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$\Sigma^2 H^3 = S^5/G$

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1. **Introduction.** We prove the following theorem.

THEOREM. *If H^3 is any homology 3-sphere, then there is a cellular, upper semicontinuous decomposition G of the 5-sphere S^5 such that the double suspension $\Sigma^2 H^3$ of H^3 and the decomposition space S^5/G are homeomorphic.*

We hope that the Theorem will supply the missing link in resolving the question of whether the double suspension of each homology 3-sphere is S^5, for R. D. Edwards (see [4] for a partial announcement) has claimed that in many special cases S^5/G and S^5 are homeomorphic. However, the decomposition G constructed in the proof of the Theorem is, in general, only approximately a product decomposition, hence is not precisely of the type studied by Edwards. [*Added in proof.* The author has just completed a proof of the double suspension theorem.]

Our construction yields many strange 5-dimensional *PL* and *TOP* ribbons (see [8] for a discussion of the Ribbon Conjecture). These ribbons undoubtedly have interesting connections with Rohlin's invariant; the 5-dimensional ribbon conjecture; triangulations of manifolds, combinatorial and noncombinatorial; and the question as to which homology 3-spheres bound contractible 4-manifolds. However, others are more competent to deal with such topics.

The construction consists in quoting a number of difficult theorems, including a surgery theorem and the 5-dimensional Poincaré Conjecture. Therefore, it may be useful to find more direct constructions or to follow the given constructions through in specific detail for special cases. In particular, for homology 3-spheres known to embed in S^4, surgery and the Poincaré Conjecture can be completely avoided.

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2. **Outline of the construction.** Let H^3 denote a homology 3-sphere (i.e., H^3 is a (smooth) compact 3-manifold-without-boundary which has the integral homology of the 3-sphere). The idea of the construction is to succeed as nearly as possible in embedding the double suspension $\Sigma^2 H^3$ of H^3 in S^5. A true embedding would be, by invariance of domain for generalized manifolds, a homeomorphism. The attempt to embed $\Sigma^2 H^3$ proceeds in the following steps, most of which are only partially or provisionally successful.

Step 1. Embed H^3 nicely in S^5.

Step 2. Extend the embedding to an embedding of ΣH^3 in S^5.

Step 3. Approximate the (possibly wild) embedding of ΣH^3 in S^5 by one with homotopically nice $(1 - ULC)$ complement.

Step 4. Bicollar the nice embedding of ΣH^3 in S^5.

Step 5. Fill in two suspension points in S^5 to obtain the double suspension $\Sigma^2 H^3$ from that product structure $(\Sigma H^3) \times (-1, 1) \subset S^5$ which constitutes the bicollared embedding of ΣH^3 in S^5.

Ultimately, Steps 1, 2 and 3 succeed completely. But, at first, Step 1 succeeds only for the three manifold $H^3 \# (-H^3)$ (the connected sum of H^3 with itself), and Steps 2, 3, 4, and 5 all produce cell-like embedding relations (see § 3) rather than true embeddings. As a consequence one obtains the decomposition G and a homeomorphism $\Sigma^2 H^3 = S^5/G$ as promised by the Theorem rather than a homeomorphism $\Sigma^2 H^3 = S^5$.

3. **Tools.** Our tools and notations are standard with the following exceptions.

Definition 3.1. A generalized n-manifold M (see [3]) is an *ENR* (Euclidean neighborhood retract = retract of an open subset of some Euclidean space E^k) such that, for each $p \in M$,

$$H_*(M, M - p; \mathbf{Z}) \approx H_*(E^n, E^n - 0; \mathbf{Z}).$$

Note that, if H^3 is a homology 3-sphere, then ΣH^3 is a generalized 4-manifold.

