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$\lambda(n, k)$-PARAMETER FAMILIES AND ASSOCIATED CONVEX FUNCTIONS

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1. **Introduction.** A family F of continuous real valued functions $f(x)$ defined on an interval I of real numbers is an *n-parameter family* on I if for every set of n distinct points $x_1, \cdots, x_n$ of I and every set of n real numbers $y_1, \cdots, y_n$ there exists a unique function $f \in F$ satisfying $f(x_i) = y_i$ for $i = 1, \cdots, n$. A function u defined on I is 'convex' with respect to an n-parameter ($n \geqq 2$) family F on I if for every set of n points, $x_1 < x_2 < \cdots < x_n$ in I, the unique element $f \in F$ defined by $f(x_i) = u(x_i)$, $i = 1, \cdots, n$ satisfies $(-1)^{n+i}(f(x) - u(x)) \leqq 0$ on (x_i, x_{i+1}), $i = 1, \cdots, n - 1$. In fact, if u is convex with respect to an n-parameter family F and $f \in F$ is determined as above, then f also satisfies [5] $f(x) - u(x) \leqq 0$ on (x_n, x_{n+1}) and $(-1)^n (f(x) - u(x)) \leqq 0$ on (x_0, x_1) where x_0 and x_{n+1} are the left and right end-points of I respectively. The above definition of convexity has been extended [3] under appropriate assumptions on F to the case when $f - u$ has n zeros counting multiplicities on I. For a discussion of these ideas we need the following definitions.

Assume $n \geqq 2$ and k is an integer $1 \leqq k \leqq n$. Let $\lambda(n, k) = (n(1), \cdots, n(k))$ be an ordered k-tuple of positive integers satisfying $n(1) + \cdots + n(k) = n$, which we call an *Ordered k-partition* of n. Let $P(n)$ denote the set of all ordered k-partitions of n with k varying such that $1 \leqq k \leqq n$. Also let $F \subset C^j(I)$ and $u \in C^j(I)$ where $j > 0$ is large enough so that the following definitions make sense.

Definition 1.1. F is said to be a *$\lambda(n, k)$-parameter family* on I if for every set of k (k fixed) distinct points $x_1 < x_2 < \cdots < x_k$ in I and every set of n real numbers y_{ir}, there exists a unique $f \in F$ satisfying

$$f^{(r)}(x_i) = y_{ir},\ r = 0, 1, \cdots, n(i) - 1,\ i = 1, \cdots, k. \tag{1.1}$$

(If F is a $\lambda(n, n)$-parameter family then we simply say F is an n-parameter family.)

For the sake of brevity of statements we shall denote

$$[\lambda(n, k)] = \{\mu(n, j) \in P(n), k \leqq j \leqq n\}.$$

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DEFINITION 1.2. If F is a $\mu(n, j)$-parameter family on I for all $\mu(n, j) \in [\lambda(n, k)]$, then we say F is a $[\lambda(n, k)]$-parameter family on I.

(If F is a $[\lambda(n, 1)]$-parameter family on I, then we refer to F as an unrestricted n-parameter family on I.)

To indicate some known results regarding $[\lambda(n, k)]$-parameter families, P. Hartman has proved [2] that in case I is an open interval then F is an unrestricted n-parameter family on I if and only if F is a $\lambda(n, k)$-parameter family on I for $k = 1$ and $k = n$. An example due to R. M. Mathsen [4] shows that Hartman's theorem cannot be extended to a closed interval I. Further results concerning the characteristic properties of n-parameter families and continuity theorems, among others can be found in [1], [4] and in some of the references contained therein.

We shall now define a $\lambda(n, k)$-convex function. Denote $M(j) = n + n(1) + \cdots + n(j)$ for $1 \leqq j \leqq k$ and $M(0) = n$.

DEFINITION 1.3. *Let* F be a $\lambda(n, k)$-parameter family on an interval I. A function u is said to be a $\lambda(n, k)$-*convex function* with respect to F on I if for every set of k points $x_1 < x_2 < \cdots < x_k$ in I the unique function $f \in F$ determined by

$$(1.2) \qquad f^{(r)}(x_i) = u^{(r)}(x_i), r = 0, 1, \cdots, n(i) - 1, i = 1, \cdots, k$$

satisfies

$$(1.3) \qquad (-1)^{M(i)}(f(x) - u(x)) \leqq 0 \text{ on } (x_i, x_{i+1}), i = 1, \cdots, k - 1.$$

(If F is an n-parameter family and u is $\lambda(n, n)$-convex with respect to F then we will simply say that u is *convex* with respect to F. If in the above definition, strict inequalities are satisfied in (1.3) then we say u is *strictly* $\lambda(n, k)$-*convex*.)

