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THE PIERCE REPRESENTATION OF AN INERTIAL COEFFICIENT RING

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ABSTRACT. Let R be a commutative ring and A be an R-algebra which is finitely generated as an R-module and has Jacobson radical N. E. C. Ingraham defined R to be an *inertial coefficient ring* if whenever A/N is a separable R-algebra, there exists a separable R-subalgebra S of A such that $A = S + N$. Let $X(R)$ denote the Pierce decomposition space of R and R_x denote the stalk of the sheaf of rings over $X(R)$ at the point $x \in X(R)$. We show R is an inertial coefficient ring if and only if R_x is an inertial coefficient ring for all $x \in X(R)$. This result is used to give new examples of inertial coefficient rings; in particular, if R is a von Neumann regular ring the polynomial ring $R[y_1, \cdots, y_n]$ and the formal power series ring $R[[y_1, \cdots, y_n]]$ are inertial coefficient rings.

Introduction. The classical Wedderburn Principal Theorem states that if A is a finite dimensional algebra over a field and N is the Jacobson radical of A then when A/N is separable, there exists a separable subalgebra S of A such that $A = S + N$. Let R be a commutative ring with identity and A be an R-algebra which is finitely generated as an R-module and has Jacobson radical N. E. C. Ingraham defined R to be an *inertial coefficient ring* if whenever A/N is a separable R-algebra, there exists a separable R-subalgebra S of A such that $A = S + N$; S is called an *inertial subalgebra* of A. The basic properties of inertial coefficient rings can be found in [6].

R. S. Pierce [11] described a representation of any commutative ring R as the ring of global sections of a sheaf of connected rings (rings with no idempotents but 0 and 1) over a totally disconnected, compact, Hausdorff space $X(R)$. W. C. Brown [3] used this representation to show that von Neumann regular rings are inertial coefficient rings. We have generalized the work of Brown by showing that R is an inertial coefficient ring if and only if for each $x \in X(R)$ the stalk R_x is an inertial coefficient ring. This result is used to construct new examples of inertial coefficient rings.

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All rings we shall consider contain an identity; all subrings contain the identity of the overring; all homomorphisms preserve the identity. Throughout R is a commutative ring and all R-algebras are finitely generated as R-modules. The Jacobson radical of the R-algebra A will be denoted by N. All unadorned tensor signs $\otimes$ will mean $\otimes_R$. The decomposition space of R will be denoted by $X(R)$ and for each $x \in X(R)$, R_x will denote the stalk of the sheaf over $X(R)$ at the point x. If M is an R-module and $m \in M$ we shall let m_x denote the image of m in $M_x = M \otimes R_x \cong M/I(x)M$, where $I(x)$ is the ideal of R generated by the elements of x.

1. **Main Theorem.** The key to the proof of our main theorem is to show that an inertial subalgebra exists if and only if a certain finite collection of equations hold. The following lemma which follows from [4, Proposition 1.1 (iii), p. 40] will be used in producing these equations.

LEMMA 1.1. *Let S be an R-algebra generated as an R-module by $s_1, \cdots, s_n$. Then S is separable if and only if there exist $b_i, b_i' \in S$ for $i = 1, \cdots, m$ such that*

(i) $\sum_{i=1}^{m} b_i b_i' = 1$, *and*

(ii) $\sum_{i=1}^{m} s_j b_i \otimes b_i' = \sum_{i=1}^{m} b_i \otimes b_i' s_j$ *holds in $S \otimes S^\circ$ for $j = 1, \cdots, n$.*

THEOREM 1.2. *A ring R is an inertial coefficient ring if and only if R_x is an inertial coefficient ring for every $x \in X(R)$.*

PROOF. If R is an inertial coefficient ring then $R_x = R/I(x)$ is an inertial coefficient ring [6, Corollary 3.4, p. 86].

Conversely, suppose R_x is an inertial coefficient ring for all $x \in X(R)$, and let A be a finitely generated R-algebra such that A/N is R-separable. Then A_x is a finitely generated R_x-algebra. Since R_x is a flat R-module, without ambiguity we can let N_x denote the image of N under the canonical homomorphism $A \to A/(I(x)A) = A_x$. Furthermore, since $0 \to N \to A \to A/N \to 0$ is an exact sequence of R-modules and R_x is R-flat, $A_x/N_x \cong (A/N)_x$ is a separable R_x-algebra. Since each R_x is an inertial coefficient ring and $N_x \subset \operatorname{rad}(A_x)$, for every $x \in X(R)$ there exists a separable R_x-algebra $S'(x)$ such that $S'(x) + N_x = A_x$ [7, Corollary, p. 3].