Definition 3.2. A cell-like embedding relation $R : X \to S^k$ from a locally compact, separable metric space X into the k-sphere S^k is the multivalued inverse $R = \pi^{-1}$ to a projection map $\pi : Y \to (X = Y/G)$, Y open or closed in S^k, G an upper semicontinuous, cell-like decomposition of Y. See [**2**, Chapters I–III and Appendix I] for details. We insist on this backward and contrary terminology of relations as opposed to the standard nomenclature of decompositions because our theorems on such relations were discovered and proved not by applying standard decomposition space techniques but by pretending that the relations were functions and applying standard taming and approximation techniques designed for functions. The reader will best understand the theorem and constructions intuitively if he too pretends that the relations are functions.

We assume the following results. One need not completely understand all of the terms (e.g., framed, 1-*LC*) in order to follow the construction which uses these results.

Surgery (M. Kervaire and J. Milnor [5, Theorem 6.6]). Let M be a smooth, compact, framed $2k + 1$ – manifold, $k > 1$, such that ∂M is a homology $2k$-sphere. By a sequence of framed modifications M can be reduced to a contractible manifold M' with $\partial M' = \partial M$.

Poincaré Conjecture (S. Smale [10]). Let M be a smooth, compact, n-manifold-without-boundary, $n \geqq 5$, having the homotopy type of S^n. Then, if M is endowed with the *PL*-structure compatible with its smooth structure, M and S^n are *PL* homeomorphic.

1-*LC* Approximation Theorem (R. Ancel and J. W. Cannon [1]). Let $R_0 : X \to S^n$, $n \geqq 5$, be a cell-like embedding relation from a compact, generalized $(n-1)$-manifold X into the n-sphere S^n. Then each neighborhood N of R_0 in $X \times S^n$ contains a cell-like embedding relation $R : X \to S^n$ such that $S^n - R(X)$ is 1-*LC* at each point image of R.

1-*LC* Taming Theorem (J. W. Cannon [3]). Let $R : X \to S^n$, $n \geqq 5$, be a cell-like embedding relation from a compact, generalized $(n-1)$-manifold X into the n-sphere S^n and suppose that $S^n - R(X)$ is 1-*LC* at each point image of R. Then there is a cell-like embedding relation $R^* : X \times (-1, 1) \to S^n$ that extends R (i.e., $(R^* \mid X \times \{0\}) = R$).

4. **The Construction (Easy Case).** We fill in the outline from § 2. Consider the following:

$$H_0 = H^3 - \operatorname{Int}\sigma,\ \sigma \text{ a smooth 3-cell in } H^3;$$
$$H^* = 2H_0 = H^3 \# (-H^3), \text{ the double of } H_0;$$
$$M = H_0 \times I \times I \text{ (smoothed)},\ I = [0, 1].$$

The following five steps prove the Theorem for the homology 3-sphere H^*. We complete the proof of the Theorem in Section 5.

Step 1. The smooth 5-manifold M is parallelizable (see [11]) since H_0 and I are ([11]); hence M has a framing by definition (see [5]). The 4-manifold ∂M is a homology 4-sphere (by duality arguments), and H^* is a (smooth) homology 3-sphere in ∂M (since $\partial M \supset H_0 \times I \times \{0\}$ and $\partial(H_0 \times I \times \{0\}) \approx H^*$). The surgery theorem of § 3 implies that M can be reduced to a (smooth) contractible 5-

manifold M', $H^* \subset \partial M = \partial M'$. The double $2M'$ of M' is simply connected by Van Kampen's Theorem and has the homology of S^5 by a Mayer-Vietoris argument. By the Hurewicz Isomorphism Theorem, $2M'$ is a homotopy 5-sphere. By the Poincaré Conjecture, $2M'$ is a PL 5-sphere. Therefore we may identify $2M'$ and S^5. The inclusion map $H^* \subset \partial M = \partial M' \subset 2M' = S^5$ is the desired (PL) embedding of H^* in S^5.

