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EXACT CLASSICAL SOLUTIONS TO THE TWO-DIMENSIONAL "SIGMA" MODEL: AN UNUSUAL APPLICATION OF INVERSE SCATTERING TECHNIQUES

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ABSTRACT. The development of semi-classical approximation methods in quantum field theories has stimulated substantial interest in localized solutions to the non-linear, partial differential equations corresponding to the "classical" limits of the quantized equations of motion. In this article we construct some exact "solitary wave" solutions to the classical field equations, in one space-one time dimension, of the "sigma" model studied in elementary particle and nuclear physics. The method of constructing these solutions provides a somewhat unusual application of inverse scattering techniques, in that the eigenfunctions of the associated linear problem appear as driving terms in the non-linear evolution equation. Our approach and results lead directly to several interesting mathematical questions, the answers to which would be very useful for further studies in this area.

1. **Introduction.** In the past two years the development of "semi-classical" techniques [1]–[8] has virtually revolutionized the study of bound state problems in quantum field theory, particularly as applied to elementary particle physics. In their execution, these semi-classical techniques are often quite involved; some of the complexities are discussed in other contributions to these proceedings [7], [8], and the full details are available in the literature [1]–[6]. In their underlying conception, however, these techniques are beautifully simple, and fortunately, for our present purposes, we can focus on the central idea of the semi-classical method: that exact solutions to the equations of the *classical* field theory—which equations, as we shall see, are "simply" coupled, non-linear partial differential equations—can be used as input to a systematic approximation procedure for calculating bound states in the (substantially more complicated) quantum field theory.

[Readers familiar with elementary quantum mechanics will recognize this as the central idea behind the WKB approximation, and, indeed, the semi-classical techniques developed in [3]–[6] are the (nontrivial) generalization to quantum field theory of the WKB approximation. To see that the quantum field theory is in fact substantially more com-

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plicated than the corresponding classical theory, consider the specific example of the sigma model. Here in the quantum case, one must solve field equations formally identical to the classical equation (5), but subject to the restrictions ("equal time commutation relations")

$$[\sigma(x, t), \sigma_t(y, t)]_- = i\hbar\, \delta(x - y)$$

$$[\pi(x, t), \pi_t(y, t)]_- = i\hbar\, \delta(x - y)$$

$$\{\psi_\alpha(x, t), \psi_\beta^{+}(y, t)\}_+ = \hbar\, \delta_{\alpha\beta}\delta(x - y)$$

where $[A, B]_- \equiv AB - BA$ and $\{A, B\}_+ \equiv AB + BA$, and $\hbar$ is Planck's constant. Thus, loosely speaking, the solutions to the quantum field equations are (infinite-dimensional) matrices! Notice that the "classical" limit is formally obtained at $\hbar \rightarrow 0$, in which case the commutation relations become trivial, and the fields can be represented by simple functions of space-time.]

More explicitly, the semi-classical method establishes that "solitary wave" solutions of the classical field equations "correspond" to bound state "particles" of the quantum system. [The "particle-like" properties of solitary waves—localization, retention of identity (exact for true "solitons")—and the possible implications for elementary particle physics were first emphasized in [9].] Whereas other articles in these proceedings [7], [8] will concentrate on the precise nature of this correspondence and on the approximation scheme to which it leads, here we have two different goals:

(1) to give an example of a specific type of "solitary wave" solution of interest in quantum field theory and elementary particle physics; and

(2) to illustrate a somewhat unusual application of inverse scattering techniques and, in particular, to discuss several interesting mathematical questions raised by the nature of the equations which arise in applications in this area.

For these purposes, we have divided the remainder of the article into several sections. In Section 2 we introduce the "sigma" model and discuss briefly the physical contexts in which the model arises. Section 3 derives and examines the classical field equations of this model. In Section 4 we illustrate how, in a particular case, these equations can be solved by inverse scattering techniques. Finally, in Section 5 we indicate and discuss three mathematical questions which are suggested by the particular form of the equations and by our method of solving them. For completeness, we include two appendices to clarify certain technical aspects of our analysis. In Appendix A we derive the "trace indentities" for the "Dirac" equation—a linear eigenvalue problem similar to the Zakharov-Shabat equation—and in Appendix B we discuss the explicit reconstruction of the solutions to the classical field equations from the scattering data of this "Dirac" equation.