In this paper one of the theorems (Theorem 3.1) concerns continuity with respect to boundary conditions of a special type for $\lambda(n, k)$-parameter families. In Theorems 4.3 and 4.4 we state sufficient conditions under which if a function u is $\mu(n, k + 1)$-convex, it is also $\lambda(n, k)$-convex. Our main result (Theorem 4.5) concerning $\lambda(n, k)$-convex functions is that if F is an unrestricted n-parameter family and u is convex with respect to F then u is also $\lambda(n, k)$-convex with respect to F where $\lambda(n, k) \in P(n)$ is arbitrary. We will further show under the hypothesis of theorem 4.5, that if $f \in F$ is determined by the conditions (1.2) then f also satisfies $f(x) - u(x) \leqq 0$ on (x_k, x_{k+1}) and

$(-1)^n(f(x) - u(x)) \leqq 0$ on (x_0, x_1). (Hereafter x_0 and x_{k+1} stand for the left and right end-points of I respectively). It will however, remain unresolved as to whether the converse of Theorem 4.5 is true or not, that is, if F is an unrestricted n-parameter family and u is $\lambda(n, 1)$-convex with respect to F for all $\lambda(n, 2) \in P(n)$, does it follow that u is $\mu(n, k)$-convex with respect to F where $\mu(n, k) \in P(n)$ and $2 < k \leqq n$ is arbitrary?

There are several results in the literature concerning smoothness properties of convex functions with respect to n-parameter families. Some of these can be found in [2], [3] and in some of the references contained therein. Of these, we shall mention only the following one due to P. Hartman [2], that is "if F is an unrestricted n-parameter family on an open interval I and u is convex with respect to F on I then u has an $(n - 2)$-nd order differential and one-sided $(n - 1)$-st order differentials at every point $x_0 \in I$". Also, relationships between convexity and $\lambda(n, n - 1)$-convexity and between different types of $\lambda(n, k)$-convexity for k-tuples $\lambda(n, k) \in P(n)$ for which $\max\{n(i), 1 \leqq i \leqq k\} \leqq 2$ have been studied by R. M. Mathsen in [3]. For instance, under the hypothesis that F is an n-parameter family and also a $\lambda(n, n - 1)$-parameter family where $\lambda(n, n - 1) \in P(n)$ is fixed, he has shown (Theorem 2.1 of [3]) that if u is convex with respect to F then u is also $\lambda(n, n - 1)$-convex with respect to F. He also proved that if F is as above and u is $\lambda(n, n - 1)$-convex with respect to F where $n(1) \neq 2$ and $n(n - 1) \neq 2$ then u is convex with respect to F.

2. **Some more definitions and notations.** In the rest of the paper we shall use the following notations for the sake of brevity of statements.

If $\lambda(n, k) \in P(n)$ is fixed and m is an integer such that $1 \leqq m \leqq k$, we shall write $\lambda(n, k; m+) = (n(1), \cdots, n(m - 1), n(m) - 1, 1, n(m + 1), \cdots, n(k))$ and $\lambda(n, k; m-) = (n(1), \cdots, n(m - 1), 1, n(m) - 1, n(m + 1), \cdots, n(k))$.

Note: In case $n(m) = 1$, the entry $n(m) - 1 = 0$ is simply deleted so that $\lambda(n, k; m+) = \lambda(n, k; m-) = \lambda(n, k)$.

Define $\{\lambda(n, k; m)\} \subset P(n)$ as $A \cup B$ where $A = \{\mu(n, j) \in P(n) :$ $\mu(n, j)$ is obtained from $\lambda(n, k)$ by writing $n(m) - 1$ in the place of $n(m)$ and inserting the integer 1 in exactly one of the $(k + 1)$ gaps formed by the elements in the ordered array $(n(1), \cdots, n(m - 1), n(m) - 1, n(m + 1), \cdots, n(k))\}$ and $B = \{\mu(n, j) \in P(n) : \mu(n, j)$ is obtained from $\lambda(n, k)$ by writing $n(m) - 1$ in the place of $n(m)$ and writing $n(i) + 1$ in the place of $n(i)$ for exactly one $i \neq m$, leaving all the other $n(i)$'s fixed$\}$.

Note: The elements of A are $(k+1)$-tuples in case $n(m) > 1$ and reduce to k-tuples in case $n(m) = 1$. The elements of B are k-tuples in case $n(m) > 1$ and reduce to $(k-1)$-tuples in case $n(m) = 1$.

