (\beta_1\alpha_1^{-1}(\ell_0), r) = (p, r)$ so that $\theta\Psi$ is the identity. For $(\ell, r) \in L \times R$, then $\Psi\theta[(\ell, r)\rho] = \Psi\phi(\ell, r) = \Psi\tau(\ell, r) = \Psi(\beta_1\alpha_1^{-1}(\ell), r) = (\beta_1\alpha_1^{-1}(\ell), q_0)\rho(\ell_0, r)\rho = (\beta_1\alpha_1^{-1}(\ell), \beta_2(t_0))\rho(\ell_0, r)\rho = (\ell, \alpha_2(t_0))\rho(\ell_0, r)\rho = (\ell, r)\rho$. Similarly, $\Psi\theta[(p, q)\rho] = (p, q)\rho$ and so $\Psi\theta$ is the identity on $X\rho \cup Y\rho$.

Case III. $\alpha_1(S) = L, \alpha_2(T) \subsetneqq R; \beta_1(S) = P, \beta_2(T) \subsetneqq Q$.

In this case it will be shown that $W/\rho \cong L \times [(R \cup Q) - \alpha_2(T)]$. The argument in this case is similar to the two arguments just given.

Thus, the same notation will be adopted, but only the definitions of the maps will be given, leaving the various calculations to the reader.

Define $\tau : L \times R \to L \times [(R \cup Q) - \alpha_2(T)]$ and $\sigma : P \times Q \to L \times [(R \cup Q) - \alpha_2(T)]$.

$$\tau_1 = 1_L \qquad\qquad \sigma_1 = \alpha_1\beta_1{}^{-1}$$

$$\tau_2(r) = \begin{cases} r & \text{if } r \notin \alpha_2(T) \\ \beta_2\alpha_2{}^{-1}(r) & \text{if } r \in \alpha_2(T) \end{cases} \qquad \sigma_2 = 1_Q.$$

Then there is a homomorphism $\phi : W \to L \times [(R \cup Q) - \alpha_2(T)]$ which restricts to τ and σ. Subsequently there is a homomorphism θ such that $\theta \circ \rho^{\natural} = \phi$.

Define $\Psi : L \times [(R \cup Q) - \alpha_2(T)] \to W/\rho$ by

$$\Psi(\ell, r) = (\ell, r)\rho \quad \text{when } \ell \in L, r \in R - \alpha_2(T)$$

$$\Psi(\ell, q) = (\ell, r_0)\rho(p_0, q)\rho \quad \text{when } \ell \in L, q \in Q \text{ where}$$

$$r_0 \in \alpha_2(T), r_0 = \alpha_2(t_0); p_0 = \beta_1(s_0).$$

It then follows that Ψ is a homomorphism and that $\Psi\theta$, $\theta\Psi$ are identity maps.

REMARK. The band K of Theorem 4.1 is also the rectangular band coproduct of X and Y amalgamating U. The same proof works; just omit the paragraph showing that W/ρ is rectangular.

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