By [12, Theorem 5, p. 5] each $S'(x)$ is a finitely generated R_x-algebra; for each $x \in X(R)$ let $s_1(x), \cdots, s_{n(x)}(x) \in A$ be such that $(s_1(x))_x, \cdots, (s_{n(x)}(x))_x$ are R_x-module generators of $S'(x)$. Let $S(x)$

be the R-submodule of A generated by $\{s_i(x) : i = 1, \cdots, n(x)\}$. Then $(S(x))_x = S'(x)$.

Let $a_1, \cdots, a_p$ be R-module generators of A. Since for each $x \in X(R)$, $(S(x))_x = S'(x)$ is a separable R_x-algebra such that $N_x + (S(x))_x = A_x$, there exist elements $r_{ijk}(x)$, $r_i(x)$, $t_{qj}(x)$, $r_{hj}(x)$, $r'_{hj}(x) \in R$, elements $z_q(x) \in N$, and elements $b_h(x)$, $b_h'(x) \in S(x)$ for $i, j, k = 1, \cdots, n(x)$, $q = 1, \cdots, p$, and $h = 1, \cdots, m(x)$ such that:

$$(1) \qquad (s_i(x))_x(s_j(x))_x = \sum_{k=1}^{n(x)} (r_{ijk}(x))_x(s_k(x))_x \text{ for } i, j = 1, \cdots, n(x).$$

$$(2) \qquad 1_x = \sum_{i=1}^{n(x)} (r_i(x))_x(s_i(x))_x.$$

$$(3) \qquad (a_q)_x = (z_q(x))_x + \sum_{j=1}^{n(x)} (t_{qj}(x))_x(s_j(x))_x \text{ for } q = 1, \cdots, p.$$

$$(4) \qquad 1_x = \sum_{h=1}^{m(x)} (b_h(x))_x(b_h'(x))_x.$$

$$(5) \qquad \sum_{h=1}^{m(x)} [(s_j(x))_x(b_h(x))_x \otimes_{R_x} (b_h'(x))_x]$$

$$= \sum_{h=1}^{m(x)} [(b_h(x))_x \otimes_{R_x} (b_h'(x))_x(s_j(x))_x]$$

$$\text{in } (S(x))_x \otimes_{R_x} (S(x))_x^o \text{ for } j = 1, \cdots, n(x).$$

$$(6) \qquad b_h(x) = \sum_{j=1}^{n(x)} r_{hj}(x)s_j(x) \text{ and } b_h'(x) = \sum_{j=1}^{n(x)} r'_{hj}(x)s_j(x)$$

$$\text{for } h = 1, \cdots, m(x).$$

Using standard sheaf theory arguments, there exists a cover $\{N'(x)\}$ of $X(R)$, where $N'(x)$ is an open-closed neighborhood of x such that equations (1) through (5) hold for all $y \in N'(x)$ when the subscript x is replaced by the subscript y. By the "partition property" of $X(R)$, there exists a disjoint open cover $\{N(x_i) : i = 1, \cdots, t\}$ of $X(R)$ refining $\{N'(x)\}$, where $N(x_i) \subseteq N'(x_i)$. Using standard "patching" arguments we obtain elements r_i, r_{ijk}, $t_{qj} \in R$, elements $z_q \in N$, and elements $s_j \in A$ for $i, j, k = 1, \cdots, n$ and $q = 1, \cdots, p$ such that $(s_j)_x = (s_j(x_k))_x$, $(r_j)_x = (r_j(x_k))_x$, $(z_j)_x = (z_j(x_k))_x$, etc., where $x \in N(x_k)$ and $n = \max n(x_i)$.

Let S be the R-submodule of A generated by $\{s_j : j = 1, \cdots, n\}$. One can check that equations (1) and (2) imply that S is an R-algebra and $S_x = (S(x_k))_x$ where $x \in N(x_k)$. Using "patching" arguments as above, elements $b_i, b_i' \in S$ can be found so that equations (5) and (6) and Lemma 1.1 demonstrate that S is a separable R-algebra. Finally equation (3) can be used to show $A = S + N$, and we can conclude that R is an inertial coefficient ring.

2. **Examples.** We shall next use Theorem 1.1 to produce several new examples of inertial coefficient rings.

PROPOSITION 2.1. *Let $S = C(X, R)$ be the ring of continuous functions from a totally disconnected, compact, Hausdorff space X to a connected, inertial coefficient ring R endowed with the discrete topology. Then S is an inertial coefficient ring.*

PROOF. By [9, pp. 40–1] for every $x \in X(S)$, $S_x \cong R$ and thus S is an inertial coefficient ring.