2. **The "Sigma" Model.** The "sigma" model in one space-one time dimension is described by the Lagrangian density (see [10]–[14])

$$
\begin{aligned}
\mathscr{L}(x, t) = {} & \frac{1}{2}[(\partial_\mu \sigma \partial^\mu \sigma) + (\partial_\mu \pi \partial^\mu \pi)] \\
& - \frac{\lambda}{4}(\sigma^2 + \pi^2 - f^2)^2 \\
& + \bar{\psi}(i\gamma^\mu \partial_\mu - g(\sigma + i\pi\gamma_5))\psi.
\end{aligned}
\tag{1}
$$

To define our notation we should mention several points:

(1) the constants λ, g, and f are parameters—of dimensions $(\text{mass})^2$, $(\text{mass})^1$, and $(\text{mass})^0$, respectively—in the model which, crudely speaking, indicate the strengths of the various interactions and determine the masses of the different particles represented by the fields; for example, the mass of the "fermion" is given by $m = gf$.

(2) $\sigma(x, t)$ and $\pi(x, t)$ can for our present purposes, be thought of as simple functions of space and time;

(3) ψ is a two-component column vector, $\psi = \binom{\psi_1}{\psi_2}$, when ψ_i is a function of space-time;

(4) the matrices γ^μ—called "Dirac γ-matrices" by elementary particle physicists—are related, in our one space-one time dimensional model, to the Pauli spin matrices. We shall use the conventions $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_1$, and $\gamma_5 \equiv \gamma_0\gamma_1 = i\sigma_3\sigma_1 = -\sigma_2$;

(5) the two-component row vector $\bar{\psi}$ is defined by $\bar{\psi} = \psi^\dagger \gamma_0$, where $(^\dagger)$ denotes Hermitian conjugation; and finally

(6) "Lorentz covariant" notation is employed: e.g., $\partial_\mu \sigma \partial^\mu \sigma \equiv (\partial_t \sigma)^2 - (\partial_x \sigma)^2$.

At this point it is perhaps appropriate to indicate briefly the model's physical relevance (see [10]–[12]). Historically, the sigma model was introduced as a description of the low energy interactions of nucleons (N)—i.e., protons and neutrons, here described by $\psi(x)$—with "mesons"—here, σ and π—which both provide the forces that bind nuclei and exist as elementary particles in their own right.

Among the more recent applications of the sigma model two are most important [13], [14]. In nuclear physics, the full sigma model has been used to suggest the possible existence of "abnormal states"—that is, phases whose macroscopic properties are very different from those of ordinary nuclei—of nuclear matter at very high nucleon density [15]–[17] and, further, as a possible phenomenological field theory of normal nuclei [18]. In elementary particle physics, the sigma model (without pions) has been used to exemplify [19] a possible mechanism for "confining" "quarks", which are the (presumed) underlying constituents of elementary particles but which apparently do not themselves exist as physical, free particles. There is thus considerable interest in understanding, for example, the bound state spectrum of the sigma model [13], [14], and the semi-classical approach offers an in-principle method for studying this problem. We should emphasize, however, that all these physically interesting contexts require the full three-space and one-time dimensional model. Hence the one-space and one-time dimensional version we discuss here must be regarded as a "toy" model, which can suggest interesting phenomena possibly present in the full (3 + 1)-dimensional model but which is itself not directly applicable to a specific physical situation. [In solid state physics, however, very similar (1 + 1)-dimensional models *may* be physically relevant. See, for example, [20].]

3. **The Classical Field Equations.** With this brief physical motivation aside, we turn to our central mathematical problem: finding exact solutions to the classical field equations of the sigma model. These field equations are just the Euler-Lagrange equations which follow from the standard variation principle for the action

$$S(T) = \int_0^T dt \int_{-\infty}^{\infty} dx \mathcal{L}(\sigma, \pi, \psi). \tag{4}$$