We shall say F is a $\{\lambda(n, k; m)\}$-*parameter family* on I in case F is a $\mu(n, j)$-parameter family for all $\mu(n, j) \in \{\lambda(n, k; m)\}$.

For $1 \leqq m < k$ with $n(m) \geqq 2$ and $n(m+1) = 1$, we shall let $L(m) =$ the largest integer $r \geqq m+1$ such that $n(i) = 1$ for all i, $m+1 \leqq i \leqq r$. Similarly for $1 < m \leqq k$ with $n(m) \geqq 2$ and $n(m-1) = 1$, we shall let $s(m) =$ the smallest integer $r \leqq m-1$ such that $n(i) = 1$ for all i, $r \leqq i \leqq m-1$.

Also for $\lambda(n, k) \in P(n)$ with $n(i) \geqq 2$ for at least one i, $1 \leqq i \leqq k$, let $p = \min\{i : 1 \leqq i \leqq k, n(i) \geqq 2\}$ and $q = \max\{i : 1 \leqq i \leqq k, n(i) \geqq 2\}$.

3. **$\lambda(n, k)$-parameter families.** The following is a continuity theorem of a special type for $\lambda(n, k)$-parameter families.

THEOREM 3.1. *Suppose F is a $\lambda(n, k)$-parameter family and also a $\{\lambda(n, k; m\}$-parameter family for some fixed $\lambda(n, k) \in P(n)$ and some fixed integer m, $1 \leqq m \leqq k$ on an interval I. Let $x_1 < x_2 < \cdots < x_k$ be k arbitrary points in I and $\{\alpha_j : 0 \leqq j < +\infty\}$ be a sequence of real numbers such that $\alpha_j \to \alpha_0$ as $j \to +\infty$. For each $j \geqq 0$, let $f_j \in F$ be determined by the condition $f_j^{n(m)-1}(x_m) = \alpha_j$ and all the conditions in* (1.1) *except for $i = m$ and $r = n(m) - 1$. Then $f_j \to f_0$ as $j \to +\infty$ uniformly on compact subsets of I.*

PROOF. Let J be a compact subset of I such that $[x_1, x_k] \subset J$. Choose a monotone subsequence $\{\alpha_{j(p)}\} \subset \{\alpha_j\}$ and denote for convenience the subsequences $\{\alpha_{j(p)}\}$ and $\{f_{j(p)}\}$ by $\{\alpha_j\}$ and $\{f_j\}$ respectively. We can assume without loss of generality that $\{\alpha_j\}$ is monotone decreasing since the proof will be similar of $\{\alpha_j\}$ is monotone increasing. Since f_0, f_j and f_{j+1} have $n(1), \cdots, n(m-1), n(m)-1, n(m+1), \cdots, n(k)$ conditions in common at $x_1, \cdots, x_k$ and F is a $\lambda(n, k)$-parameter family if $\alpha_j = \alpha_{j+1}$ for any j then $f_j \equiv f_{j+1}$. Therefore we can assume without loss of generality that $\{\alpha_j\}$ is strictly decreasing. Consequently,

$$\begin{aligned} f_0^{(n(m)-1)}(x_m) &< f_{j+1}^{(n(m)-1)}(x_m) \\ &< f_j^{(n(m)-1)}(x_m) \\ &< f_1^{(n(m)-1)}(x_m) \end{aligned}$$

for all $j \geqq 2$. These inequalities together with the hypothesis that F is a $\{\lambda(n, k; m)\}$-parameter family imply for all $j \geqq 1$

$$
\begin{aligned}
\operatorname{Sgn}[f_0(x) - f_{j+1}(x)] &= \operatorname{Sgn}[f_{j+1}(x) - f_j(x)] \\
&= \operatorname{Sgn}[f_j(x) - f_1(x)] \\
&= -1 \text{ on } (x_m, x_{m+1}) \\
&= (-1)^{n(m+1)+1} \text{ on } (x_{m+1}, x_{m+2}) \\
&\quad \vdots \\
&= (-1)^{n(m+1)} + \cdots + n(k) + 1 \text{ on } (x_k, x_{k+1}) \\
&= (-1)^{n(m)} \text{ on } (x_{m-1}, x_m) \\
&\quad \vdots \\
&= (-1)^{n(1)+\cdots+n(m)} \text{ on } (x_0, x_1).
\end{aligned}
$$

