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DIFFERENTIAL INEQUALITIES AND THE ASYMPTOTICS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** In this paper we consider equations of the form

$$(1.1) \qquad Ly \equiv y'' + h(t)y' + r(t)y = f(t, y, y')(' \equiv d/dt \equiv D),$$

where h, r and f are continuous function on $a < t < b$, $|y| < \infty$, $|y'| < \infty$. We allow the open interval (a, b) to be bounded or unbounded and envisage the situation where f is a small perturbation in some sense in the differential equation (1.1) near the endpoints a and b.

The asymptotic nature of solutions of (1.1) depends critically upon whether solutions of $Ly = 0$ are oscillatory or nonoscillatory. This is clearly illustrated by the results in [1], [2], [4], [8] and [11]. For example, solutions of $y'' = r(t)y$ behave at ∞ like solutions of the nonoscillatory equation $y'' = 0$ if $tr(t)$ is integrable at ∞, whereas solutions of $y'' + y = r(t)y$ behave at ∞ like solutions of the oscillatory equation $y'' + y = 0$ if just $r(t)$ is integrable at ∞. In this investigation we assume that L is disconjugate on (a, b), i.e., no nontrivial solution of $Ly = 0$ has more than one zero in (a, b). For conditions on h and r which imply L is disconjugate, see [12]-[15]. The end results of our investigation provide conditions on f which imply (1.1) has solutions which are asymptotic to the maximal solutions of $Ly = 0$ at the endpoints of (a, b).

A nontrivial solution u of $Ly = 0$ is said to be a *minimal solution* at b if

$$\lim_{t \to b-} \frac{u(t)}{v(t)} = 0$$

for all solutions v linearly independent of u. Minimal solutions are unique up to multiplication by nonzero constants. Any nontrivial solution which is not a minimal solution at b is called a *maximal solution* at b. By a positive solution at b, we shall mean a solution which is positive in some left neighborhood (a_1, b) of b. The assumption that L is disconjugate on (a, b) implies the existence of minimal and maxi-

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mal solutions at a and b. We will say that L is *disconjugate on the closed interval* $[a, b]$, even though L may be singular at a or b, provided L is disconjugate on (a, b) in the sense described above and provided the minimal solutions u_1 and u_2 at b and a, respectively, are linearly independent. Thus, $L \equiv D^2$ is disconjugate on $[0, \infty]$ because the minimal solutions at ∞ and 0 are (multiples of) 1 and t, respectively, which are linearly independent. But D^2 is not disconjugate on $[-\infty, \infty]$ because the constant solutions are minimal at both $-\infty$ and ∞. Finally, we note that disconjugacy on $[a, b]$ implies the existence of minimal and maximal solutions at a and b which are positive throughout (a, b).

THEOREM 1.1. *Assume that* $h, r \in C(a, b)$, $f \in C((a, b) \times R^2)$, *$L$ is disconjugate on* $[a, b]$ *and there exist functions* $\alpha, \beta \in C^2(a, b)$ *such that* $L\beta \leqq f(t, \beta, \beta')$, $L\alpha \geqq f(t, \alpha, \alpha')$ *and* $\beta \geqq \alpha$. *Let*

$$\psi(t) = \sup\{|f(t, y, z)| : |z| < \infty \text{ and } \alpha(t) \leqq y \leqq \beta(t)\} \tag{1.2}$$

and u_1 and u_2 be positive minimal solutions of $Ly = 0$ *at b and a, respectively. If* $\psi(t)u_2(t)\exp(\int^t p(s)\, ds)$ *is integrable at a and* $\psi(t)u_1(t)\exp(\int^t p(s)\, ds)$ *is integrable at b, then for any A and B such that*

$$\lim_{t \to a+} \frac{\alpha(t)}{u_1(t)} \leqq A \leqq \lim_{t \to a+} \frac{\beta(t)}{u_1(t)}, \quad \lim_{t \to b-} \frac{\alpha(t)}{u_2(t)} \leqq B \leqq \lim_{t \to b-} \frac{\beta(t)}{u_2(t)} \tag{1.3}$$

the boundary value problem

$$Ly = f(t, y, y'), \quad \lim_{t \to a+} \frac{y(t)}{u_1(t)} = A, \quad \lim_{t \to b-} \frac{y(t)}{u_2(t)} = B, \tag{1.4}$$

has a solution $y \in C^2(a, b)$.