We note that one can construct rings R of certain functions on particular totally disconnected, compact, Hausdorff spaces so that R_x is an inertial coefficient ring for all $x \in X(R)$ [see 2, examples 5.2.1, 5.2.3, and 5.3, p. 224], providing further examples of inertial coefficient rings.

COROLLARY 2.2. *Let ΠR denote the direct product of a family of rings $\{R_\alpha : \alpha \in I\}$ where each R_α is isomorphic to a fixed finite, connected ring R. ΠR is an inertial coefficient ring.*

PROOF. If I and R are given the discrete topology, $\Pi R \cong C(I, R)$. Let $\beta(I)$ denote the Stone-Čech compactification of I. It is not hard to show $\beta(I)$ is a totally disconnected, compact, Hausdorff space, and hence by Proposition 2.1 $C(\beta(I), R)$ is an inertial coefficient ring, since any finite ring is an inertial coefficient ring. The natural ring homomorphism $\phi : C(\beta(I), R) \to C(I, R)$ given by restriction is surjective since R is compact. Then $C(I, R)$, being a homomorphic image of an inertial coefficient ring, is itself an inertial coefficient ring.

It is unknown if the direct product of any family of inertial coefficient rings is an inertial coefficient ring. We next show that the infinite direct sum (with identity adjoined) of a family of inertial coefficient rings is an inertial coefficient ring.

PROPOSITION 2.3. *Let $R = \oplus \sum R_\alpha$ be the direct sum of a collection of inertial coefficient rings $\{R_\alpha\}$. Embed R in a ring with an identity element in the usual manner (let $R^* = R \oplus Z$ where Z denotes the* *integers; define addition coordinatewise and multiplication by* $(a, i) \cdot (b, j) = (ab + ib + ja, ij)$). *Then* R^* *is an inertial coefficeent ring.*

PROOF. If $x \in X(R^*)$ then one can check that $R_x^* = (R_\alpha)_{x(\alpha)}$ for some $x(\alpha) \in X(R_\alpha)$ or $R_x^* = Z$. Since all the stalks of R^* are inertial coefficient rings, R^* is an inertial coefficient ring.

Our next example is the polynomial ring $R[y_1, \cdots, y_n]$ over a von Neumann regular ring R.

PROPOSITION 2.4. *If* R *is a ring such that* $R_x[y_1, \cdots, y_n]$ *is an inertial coefficient ring for all* $x \in X(R)$ *then* $S = R[y_1, \cdots, y_n]$ *is an inertial coefficient ring. In particular, if* R *is a von Neumann regular ring,* $R[y_1, \cdots, y_n]$ *is an inertial coefficient ring.*

PROOF. It is not difficult to show that any idempotent of S is contained in R, that if $x \in X(S)$ and I is the ideal of R generated by elements of x then $I(x) = I \cdot S$, and that $S_x = R_x[y_1, \cdots, y_n]$. If R is a von Neumann regular ring, R_x is a field and hence $R_x[y_1, \cdots, y_n]$ is an inertial coefficient ring by [8, Corollary 2, p. 553].

As our final example we consider a class of rings studied by U. Oberst and H. Schneider [10]. They call a ring R *F-semiperfect* if the factor ring $R/J(R)$ of R modulo its Jacobson radical is von Neumann regular and idempotents can be lifted from $R/J(R)$ to R. An F-semiperfect ring R is Hensel (i.e., $(R, J(R))$ is a Hensel pair, see [5]), if and only if the local ring R_m is a Hensel local ring for every maximal ideal m of R [10, Theorem 4.4, p. 346].

PROPOSITION 2.5. *A Hensel, F-semiperfect ring* R *is an inertial coefficient ring.*

PROOF. By [10, Lemma 3.2, p. 336] and [10, Proposition 3.5, p. 340] for every maximal ideal m of R, $R_m \cong R/m'$ where m' is the ideal of R generated by the idempotents of m. By [9, Proposition II.6, p. 27] every stalk of R is of the form R/m' for some maximal ideal m of R. Since for each $x \in X(R)$, $R_x \cong R_m$ for some maximal ideal m of R, R_x is a Hensel local ring and hence an inertial coefficient ring [1]. Thus R is an inertial coefficient ring.

Oberst and Schneider show that F-linear compact and topologically coherent linear topological rings ("FKK rings") are Hensel, F-semiperfect rings ([10, Proposition 2.2, p. 328] and [10, Theorem 4.3, p. 346]). By the preceding proposition FKK-rings are inertial coefficient rings. If $S = R[[y_1, \cdots, y_n]]$ is the ring of formal power series over a von Neumann regular ring R then S is a Hensel ring

[5, Example 1.13, p. 49]; since $S/J(S) \cong R$, S is an F-semiperfect ring and hence an inertial coefficient ring.

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