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ON A. LAX'S CONDITION OF HYPERBOLICITY

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ABSTRACT. We give an elementary proof of A. Lax's condition of hyperbolicity for polynomials with constant coefficients (see [3]). Our method is independent of Puiseux series. In a second paper, we shall show how this method can be adapted to obtain other criteria of hyperbolicity, including Svensson's criterion (see [4]).

DEFINITION. Let P_k $(k = 0, \cdots, m)$ be a polynomial in n variables, homogeneous of degree k and with constant coefficients.

The polynomial $P = \sum_{k=0}^{m} P_{m-k}$ is said to be hyperbolic with respect to $N \in R^n \setminus \{0\}$ if $P_m(N) \neq 0$ and if there exists a constant c such that

$$\left.\begin{array}{r} P(ix + \tau N) = 0 \\ x \in R^n, \tau \in C \end{array}\right\} \Rightarrow |\mathcal{R}\tau| \leqq c.$$

THEOREM (A. LAX [3]). *Let P be hyperbolic with respect to N. If, for x_0 in R^n, τ_0 is a root of $P_m(ix_0 + \tau N)$ with multiplicity $\alpha_0 \in [2, m]$, then, τ_0 is a root of $P_{m-k}(ix_0 + \tau N)$ with multiplicity $\geqq \alpha_0 - k$, for $k = 0, \cdots, \alpha_0 - 1$.*

Conversely, if $n = 2$, if P satisfies the quoted condition and if P_m is hyperbolic with respect to N, then P is also hyperbolic with respect to N.

PROOF. (a) *Necessity.* If P is hyperbolic with respect to N, we have, for any α and $t > 0$,

$$\alpha^m P \left[\frac{itx_0 + (\tau + t\tau_0)N}{\alpha}\right] \equiv$$

$$\sum_{k=0}^{m} \alpha^k \sum_{j=0}^{m-k} \frac{\tau^j}{j!} t^{m-k-j} \{D_\tau^j [P_{m-k}(ix_0 + \tau N)]\}_{\tau=\tau_0} = 0 \Rightarrow |\mathcal{R}\tau| \leqq c\alpha,$$

since $\mathcal{R}\tau_0 = 0$ (see [2] p. 90).

We shall prove the necessity by induction. For $k = 0$, it is true by hypothesis. Let us suppose the condition is true for $k < k_0$ $(1 \leqq k_0 \leqq \alpha_0 - 1)$ and show that $P_{m-k_0}(ix_0 + \tau_0 N) = 0$.

By the inductive hypothesis we have

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$$\left(\sum_{k=0}^{k_0-1}\sum_{j=\alpha_0-k}^{m-k}+\sum_{k=k_0}^{m}\sum_{j=0}^{m-k}\right)\alpha^k\frac{\tau^j}{j!}t^{m-k-j}$$

$$\{D_\tau^j\,[P_{m-k}(ix_0+\tau N)]\,\}_{\tau=\tau_0}=0\Longrightarrow|\mathcal{R}\tau|\leqq c\alpha.$$

Choose α so that the coefficient of $P_{m-k_0}\ (ix_0+\tau_0 N)$ is equal to that of

$$\frac{\tau^{\alpha_0}}{\alpha_0!}\{D_\tau^{\alpha_0}\,[P_m(ix_0+\tau N)]\,\}_{\tau=\tau_0},$$

that is

$$\alpha=\frac{1}{t^{(\alpha_0-k_0)/k_0}}$$

After division of the equation by $t^{m-\alpha_0}$, the implication becomes

$$\left(\sum_{k=0}^{k_0-1}\sum_{j=\alpha_0-k}^{m-k}+\sum_{k=k_0}^{m}\sum_{j=0}^{m-k}\right)\frac{\tau^j}{j!}t^{\alpha_0-j-k(\alpha_0/k_0)}$$

$$\{D_\tau^j[P_{m-k}(ix_0+\tau N)]\,\}_{\tau=\tau_0}=0\Longrightarrow|\mathcal{R}\tau|\leqq c/t^{(\alpha_0-k_0)/k_0}.$$

In this equation, each power of t is $\leqq 0$; it is $=0$ only for (k,j) $=(0,\alpha_0)$ and $(k_0,0)$.

Letting $t\rightarrow\infty$, we get, by Hurwitz' theorem (see, for instance, [5] p. 119),

$$\frac{\tau^{\alpha_0}}{\alpha_0!}\{D_\tau^{\alpha_0}[P_m(ix_0+\tau N)]\,\}_{\tau=\tau_0}+P_{m-k_0}(ix_0+\tau_0 N)=0$$

$$\Longrightarrow\mathcal{R}\tau=0.$$

If α_0 is >2, this trivially implies $P_{m-k_0}\ (ix_0+\tau_0 N)=0$. If $\alpha_0=2$ (thus $k_0=1$), it reads as follows:

$$\frac{\tau^2}{2}\{D_\tau{}^2[P_m(ix_0+\tau N)]\,\}_{\tau=\tau_0}+P_{m-1}(ix_0+\tau_0 N)=0\Longrightarrow\mathcal{R}\tau=0,$$

so that

$$\frac{P_{m-1}(ix_0+\tau_0 N)}{\{D_\tau{}^2[P_m(ix_0+\tau N)]\,\}_{\tau=\tau_0}}\geqq 0.$$

This inequality is valid whenever τ_0 is a double root of $P_m(ix_0+\tau N)$; so it remains true if we replace x_0 by $-x_0$ and τ_0 by $-\tau_0$. But then, the quotient is multiplied by -1 and $P_{m-1}(ix_0+\tau_0 N)=0$.