THEOREM 1.2. *Assume that* $h, r \in C(a, b)$, $f \in C((a, b) \times R^2)$, $f(t, 0, 0) \leqq 0$ *and L is disconjugate on* (a, b).

Let u_1 and u_2 be positive minimal and maximal solutions at b of $Ly = 0$, *respectively. If there exist a constant c and a continuous function* $g(t, y)$, *which is nondecreasing in y for* $y > 0$, *such that*

$$|f(t, y, y')| \leqq g(t, y), \; a < t < b, \; y > 0, \; -\infty < y' < \infty, \tag{1.5}$$

and

$$\int^b u_1(s) g(s, c u_2(s)) \exp\left(\int^s p(\tau)\, d\tau\right) ds < \infty, \tag{1.6}$$

then for each B, $0 \leqq B < c$, equation (1.1) *has a solution y defined in some left neighborhood of b such that*

$$\lim_{t \to b-} \frac{y(t)}{u_2(t)} = B. \tag{1.7}$$

Theorem 1.1 will be proven in section 2 and Theorem 1.2, which is a consequence of Theorem 1.1, will be proven in § 3. Of course, the companion result to Theorem 1.2 emphasizing the asymptotic behavior at a instead of b would also hold.

As an existence theorem for boundary value problems of the type (1.4), Theorem 1.1 extends the main result of Lee and Willett [**6**], who assume that $\psi(t)u_i(t)\exp(\int^t p(s)\,ds)$, $i = 1, 2$, is integrable on the whole interval $[a, b]$. However, the results in [**6**] allow more general boundary conditions than (1.4) and more general functions $f(t, y, y')$ with respect to y'. A simple useful consequence of Theorem 1.1 is the following.

Corollary 1.1. *If $f \in C((0, \infty) \times R^2)$, there exist constants c_1 and c_2 such that $f(t, c_2, 0) \geqq 0 \geqq f(t, c_1, 0)$ and $c_1 < c_2$, and*

$$\psi(t) = \sup\{|f(t, y, z)| : |z| < \infty, c_1 < y < c_2\}$$

is integrable at ∞ and $t\psi(t)$ is integrable at 0, then for each A, $c_1 \leqq A \leqq c_2$, the problem

$$y'' + f(t, y, y'), y(0) = A,$$

has a solution $y \in C^2(0, \infty)$ such that $c_1 \leqq y(t) \leqq c_2$, $0 < t < \infty$.

Many (cf., e.g., [3]; [5], [7], [9], [**10**] and [**16**]) results in the literature when applied to (1.1) follow directly from Theorems 1.1 and 1.2. A simple example is the following:

Corollary 1.2. *If for each $i = 0, 1, \cdots, M$, the functions $a_i(t)t^i$ are continuous and integrable on some neighborhood of ∞, then for each positive constant B, there exists a neighborhood N of ∞ such that*

$$y'' + \sum_{i=0}^{M} a_i(t)y^i = 0$$

has a solution $y \in C^2(N)$ such that $\lim_{t \to \infty} y(t)/t = B$.

The converse of Corollary 1.2 obviously holds in the case the functions $a_i(t)$ are of constant and identical sign.

2. **Proof of Theorem 1.1.** With u_1 and u_2 positive minimal solutions of $Ly = 0$ at b and a, respectively, define $u = u_1 + u_2$ and

$$(2.1)\quad D^0y(t) = \lim_{s\to t}\frac{y(s)}{u(s)},\ D^1y(t) = \lim_{s\to t}\frac{u(s)y'(s) - u'(s)y(s)}{W(s)},\ a \leqq t \leqq b,$$

where

$$(2.2)\quad \begin{aligned} W(s) &= u_1(s)u_2'(s) - u_2(s)u_1'(s) \\ &\equiv \text{const}\cdot\exp\left(-\int^s h(\tau)\,d\tau\right). \end{aligned}$$

Let

$$(2.3)\quad F(t, y, z) = f\left(t, y, \frac{u'(t)y + W(t)z}{u(t)}\right),$$

so that $F(t, y, D^1y) = f(t, y, y')$.