Thus $\{f_j\}$ is pointwise monotone on each of the intervals (x_i, x_{i+1}), $i = 0, 1, \cdots, k$ and pointwise bounded by the functions f_0 and f_1 on I. Moreover by the continuity of the functions f_0 and f_1 it follows that $\{f_j\}$ is uniformly bounded on J. Now we will show $\{f_j\}$ converges to f_0 uniformly on J. By virtue of Dini's theorem it suffices to show that $\{f_j\}$ converges to f_0 pointwise on J. If not, there exists some $x' \in J$, an $\epsilon > 0$ and a subsequence $\{f_{j(q)}\}$, which we again denote by $\{f_j\}$ such that $|f_j(x') - f_0(x')| > \epsilon$ for all $j \geqq 1$. Clearly $x' \neq x_i$ for any i, $1 \leqq i \leqq k$. Let us suppose $x' \in (x_s, x_{s+1})$ where s is some integer, $0 \leqq s \leqq k$. Also we can suppose without loss of generality that $s \geqq m$, since the proof will be similar if $s < m$.

Now $\operatorname{Sgn}[f_j(x') - f_0(x')] = (-1)^{M(s)-M(m)}$ and consequently $f_j(x') > f_0(x') + \epsilon$ or $f_j(x') < f_0(x') - \epsilon$ according as $M(s) - M(m)$ is even or odd. In case $M(s) - M(m)$ is even, let $h \in F$ be the unique function determined by the condition $h(x') = f_0(x') + \epsilon/2$ and all the conditions of (1.1) except for $i = m$ and $r = n(m) - 1$. (h exists since F is a $\{\lambda(n, k; m)\}$-parameter family.) Then since h and $f_j (j \geqq 0)$ have $n(1), \cdots, n(m-1), n(m) - 1, n(m), \cdots, n(k)$ conditions in common at $x_1, \cdots, x_k$ respectively, and since F is a $\lambda(n, k)$ as well as $\{\lambda(n, k; m)\}$-parameter family, we must have

$$f_0^{(n(m)-1)}(x_m) < h^{(n(m)-1)}(x_m) < f_j^{(n(m)-1)}(x_m)$$

for all $j \geqq 1$. This is a contradiction to the fact that α_0 is the limit of the decreasing sequence $\{\alpha_j\}$. In case $M(s) - M(m)$ is odd, the argument is similar if we choose $h \in F$ satisfying the condition $h(x') = f_0(x') - \epsilon/2$ and all the conditions of (1.1) except for $i = m$ and $r = n(m) - 1$. Thus the sequence $\{f_j\}$ converges to f_0 pointwise on J.

Since from every subsequence of the original sequence $\{f_j\}$, we can by the above process, obtain a further subsequence which converges to f_0 uniformly on J, it follows that the original sequence must converge to f_0 uniformly on J. This completes the proof of the theorem.

COROLLARY 3.2. *Suppose F is a $\lambda(n, k)$-parameter family for some fixed $\lambda(n, k) \in P(n)$ and also a $\{\lambda(n, k; i)\}$-parameter family for each $i, 1 \leqq i \leqq k$ on $I = [a, b]$. For each i, $1 \leqq i \leqq k$, let $\{\alpha_{ij_i} : 0 \leqq j_i < +\infty\}$ be k sequences of real numbers such that $\alpha_{ij_i} \to \alpha_{i0}$ as $j_i \to +\infty$. Also for each $j_i \geqq 0$, $1 \leqq i \leqq k$, let $f_{j_1 \cdots j_k} \in F$ be the unique function determined by the conditions* (1.1) *with $y_{i,n(i)-1}$ replaced by α_{ij_i} for $1 \leqq i \leqq k$. Then there exists a sequence of functions $\{f_p : 1 \leqq p < +\infty\}$ where $f_p = f_{j_1(p) \cdots j_k(p)}$ with $j_i(p) \to +\infty$ as $p \to +\infty$, $1 \leqq i \leqq k$, such that $f_p(x) \to f_{0 \cdots 0}(x)$ as $p \to +\infty$ uniformly on I.*

PROOF. For fixed values for $j_1, \cdots, j_{k-1}$, consider the sequence $\{f_{j_1 \cdots j_k} : 0 \leqq j_k < +\infty\} \subset F$. This sequence by Theorem 3.1 converges to $f_{j_1 \cdots j_{k-1}0}$ as $j_k \to +\infty$ uniformly on I. Again for fixed values of $j_1, \cdots, j_{k-2}$ and for $j_k = 0$ the sequence $\{f_{j_1 \cdots j_{k-1}0} : 0 \leqq j_{k-1} < +\infty\}$ converges to $f_{j_1 \cdots j_{k-2}00}$ as $j_{k-1} \to +\infty$ uniformly on I. Continuing in this way, we obtain that the sequence $\{f_{j_1 0 \cdots 0} : 0 \leqq j_1 < +\infty\}$ converges to $f_{0 \cdots 0}$ as $j_1 \to +\infty$ uniformly on I. (Note that the number of zeros in the subscripts in each case here is such that the total number of subscripts is k.) Now by the standard diagonalisation process we can obtain sequences $\{j_i(p)\}$ with $j_i(p) \to +\infty$ as $p \to +\infty$, $1 \leqq i \leqq k$ such that $\{f_{j_1(p) \cdots j_k(p)}\}$ converges to $f_{0 \cdots 0}$ as $p \to +\infty$ uniformly on I. This completes the proof of the corollary.