Step 2. Let $h : \partial M \times [-1, 1] \to S^5$ denote a PL bicollar on $\partial M = \partial M' \subset 2M' = S^5$. Let C_1 and C_2 denote the two components of $S^5 - h(\partial M \times (-1, 1))$, and let G_0 denote the *cell-like* decomposition of S^5 having as its only nondegenerate elements the contractible polyhedra C_1 and C_2. Let $\pi : S^5 \to S^5/G_0$ denote the projection map, and note that S^5/G_0 is homeomorphic with the suspension $\Sigma(\partial M)$. We identify S^5/G_0 with $\Sigma(\partial M)$ and note that, by Step 1, we have $\Sigma H^* \subset \Sigma(\partial M)$. Then $R_0 = (\pi^{-1} \mid \Sigma H^*) : \Sigma H^* \to S^5$ is a cell-like embedding relation and is the desired extension of the embedding of Step 1.

Step 3. The suspension ΣH^* is a generalized 4-manifold since H^* is a homology 3-sphere. By the 1-LC Approximation Theorem of § 3, there is a cell-like embedding relation $R : \Sigma H^* \to S^5$ (approximating R_0) such that $S^n - R(\Sigma H^*)$ is 1-LC at each point image of R. The embedding (relation) R is the approximation required by Step 3.

Step 4. By the 1-LC Taming Theorem of § 3, there is a cell-like embedding relation $R^* : \Sigma H^* \times (-1, 1) \to S^5$ (extending R). This is the embedding (relation) required by Step 4.

Step 5. Since $(\Sigma H^*) \times (-1, 1)$ had the homology of S^4, a homology duality argument shows that $S^5 -$ Image R^* has precisely two components D_1 and D_2, each having the Čech homology of a point. Since ΣH^* has the homotopy type of S^4, it is easy to show that D_1 and D_2 are also simply connected in the Čech sense, hence actually cell-like. If we make the natural identification of $(\Sigma H^*) \times (-1, 1)$ with $\Sigma(\Sigma H^*) - \{$suspension points$\}$, then R^* has a natural extension $R^* : \Sigma^2 H^* \to S^5$ taking one suspension point to D_1, the other to D_2. This completes Step 5.

The relation $R^* : \Sigma^2 H^* \to S^5$ of Step 5 induces a cell-like decomposition G^* of S^5,

$$G^* = \{g \subset S^5 \mid \text{there exists } x \in \Sigma^2 H^*,\ R^*(x) = g\};$$

and $\Sigma^2 H^*$ and S^5/G^* are homeomorphic. This essentially proves the Theorem for H^*. We will indicate in § 6 why G^* is actually cellular and not just cell-like.

5. **Completing the construction.** In § 4 we constructed a cell-like

embedding relation $R^*: \Sigma^2(H^3 \# (-H^3)) \to S^5$. We use one-sided versions of the 1-*LC* Approximation and Taming Theorems of § 3 to adjust R^* and thereby replace $\Sigma^2(H^3 \# (-H^3))$ by $\Sigma^2 H^3$ itself. The idea of the construction is due to R. D. Edwards in the analogous function (as opposed to relation) case.

Recall that $H^3 \# (-H^3) = 2H_0$, and examine the 2-sphere $\partial H_0 \subset 2H_0$. The 4-sphere $\Sigma^2(\partial H_0)$ bounds two closed complementary domains E_1 and E_2 in $\Sigma^2(H^3 \# (-H^3))$; and, if the 5-ball B^5 is sewn to either E_1 or E_2 along the 4-sphere boundary $\partial E_i = \Sigma^2(\partial H_0)$, one obtains $\Sigma^2 H^3$. We consider $\Sigma^2 H^3 = E_1 \cup \partial B^4$. We wish to adjust $R^* \mid E_1$ so that R^* can be extended to all of $\Sigma^2 H^3$. By the one-sided version of the 1-*LC* Approximation Theorem of § 3 (see [1], there is a cell-like embedding relation $R^{**}: E_1 \to S^5$ (approximating $R^* \mid E_1$) such that S^5-Image R^{**} is 1-*LC* at each point image of R^{**}. By the one-sided version of the 1-*LC* Taming Theorem of § 3 (see the union of [2, 3] and [1]), R^{**} can be extended to all of $\Sigma^2 H^3$. Just as in § 4, the extended R^{**} induces a cell-like decomposition $G = G^{**}$ of S^5 such that $\Sigma^2 H^3$ and S^5/G are homeomorphic. The fact that G is cellular (§ 6) completes the Theorem.