Let $k(t)$ be a positive continuous function on (a, b) for which the combination $k(t)u(t)\exp(\int^t h(s)\,ds)$ is integrable on $[a, b]$. With α and β as in the assumptions, define

$$(2.4)\quad F^*(t, y, z) = \begin{cases} F(t, \beta(t), z) + k(t)\mathrm{Arctan}(y - \beta(t)), & \text{when } y > \beta(t), \\ F(t, y, z), & \text{when } \alpha(t) \leqq y \leqq \beta(t), \\ F(t, \alpha(t), z) - k(t)\arctan(\alpha(t) - y), & \text{when } \alpha(t) > y. \end{cases}$$

Then $F^* \in C((a, b) \times R^2)$ and (1.2) implies

$$(2.5)\quad \sup\{|F^*(t, y, z)| : |y| < \infty, |z| < \infty\} \leqq \psi(t) + \pi k(t).$$

The first part of the proof consists of showing the existence of a solution y to the boundary value problem

$$(2.6)\quad Ly = F^*(t, y, D^1y),\ D^0y(a) = A,\ D^0y(b) = B,$$

for A and B satisfying (1.3). The second part of the proof consists of showing that (1.3) is sufficient in this case to imply that the solution y satisfies $\alpha \leqq y \leqq \beta$ and is thus a solution of $Ly = F(t, y, D^1y) = f(t, y, y')$.

Let $a < c < b$ and consider c fixed in what follows. For $0 < \epsilon \leqq \epsilon_0 \leqq \min(b - c, c - a)$ and $p, q \in (-\infty, \infty)$, define $z(t)$ so that

$$(2.7)\quad z(t) - pu_1(t) - qu_2(t) = \begin{cases} 0, & \text{when } c - \epsilon < t < c + \epsilon, \\ \displaystyle\int_{c+\epsilon}^{t} g(t,s)F^*(s, z(s-\epsilon), D^1z(s-\epsilon))\,ds, & \text{when } c + \epsilon \leqq t < b \\ \displaystyle\int_{c-\epsilon}^{t} g(t,s)F^*(s, z(s+\epsilon), D^1z(s+\epsilon))\,ds, & \text{when } a < t \leqq c - \epsilon \end{cases}$$

where

$$(2.8)\qquad g(t,s) = [u_2(t)u_1(s) - u_2(s)u_1(t)]/W(s)$$

is the Cauchy function for the operator L. If $\varphi(t) = \psi(t) + \pi k(t)$, then the assumptions and (2.2) imply that $u_2(t)\varphi(t)W^{-1}(t)$ is integrable at a and $u_1(t)\varphi(t)W^{-1}(t)$ is integrable at b. It follows from this observation and (2.7) that $D^0z(a)$ and $D^0z(b)$ exist and

$$(2.9)\quad D^0z(a) = p + \int_a^{c-\epsilon} u_2(s)W^{-1}(s)F^*(s, z(s+\epsilon), D^1z(s+\epsilon))\,ds,$$

$$(2.10)\quad D^0z(b) = q + \int_{c+\epsilon}^{b} u_1(s)W^{-1}(s)F^*(s, z(s-\epsilon), D^1z(s-\epsilon))\,ds.$$

In what follows we will emphasize the dependence of $z(t)$ on its various parameters ϵ, p and q by using additional arguments, thus, $z(t) \equiv z(t; \epsilon, p, q)$.