4. **$\lambda(n, k)$-convex functions.** In this section we shall assume one or the other of the following hypotheses and so it will be convenient to assign them the abbreviations as follows:

H: F is a $\lambda(n, k)$-parameter family on I.
H_i: F is a $\lambda(n, k; i+)$-parameter family on I.
H_i': F is a $\lambda(n, k; i-)$-parameter family on I.
G_i: u is $\lambda(n, k; i+)$-convex with respect to F on I.
G_i': u is $\lambda(n, k; i-)$-convex with respect to F on I.

The following lemmas are consequences of the Definitions (1.1) and (1.3).

LEMMA 4.1. *Suppose for some fixed $\lambda(n, k) \in P(n)$ with $n(m) \geqq 2$ for some m, $1 \leqq m \leqq k$, F and u satisfy the hypothesis H, H_m and G_m. Then $f \in F$ determined by the conditions* (1.1) *satisfies*

(i) $(-1)^{M(m)}(f(x) - u(x)) \leqq 0$ *on* (x_m, x_{m+1})

(ii) $(-1)^{M(i)}(f(x) - u(x)) \leqq 0$ *on* (x_i, x_{i+1}) *for all* i. $m \leqq i \leqq L(m)$, *in case* $n(m + 1) = 1$,

(iii) $f(x) \equiv u(x)$ *on* $[x_m, z]$ *if equality holds in* (i) *for some* $z \in (x_m, x_{m+1})$

(iv) $f(x) \equiv u(x)$ *on* $[x_m, \max\{z, x_{L(m)}\}]$ *if equality holds in* (ii) *for some* $z \in (x_i, x_{i+1})$ *for some* i, $m \leqq i \leqq L(m)$, *in case* $n(m+1) = 1$.

LEMMA 4.2. *Suppose for some fixed* $\lambda(n, k) \in P(n)$ *with* $n(m) \geqq 2$ *for some* m, $1 \leqq m \leqq k$, F *and* u *satisfy the hypothesis* H, $H_m{}'$ *and* $G_m{}'$. *Then* $f \in F$ *determined by the conditions* (1.1) *satisfies*

(i) $(-1)^{M(m-1)}(f(x) - u(x)) \leqq 0$ *on* (x_{m-1}, x_m)

(ii) $(-1)^{M(i-1)}(f(x) - u(x)) \leqq 0$ *on* (x_{i-1}, x_i) *for all* i, $s(m) \leqq i \leqq m$, *in case* $n(m-1) = 1$.

(iii) $f(x) \equiv u(x)$ *on* $[z, x_m]$, *in case equality holds in* (i) *for some* $z \in (x_{m-1}, x_m)$

(iv) $f(x) \equiv u(x)$ *on* $[\min\{z, x_{s(m)}\}, x_m]$ *if equality holds in* (ii) *for some* $z \in (x_{i-1}, x_i)$ *for some* i, $s(m) \leqq i \leqq m$, *in case* $n(m-1) = 1$.

We shall prove only Lemma (4.1) since the proof of Lemma (4.2) is analogous. In the proof of Lemma (4.1) we shall consider only the general case $1 \leqq m < k$. The case $m = k$ can be treated by appropriate modifications of notation in the general proof.

PROOF. (i) Suppose $(-1)^{M(m)}(f(z) - u(z)) > 0$ for some $z \in (x_m, x_{m+1})$. Let $g \in F$ be determined by the condition $g(z) = u(z)$ and all the conditions of (1.1) except for $i = m$ and $r = n(m) - 1$. (g exists by the hypothesis H_m.). Now $g - f$ has $n(1), \cdots, n(m-1), n(m) - 1, n(m+1), \cdots, n(k)$ zeros at $x_1, \cdots, x_k$ respectively with $n(1) + \cdots + n(k) = n$ and hence must keep a constant sign on (x_m, x_{m+1}), namely that of $g(z) - f(z) = u(z) - f(z)$, for if otherwise we will have a contradiction to the hypothesis H_m. Consequently $(-1)^{M(m)}(f(x) - g(x)) > 0$ on (x_m, x_{m+1}). Then by the hypothesis H, it follows that $(-1)^{M(m)}(f^{(n(m)-1)}(x_m) - g^{(n(m)-1)}(x_m)) > 0$, that is