To show that each derivative of order $\leqq\alpha_0-k_0-1$ of $P_{m-k_0}(ix_0$.

$+ \tau N)$ is null at $\tau = \tau_0$, it remains to use the well-known fact that each derivative of P is hyperbolic with respect to N.

To be complete, let us recall the proof of this last fact: for x fixed in R^n, if τ_k $(k = 1, \cdots, p)$ denote the different roots of $P(ix + \tau N)$ and α_k their multiplicity, we have

$$\mathcal{R}\left[\frac{D_\tau P(ix + N)}{P(ix + N)}\right] = \mathcal{R} \sum_{k=1}^{p} \frac{\alpha_k}{\tau - \tau_k} = \sum_{k=1}^{p} \alpha_k \frac{\mathcal{R}(\tau - \tau_k)}{|\tau - \tau_k|^2}$$

and this expression is > 0 (< 0) for $\mathcal{R}\tau > c$ $(< -c)$ if $|\mathcal{R}\tau_k| \leqq c$.

(b) *Sufficiency*. Suppose now, for $n = 2$, that P satisfies the quoted conditions. We shall prove the existence of a constant K such that

$$|\mathcal{R}\tau| \geqq 1 \Rightarrow \left| \frac{P_{m-k}(ix + \tau N)}{P_m(ix + \tau N)} \right| \leqq K \, (k = 1, \cdots, m).$$

This implies, by the properties of homogeneity of P_{m-k} and P_m,

$$\left| \frac{P_{m-k}(ix + \tau N)}{P_m(ix + \tau N)} \right| \leqq \frac{K}{|\mathcal{R}\tau|^k} \text{ for } \mathcal{R}\tau \neq 0,$$

so that

$$|P(ix + \tau N)| \geqq \tfrac{1}{2} \, |P_m(ix + \tau N)| \neq 0$$

for $|\mathcal{R}\tau|$ large enough.

Fix arbitrarily a point $x_0 \in R^2$ linearly independent of N. If

$$x = \lambda x_0 + \mu N \ (\lambda, \mu \in R),$$

we have, for $\lambda \neq 0$,

$$\frac{P_{m-k}(ix + \tau N)}{P_m(ix + \tau N)} = \frac{1}{\lambda^k} \frac{P_{m-k}(ix_0 + (\tau'/\lambda)N)}{P_m(ix_0 + (\tau'/\lambda)N)},$$

with $\mathcal{R}\tau' = \mathcal{R}\tau$.

If

$$P_m(ix_0 + \tau N) \equiv P_m(N) \prod_{j=1}^{p} (\tau - \tau_j)^{\alpha_j},$$

then,

$$P_{m-k}(ix_0 + \tau N) = \prod_{j=1}^{p} (\tau - \tau_j)^{(\alpha_j - k)_+} \cdot Q_k(\tau),$$

where Q_k is a polynomial of degree $\leqq m - k - \sum_{j=1}^{p} (\alpha_j - k)_+$. Therefore,

$$\frac{P_{m-k}(ix+\tau N)}{P_m(ix+\tau N)} = \frac{1}{\lambda^k} \frac{Q_k\left(\frac{\tau'}{\lambda}\right)}{P_m(N)\prod_{j=1}^{p}\left(\frac{\tau'}{\lambda}-\tau_j\right)^{\inf(\alpha_j,k)}}.$$

Let us first examine the case $|\lambda| \geqq 1$. There exist constants $C_{j,\beta}^{(k)}$ such that

$$\frac{Q_k(z)}{P_m(N)\prod_{j=1}^{p}(z-\tau_j)^{\inf(\alpha_j,k)}}$$

$$= \sum_{j,\beta} \frac{C_{j,\beta}^{(k)}}{(z-\tau_j)^{\beta}}, \quad 1 \leqq \beta \leqq \inf(\alpha_j, k).$$

Therefore, there exists a constant K_1 such that

$$\left|\frac{P_{m-k}(ix+\tau N)}{P_m(ix+\tau N)}\right| = \left|\frac{1}{\lambda^k}\sum_{j,\beta}\frac{C_{j\beta}^{(k)}}{\left(\frac{\tau'}{\lambda}-\tau_j\right)^{\beta}}\right| \leqq K_1,$$

for $|\mathscr{R}\tau'| = |\mathscr{R}\tau| \geqq 1$, because $\mathscr{R}\tau_j = 0$ and $\beta \leqq k$.

The case $0 < |\lambda| \leqq 1$ may be treated in the same way, starting from the decomposition

$$\frac{z^k Q_k(z)}{P_m(N)\prod_{j=1}^{p}(z-\tau_j)^{\inf(\alpha_j,k)}} = C^* + \sum_{j,\beta}\frac{C_{j\beta}^{(k)*}}{(z-\tau_j)^{\beta}}.$$

The inequality is then proved for $\lambda \neq 0$. For $\lambda = 0$, it is obvious and the proof is complete.

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