For ϵ fixed define a mapping $T : R^2 \to R^2$ by

$$T(p,q) = (P,Q),$$

where

$$P = A - \int_a^{c-\epsilon} u_2(s)W^{-1}(s)F^*(s, z(s+\epsilon, p, q), D^1z(s+\epsilon, p, q))\,ds,$$

$$Q = B - \int_{c+\epsilon}^{b} u_1(s)W^{-1}(s)F^*(s, z(s-\epsilon, p, q), D^1z(s-\epsilon, p, q))\,ds.$$

Since

$$\int_a^{c-\epsilon} |u_2(s)W^{-1}(s)F^*(s, z, D^1z)|\,ds \leqq \int_a^c u_2(s)W^{-1}(s)\varphi(s)\,ds < \infty$$

and

$$\int_{c+\epsilon}^{b} |u_1(s)W^{-1}(s)F^*(s, z, D^1z)|\,ds \leqq \int_c^b u_1(s)W^{-1}(s)\varphi(s)\,ds < \infty,$$

TR^2 is bounded uniformly in ϵ, $0 < \epsilon < \epsilon_0$. Thus, there exists a compact subset K of R^2 independent of ϵ such that $TK \subset K$ for all $0 < \epsilon < \epsilon_0$. Since T is continuous, the Browder Fixed Point Theorem implies that for each ϵ, $0 < \epsilon < \epsilon_0$, T has a fixed point $(p_\epsilon, q_\epsilon) \in K$.

Let $z_\epsilon(t) \equiv z(t;\ p_\epsilon,\ q_\epsilon)$ and consider $\chi_0 = \{D^0 z_\epsilon(t) : 0 < \epsilon \leqq \epsilon_0\}$. On compact subsets of (a, b), χ_0 is a uniformly bounded and equicontinuous set. Hence by the Ascoli-Arzela theorem and the compactness of K, there exists a sequence $\epsilon_k \downarrow 0$, such that

$$(p_k, q_k) \equiv (p_{\epsilon_k}, q_{\epsilon_k}) \rightarrow (p_*, q_*) \in k,$$
$$w_k \equiv D^0 z_{\epsilon_k} \rightarrow w_* \in C(a, b),$$

and the convergence of w_k to w_* is uniform on compact subsets of (a, b). Furthermore, $w_*(a) = A$ and $w_*(b) = B$ since $w_k(a) = A$ and $w_k(b) = B$ for all values of k.

Let $z_k \equiv z_{\epsilon_k} = u w_k$ and $z_* = u w_*$, and consider $\chi_1^* = \{D^1 z_k : k = 1, 2, \cdots\}$. From (2.7) we obtain

$$D^1 z_k(t) = \begin{cases} q_k - p_k, \text{ when } c - \epsilon_k < t < c + \epsilon_k \\ q_k - p_k + \displaystyle\int_{c+\epsilon_k}^{t} \frac{u(s)}{W(s)} F^*(s, z_k(s - \epsilon_k), D^1 z_k(s - \epsilon_k))\, ds, \\ \qquad \text{when } c + \epsilon_k \leqq t < b \\ q_k - p_k + \displaystyle\int_{c-\epsilon_k}^{t} \frac{u(s)}{W(s)} F^*(s, z_k(s + \epsilon_k), D^1 z_k(s + \epsilon_k))\, ds, \\ \qquad \text{when } a < t \leqq c - \epsilon_k. \end{cases} \tag{2.13}$$

Hence, χ^*_1 is a uniformly bounded and equicontinuous set, and so the Ascoli-Arzela theorem implies there exists a subsequence, without loss of generality assume it to be $\{D^1 z_k\}$, which converges uniformly on compact subsets of (a, b) to a function $z_*{}^1 \in C(a, b)$. So from (2.7) and (2.13), we obtain

$$z_*(t) = p_* u_1(t) + q_* u_2(t) + \int_c^t g(t, s) F^*(s, z_*(s), z_*{}^1(s))\, ds, \tag{2.14}$$

$$z_*{}^1(t) = q_* - p_* + \int_c^t u(s) W^{-1}(s) F^*(s, z_*(s), z_*{}^1(s))\, ds. \tag{2.15}$$

But (2.14) implies

$$D^1 z_*(t) = q_* - p_* + \int_c^t u(s)W^{-1}(s)F^*(s, z_*(s), z_*{}^1(s))\, ds;$$

hence, $D^1 z_*(t) = z_*{}^1(t)$ from (2.15).