6. **Refinements in the Construction.**

Cellularity. We give only an indication of why the decompositions G^* of § 4 and $G = G^{**}$ of § 5 are cellular. If X is a generalized n-manifold, $n > 1$, then each point $(x, t) \in X \times E^1$ satisfies the Cellularity Criterion (∗) of McMillan in $X \times E^1$:

(∗) If U is a neighborhood of (x, t) in $X \times E^1$, then there is a neighborhood V of (x, t) in $X \times E^1$ such that each loop in $V - (x, t)$ is nullhomotopic in $U - (x, t)$.

The proof of (∗) is precisely the standard proof given, for example, in [7, Theorems 7 and 8, pp. 335–337] for the case where X is a manifold and x is not a point but a cell-like subset of X. Thus, if $R: X \times E^1 \to M^{n+1}$ is a cell-like embedding relation into an $(n+1)$-manifold M^{n+1}, an argument discovered by many people, the key ideas of which are presented very simply, for example, in [6, Lemma 5, p. 149], shows that the cellularity criterion is satisfied for $R(x, t)$ in M^{n+1}. If $n + 1 \geqq 5$, then $R(x, t)$ is thus cellular by [7, Theorem 1]. (Note that both McMillan [7] and Martin [6] state their theorems for *CAR*'s but that the Arguments apply equally well for cell-like sets; McMillan noted this fact in [7, p. 327].) Since $\Sigma^2 H^3$ is locally of the form $X \times E^1$, the desired cellularity of G and G^* follows.

Shrinking Certain Subdecompositions of G. Since $\Sigma^2 H^3$ is a manifold except on the suspension circle Σ^2, we may assume that $R^{**}: \Sigma^2 H^3$

$\to S^5$ is actually a (single-valued) function except on the suspension circle Σ^2; this fact follows easily from the homeomorphic approximability of cell-like maps between n-manifolds ($n \geqq 5$) as follows: by [**2**, Appendix I], there is a neighborhood U of $R^{**} \mid (\Sigma^2 H^3 - \Sigma^2)$ in $(\Sigma^2 H^3 - \Sigma^2) \times S^5$ such that $U \cup (R^{**} \mid \Sigma^2) : \Sigma^2 H^3 \to S^5$ is continuous at each point of Σ^2; by [**9**, Approximation Theorem A] (see also [**2**, Chapters IV, V, and VI]) there is a homeomorphism $h : (\Sigma^2 H^3 - \Sigma^2) \to R^{**}(\Sigma^2 H^2 - \Sigma^2)$ in U; then $h \cup (R^{**} \mid \Sigma^2) : \Sigma^2 H^3 \to S^5$ is a cell-like embedding relation that is a function except on the suspension circle Σ^2.

Ribbons. Let CH^* denote the cone on H^*. Examination of the proofs involved in Steps 1, 2, and 3 of § 4 show that one may assume that the image $R^*((CH^*) \times (-1, 1))$ of $(CH^*) \times (-1, 1)$ is a codimension-zero PL submanifold of S^5. This image is the strange PL ribbon discussed in the introduction. The moves described in § 5 however, are in no way obviously PL and one obtains thereby strange TOP ribbons $R^{**}((CH^3) \times (-1, 1))$ which have no obvious PL structure. The Rohlin invariant may play some strange role in the determination of whether these TOP ribbons have a PL structure.

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