$$(4.1) \qquad (-1)^{M(m)}(u^{(n(m)-1)}(x_m) - g^{(n(m)-1)}(x_m)) > 0.$$

Also by the hypothesis G_m, g must satisfy $(-1)^{M(m)-1}(g(x) - u(x)) \leqq 0$ on (x_m, z) and consequently $(-1)^{M(m)-1}(g^{(n(m)-1)}(x_m) - u^{(n(m)-1)}(x_m)) \leqq 0$. This contradicts the inequality (4.1). Thus we must have $(-1)^{M(m)}(f(x) - u(x)) \leqq 0$ on (x_m, x_{m+1}).

(ii) The proof is by induction on i, $m \leqq i \leqq L(m)$. By conclusion (i), the inequality is true for $i = m$. Now assume the inequalities are true for all i, $m \leqq i \leqq J (< L(m))$. We will show that the inequality holds for $i = J + 1$. Suppose if possible

$$(4.2) \qquad (-1)^{M(J+1)}(f(z) - u(z)) > 0 \text{ for some } z \in (x_{J+1}, x_{J+2}).$$

Let $g \in F$ be determined by the condition $g(z) = u(z)$ and all the conditions of (1.1) except for $i = J + 1$ and $r = 0$. (g exists by the hypothesis H, since $n(J + 1) = 1$). Now $g - f$ has $n(1), \cdots, n(J), n(J + 2), \cdots, n(k)$ zeros at $x_1, \cdots, x_J, x_{J+2}, \cdots, x_k$ respectively and hence by the hypothesis H, cannot have any more zeros on (x_J, x_{J+2}). Therefore $g - f$ must keep a constant sign on (x_J, x_{J+2}) and in view of the inequality (4.2) it must satisfy

$$(4.3) \qquad (-1)^{M(J+1)}(f(x_{J+1}) - g(x_{J+1})) > 0.$$

However $g \in F$ is determined by the conditions (1.1) with x_{J+1} replaced by z and hence by the induction hypothesis it must satisfy $(-1)^{M(J)}(g(x) - u(x)) \leqq 0$ on (x_J, z). In particular $(-1)^{M(J)}(g(x_{J+1}) - u(x_{J+1})) \leqq 0$ and this together with the assumption that $n(J + 1) = 1$ contradicts the inequality (4.3). This completes the proof of (ii).

(iii) Since f satisfies the condition $f(z) = u(z)$ and all the conditions of (1.1) (we can ignore the condition with $i = m$ and $r = n(m) - 1$), by the hypothesis G_m we must have $(-1)^{M(m)-1}(f(x) - u(x)) \leqq 0$ on (x_m, z). This inequality together with (i) implies $f(x) \equiv u(x)$ on $[x_m, z]$.

(iv) We shall first consider the case $z \in (x_m, x_{m+1})$. Now f satisfies the condition $f(z) = u(z)$ and all the conditions of (1.1) (we shall now ignore the condition with $i = L(m)$ and $r = 0$). Consequently by (ii) we must have

$$(-1)^{M(m+1)}(f(x) - u(x)) \leqq 0 \text{ on } (z, x_{m+1}),$$

$$(-1)^{M(i+1)}(f(x) - u(x)) \leqq 0 \text{ on } (x_i, x_{i+1}),\ m + 1 \leqq i \leqq L(m) - 2$$

$$\text{and } (-1)^{M(L(m))}(f(x) - u(x)) \leqq 0 \text{ on } (x_{L(m)-1}, x_{L(m)+1}).$$

These inequalities together with (i), (ii), and (iii) imply $f(x) \equiv u(x)$ on $[x_m, x_{L(m)}]$.

In case $z \in (x_{m+1}, x_{m+2})$, we can interchange the roles of z and x_{m+1} in the preceding argument and then using (iii) as before we can obtain in the same way $f(x) \equiv u(x)$ on $[x_m, x_{L(m)}]$.

The proof for the cases $z \in (x_i, x_{i+1})$ for some i, $m + 2 \leqq i \leqq L(m)$ does not involve any new ideas and hence is omitted.

In the next two theorems we state sufficient conditions under which $\lambda(n, k; m+)$ and $\lambda(n, k; m-)$ convexity implies $\lambda(n, k)$-convexity.