Furthermore,

$$p_* = A - \int_a^c u_2(s)W^{-1}(s)F^*(s, z_*(s), D^1 z_*(s))\, ds,$$

$$q_* = B - \int_c^b u_1(s)W^{-1}(s)F^*(s, z_*(s), D^1 z_*(s))\, ds,$$

and hence by substituting into (2.14)–(2.15), we conclude that

$$\begin{aligned} u(t)z_*(t) = {} & Au_1(t) + Bu_2(t) \\ & - u_2(t)\int_t^b u_1(s)W^{-1}(s)F^*(s, u(s)z_*(s), z_*{}^1(s))\, ds \\ & - u_1(t)\int_a^t u_2(s)W^{-1}(s)F^*(s, u(s)z_*(s), z_*{}^1(s))\, ds, \\ z_*{}^1(t) = {} & B - A + \int_a^t u_2(s)W^{-1}(s)F^*(s, u(s)z_*(s), z_*{}^1(s))\, ds \\ & - \int_t^b u_1(s)W^{-1}(s)F^*(s, u(s)z_*(s), z_*{}^1(s))\, ds = D^1(uz_*). \end{aligned}$$

Hence, $y = uz_*$ satisfies (2.6), and the first part of the proof is complete.

The second part of the proof consists of showing the solution y of (2.6) satisfies

$$\alpha(t) \leqq y(t) \leqq \beta(t),\ a < t < b; \tag{2.18}$$

hence, y is a solution of (1.4) since $F^*(t, y(t), D^1y(t))$ then agrees with $f(t, y(t), y'(t))$ for $a < t < b$. Inequality (2.18) can be established by means of the following elementary maximal principle.

LEMMA. *If $y \in C(t_0, t_1)$, $y(t) > 0$ for $t_0 < t < t_1$ and $D^0y(t_0) = 0 = D^0y(t_1)$, then there exists $E \in (t_0, t_1)$ such that*

$$D^1y(E) = 0 \quad \text{and} \quad Ly(E) \leqq 0. \tag{2.19}$$

PROOF. Since $D^0y(t) = y(t)/u(t) > 0$ in (t_0, t_1) and $\lim_{t\to t_2-} D^0y(t) = 0 = \lim_{t\to t_1+} D^0y(t)$, there exists a point E in (t_0, t_1) at which D^0y is maximal. At this point, it must be the case that $(y/u)'(E) = 0$ and $(y/u)''(E) \leqq 0$. But

$$D^1y(E) = \frac{u^2(E)}{W(E)}\left(\frac{y}{u}\right)'(E) = 0$$

$$Ly(E) = \frac{W}{u}D\frac{u^2}{W}D\frac{y}{u}\Big|^{E} = u(E)\left(\frac{y}{u}\right)''(E) \leqq 0.$$

To complete the proof of Theorem 1.1, assume $y(t) > \beta(t)$ for some $t \in (a, b)$; the proof is similar if $y(t) < \alpha(t)$ for some t. Then there exists $[t_1, t_2] \subset [a, b]$ such that $z(t) = y(t) - \beta(t) > 0$ for $t \in (t_1, t_2)$ and $D^0z(t_1) = 0 = D^0z(t_2)$, since $D^0y(a) \leqq D^0\beta(a)$ and $D^0y(b) \leqq D^0\beta(b)$. Hence, the Lemma implies there exists a point $E \in (t_0, t_1) \subset (a, b)$ such that

(2.20) $$D^1z(E) = 0$$

and

(2.21) $$Lz(E) \leqq 0.$$

But (2.20) implies $D^1y(E) = D^1\beta(E)$; hence,

$$\begin{aligned} Lz(E) &= Ly(E) - L\beta(E) \\ &\geqq F^*(E, y(E), D^1y(E)) - f(E, \beta(E), \beta'(E)) \\ &= F(E, \beta(E), D^1\beta(E)) + k(E)\operatorname{Arctan} z(E) - f(E, \beta(E), \beta'(E)) \\ &= k(E) \operatorname{Arctan} z(E) > 0, \end{aligned}$$

which contradicts (2.21).

3. **Proof of Theorem 1.2.** Choose $a_1, a \leqq a_1 < b$, sufficiently close to b so that $u_2(t) > 0$ on (a_1, b) and

(3.1) $$B + \int_{a_1}^{b} u_1(s)g(s, cu_2(s))/W(s)\, ds \leqq c,$$

where $W(t)$ is defined by (2.2).

We will show that for any A such that

$$0 \leqq A \leqq c \lim_{t \to a_1+} u_2(t)/u_1(t),$$

Theorem 1.1 implies the boundary value problem

$$Ly = f(t, y, y'), \lim_{t \to a_1+} y(t)/u_1(t) = A, \lim_{t \to b^-} y(t)/u_2(t) = B,$$

has a solution $y \in C^2(a, b)$, which will be sufficient to prove Theorem 1.2. Note that $u_2(t)$ is not in general a minimal solution of $Ly = 0$ at

a_1, which is required in Theorem 1.1. However, in the present context this will not be important for if $u_2(t)$ is not a minimal solution at a_1, then

$$\bar{u}_2(t) = u_2(t) - u_2(a_1)u_1(t)/u_1(a_1) > 0, a_1 < t < b,$$

is such, and can be used in place of $u_2(t)$ since

$$\lim_{t\to b-} \frac{y(t)}{u_2(t)} = \lim_{t\to b-} \frac{y(t)}{\bar{u}_2(t)}. \tag{3.2}$$

Let $\alpha \equiv 0$ so that the boundary inequalities for α are automatically satisfied, because $A, B \geqq 0$, and

$$L\alpha(t) = 0 \geqq f(t, 0, 0) = f(t, \alpha(t), \alpha'(t)), a_1 < t < b,$$

by assumption.

Let $\beta(t) = u_2(t)z(t)$, where

$$z(t) = c - \int_{a_1}^{t} \left(\int_{s}^{t} u^{-2}(r)\exp\left(- \int^{r} h(\tau)\, d\tau\right) dr \right)$$
$$u_2(s)g(s, cu_2(s)) \exp \left(\int^{s} h(\tau)\, d\tau\right) ds,$$

so that

$$z(t) \leqq z(a_1) = c,\ a_1 < t < b.$$

Since minimal solutions are unique up to a constant factor,

$$u_2(t) \int_{t}^{b} u_2{}^{-2}(s)\exp\left(- \int^{s} h(\tau)\, d\tau\right) ds$$

$$= c_0 u_1(t) = \frac{u_1(t)\exp(- \int^{t} h(t)\, d\tau)}{W(t)}.$$

Thus, (3.1) implies

$$z(t) \geqq z(b) = c - \int_{a_1}^{b} \frac{u_1(s)g(s, cu_2(s))}{W(s)}\, ds \geqq B.$$

Hence,

$$\beta(t) = u_2(t)z(t) \geqq 0 \equiv \alpha(t), a_1 < t < b,$$

and

$$\lim_{t\to b-} \frac{\beta(t)}{u_2(t)} = z(b) \geqq B.$$

Finally, from the monotonicity of g and (1.5), we conclude that

$$\begin{aligned} L\beta(t) &= u_2(t)z''(t) + [2u'(t) + h(t)u_2(t)]\, z'(t) \\ &= -g(t, cu_2(t)) \\ &\leqq -g(t, u_2(t)z(t)) = -g(t, \beta(t)) \\ &\leqq f(t, \beta(t), \beta'(t)),\, a_1 < t < b. \end{aligned}$$

There remains to show the appropriate integrability conditions hold at b and a_1. If

$$\psi(t) = \sup\{|f(t, y, z)| : |z| < \infty, 0 \leqq y \leqq u(t)z(t)\},$$

then

$$\psi(t) \leqq g(t, u_2(t)z(t)) \leqq g(t, cu_2(t)).$$

Since $u_1(t)g(t, cu_2(t))/W(t)$ is integrable at b by assumption,

$$u_1(t)\psi(t)\exp\left(- \int^t \quad h(\tau)\, d\tau \right) = c_0 u_1(t)\psi(t)/W(t)$$

is integrable at b. The appropriate integrability condition at a_1 will also hold in the present context because of (3.1), that is (1.6).

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