THEOREM 4.3. *Let $\lambda(n, k) \in P(n)$ be such that $n(i) \geqq 2$ for at least one i, $1 \leqq i < k$. Suppose F and u satisfy the hypotheses H, H_i and G_i for all i, $p \leqq i < k$ for which $n(i) \geqq 2$ and $f \in F$ is as in Lemma 4.1. Then*

(i) *f satisfies $(-1)^{M(i)}(f(x) - u(x)) \leqq 0$ on (x_i, x_{i+1}), $p \leqq i < k$.*

(ii) *If either* (a) *$p = 1$ or* (b) *$p > 1$ and F and u satisfy the hypotheses $H_p{}'$ and $G_p{}'$, then u is $\lambda(n, k)$-convex with respect to F on I.*

(iii) *If either* (a) *$n(k) = 1$ or* (b) *$n(k) \geqq 2$ and H_k holds then $f(x) - u(x) \leqq 0$ on (x_k, x_{k+1}).*

The next theorem is an analogue of Theorem 4.3.

THEOREM 4.4. *Let $\lambda(n, k) \in P(n)$ be such that $n(i) \geqq 2$ for at least one i, $1 < i \leqq k$. Suppose F and u satisfy the hypotheses H, $H_i{}'$, and $G_i{}'$ for all i, $1 < i \leqq q$ for which $n(i) \geqq 2$ and $f \in F$ is as in Lemma 4.1. Then*

(i) *f satisfies $(-1)^{M(i-1)}(f(x) - u(x)) \leqq 0$ on (x_{i-1}, x_i), $1 < i \leqq q$.*

(ii) *If either* (a) *$q = k$ or (b) $q < k$ and F and u satisfy the hypotheses H_q and G_q, then u is $\lambda(n, k)$-convex with respect to F on I.*

(iii) *If either* (a) *$n(1) = 1$ or* (b) *$n(1) \geqq 2$ and $G_1{}'$ holds, then $(-1)^n(f(x) - u(x)) \leqq 0$ on (x_0, x_1).*

We shall prove only Theorem (4.3) since the proof of Theorem (4.4) is analogous.

PROOF (i). Since $n(p) \geqq 2$, by (i) of Lemma (4.1) f satisfies $(-1)^{M(p)}(f(x) - u(x)) \leqq 0$ on (x_p, x_{p+1}). In case $p + 1 = k$, there is nothing to prove. In case $p + 1 < k$, we can have either (a) $n(p + 1) \geqq 2$ or (b) $n(p + 1) = 1$. We now use (i) or (ii) of Lemma 4.1 according as (a) or (b) occurs to get the required inequality on (x_{p+1}, x_{p+2}). Again the procedure stops if $p + 2 = k$. Otherwise repeating the previous argument a finite number of times we arrive at the conclusion (i).

(ii) (a) Obvious

(b) This follows from (ii) of Lemma (4.2) and (i) above.

(iii) (a) This follows from (ii) of Lemma (4.1) by letting $m = q$ in that lemma.

(b) Obvious.

THEOREM 4.5. *Suppose F is an unrestricted n-parameter family on I and u is convex with respect to F* on *I. Let $\lambda(n, k) \in P(n)$ be arbitrary and $f \in F$ be as in Lemma* 4.1. *Then*

(i) *u is $\lambda(n, k)$-convex with respect to F on I.*

(ii) *f satisfies* (a) $f(x) - u(x) \leqq 0$ *on* (x_k, x_{k+1}) *and* (b) $(-1)^n(f(x) - u(x)) \leqq 0$ *on* (x_0, x_1).

(iii) *In case* $f(z) = u(z)$ *for some* $z \in (x_i, x_{i+1})$, $0 \leqq i \leqq k$ *then* $f(x) \equiv u(x)$ *on* $[\min\{z, x_1\}, \max\{z, x_k\}]$.

PROOF (i). We can assume $\lambda(n, k) \in P(n)$ is such that $n(i) \geqq 2$ for some i, $1 \leqq i \leqq k$ for if otherwise there is nothing to prove. Now one of the following two cases must occur.

Case I. $n(i) = 1$ for all i, $1 \leqq i < k$,

Case II. $n(i) \geqq 2$ for some i, $1 \leqq i < k$.

In case I, we have $n(k) \geqq 2$, so the $\lambda(n, k)$-convexity of u follows from the convexity of u by (ii) (a) of Theorem 4.4 and induction on $n(k)$.

In case II, the proof is by induction on k. Since u is convex we obtain by (ii) of Theorem 4.3 that for all $\mu(n, n-1) \in P(n)$ with $n(i) \geqq 2$ for some i, $1 \leqq i < n - 1$, u is $\mu(n, n-1)$-convex. Now assume that for all $j \geqq k + 1$ and for all $\mu(n, j) \in P(n)$ with $\mu(i) \geqq 2$ for some i, $1 \leqq i < j$, u is $\mu(n, j)$-convex. We will show u is $\lambda(n, k)$-convex. By the induction hypothesis u is $\mu(n, k+1)$-convex for all $\mu(n, k+1) \in P(n)$ with $n(i) \geqq 2$ for some i, $1 \leqq i < k + 1$. Now by our choice of $\lambda(n, k)$ in case II we have for all i, $1 \leqq i < k$ for which $n(i) \geqq 2$ that $\lambda(n, k; i+)$ and $\lambda(n, k; i-)$ are $(k+1)$-tuples satisfying either $n(i) = 1$ for all i, $1 \leqq i < k + 1$ or $n(i) \geqq 2$ for some i, $1 \leqq i < k + 1$. Hence we have either by case I or by induction hypothesis that u is $\lambda(n, k; i+)$ and $\lambda(n, k; i-)$-convex for all i, $1 \leqq i < k$ with $n(i) \geqq 2$. Hence by (ii) of Theorem (4.3) it follows that u is $\lambda(n, k)$-convex.

(ii) If $n(i) = 1$ for all i, $1 \leqq i \leqq k$ these inequalities are known [5] to be true. So we shall assume $n(i) \geqq 2$ for some i, $1 \leqq i \leqq k$ and consider the two cases I and II as in the proof of (i). If case I occurs then the inequalities follow by (i) of Lemma 4.1 and (ii) of Lemma (4.2). If case II occurs then (a) follows from (iii) of Theorem (4.3) and (b) follows either from (ii) of Lemma (4.2) with $m = p$ or from (i) of Lemma (4.2).

(iii) If $n(i) = 1$ for all i, $1 \leqq i \leqq k$, then the desired identity follows from the definition of convexity and (ii). If $n(i) \geqq 2$ for some i, $1 \leqq i \leqq k$, again consider the two cases I and II as in the proof of (i). If case I occurs and $z > x_k$ then we have by (iii) of Lemma (4.1) that $f(x) \equiv u(x)$ on $[x_k, z]$. Now choose n points $x_k = t_1 < t_2 < \cdots < t_n = z$. Then f satisfies the conditions $f(t_i) = u(t_i)$, $1 \leqq i \leqq n$ together with $f(x_1) = u(x_1)$. Hence by the first assertion in the proof of (iii) we will have $f(x) \equiv u(x)$ on $[x_1, z]$. If $z < x_k$ then the required identity follows from (iv) of Lemma (4.2) with $m = k$.

If case II occurs then one of the following must hold. (a) $z > x_q$, (b) $z < x_p$ and (c) $x_p < z < x_q$ if $p < q$. If (a) holds then by (iii) or (iv) of Lemma (4.1) we will have $f(x) \equiv u(x)$ on $[x_q, \max\{z, x_k\}]$. If $q = 1$, we are done. If $q > 1$, choose points t_i, $1 \leqq i \leqq n$ so that $x_q = t_1 < t_2 < \cdots < t_n = z$. Since u is convex and f satisfies $f(t_i) = u(t_i)$, $1 \leqq i \leqq n$ with $f(x_1) = u(x_1)$ it must follow from the first assertion in the proof of (iii) that $f(x) \equiv u(x)$ on $[x_1, z]$. Consequently $f(x) \equiv u(x)$ on $[x_1, \max\{z, x_k\}]$.

If (b) holds (note $p < k$) then the proof is analogous to that for (a).

If $p = q$, then the proof is complete. Otherwise if (c) holds then it suffices to consider the case $x_p < z < x_{p+1}$ since the proof for the cases of other intervals will be similar. Now by (iii) of Lemma (4.1) we have $f(x) \equiv u(x)$ on $[x_p, z]$. Then choosing points v_i, $1 \leqq i \leqq n$ such that $x_p \leqq v_1 < v_2 < \cdots < v_n = z$ and using the hypothesis of convexity of u along with $f(x_k) = u(x_k)$, we obtain $f(x) \equiv u(x)$ on $[x_p, x_k]$. If $p = 1$ we are done. If $p > 1$, the fact that $f(x_1) = u(x_1)$ yields also $f(x) \equiv u(x)$ on $[x_1, z]$. Consequently $f(x) \equiv u(x)$ on $[x_1, x_k]$. This completes the proof of (iii).

COROLLARY 4.6. *Let F, $\lambda(n, k)$ and f be as in Theorem* 4.5. *Suppose u is strictly convex with respect to F on I. Then*

(i) *u is strictly $\lambda(n, k)$-convex with respect to F on I.*

(ii) *f satisfies strict inequalities in* (ii) *of Theorem* 4.5.

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