Requiring $\delta S = 0$ for functional variations of the fields $\sigma(x, t)$ $\pi(x, t)$ and $\psi(x, t)$ (or $\bar{\psi}(x, t)$) yields the equations

$$(\partial_t^2 - \partial_x^2)\sigma + \lambda\sigma(\sigma^2 + \pi^2 - f^2) = -g\bar{\psi}\psi \tag{5a}$$

$$(\partial_t^2 - \partial_x^2)\pi + \lambda\pi(\sigma^2 + \pi^2 - f^2) = -ig\bar{\psi}\gamma_5\psi \tag{5b}$$

$$[i\gamma_0\partial_t + i\gamma_1\partial_x - g(\sigma + i\pi\gamma_5)]\psi = 0. \tag{5c}$$

Assuming σ, π, and ψ are ordinary functions of (x, t) as described above, these equations are "simply" coupled, non-linear, partial differential equations. From the inverse scattering point of view, however, these equations have an unusual twist. To illustrate this most simply, we consider the case when only the sigma field is present. Then equations (5) reduce to

$$(\partial_t^2 - \partial_x^2)\sigma + \lambda\sigma(\sigma^2 - f^2) = -g\bar{\psi}\psi \tag{6a}$$

and

$$[i\gamma_0\partial_t + i\gamma_1\partial_x - g\sigma]\psi = 0. \tag{6b}$$

Recalling the definitions of the γ-matrices, we observe that equation (6b)—called the Dirac equation by elementary particle physicists—looks very much like the Zakharov-Shabat equation one would set up as an associated linear eigenvalue problem to solve (6a), the non-linear evolution equation for $\sigma(x, t)$. But notice that the wave function, ψ, of this associated "Zakharov-Shabat" equation *appears itself as a driving term in the non-linear equation.* This, then, is one central mathematical question which we wish to raise by our discussion: can one develop a variant of the inverse scattering technique which can solve non-linear PDE's coupled in the sense of equations (5) and (6)? At present, we can supply only the very incomplete answer that for *time-independent* σ and π—so that the PDE's become ODE's—and subject in our case to a certain important restriction, one *can* solve equations (5) and (6) using in fact only a small fraction of the full inverse scattering approach. This partial result, of course, makes the general question all the more tantalizing. By "time-independent σ and π" we mean fields which in their rest frame have no time dependence. The Lorentz invariance of the model, of course, allows us to construct moving solitary waves, which depend on time via the (characteristic) variable $\xi = (x - vt)/(1 - v^2)^{1/2}$.

4. **The Application of Inverse Scattering Techniques.** To see how to solve these equations in the time-independent case we consider first the simpler, σ only example, for which the equations become, upon assuming

(7) $$\psi(x, t) \equiv \psi_0(x)e^{-i\omega_0 t},$$

(8a) $$-\partial_x{}^2\sigma + \lambda\sigma(\sigma^2 - f^2) = -g\bar{\psi}_0\psi_0$$

and

(8b) $$[\omega_0\gamma_0 + i\gamma_1 \frac{d}{dx} - g\sigma]\psi_0 = 0.$$

To motivate our next steps, let us indicate briefly where they will lead. First, we recall that equations (6) and (8) followed from a variational principle on the action; formally, this variational principle is just the functional extremal condition,

(9) $$\frac{\delta S}{\delta\sigma(x, t)} = 0 = \frac{\delta S}{\delta\bar{\psi}(x, t)}\,.$$

Our approach will be to show that, subject to an important restriction, it is possible to "change variables" in the action from the fields σ and ψ to the scattering data—as usual, bound state energies and normalizations and reflection coefficient—associated with the Dirac equation (8b). This change of variables is identical in spirit to the canonical transformation to scattering data carried out, for example, by Zakharov and Faddeev [21] for the case of the *KdV* equation. When S is expressed in terms of the scattering data, the variational conditions become, as we shall see, *algebraic* rather than differential equations. It is thus straightforward to solve for the scattering data and thence, using Frolov's solution [22] to the inverse problem for the Dirac equation, to reconstruct σ and ψ.

To implement this approach, we begin by noting that for time-independent σ and for wave functions ψ as in (7)—called "stationary state" wave functions—$\mathscr{L}$ is independent of time and, indeed, the action is given simply by

$$S(T) = -E \cdot T \tag{10}$$

where E is the energy of the full field configuration. The energy can be written as an integral over an energy density, $\mathscr{E}(x)$, which we can separate into a piece coming from the σ field alone,

$$\mathscr{E}_\sigma(x) \equiv \left\{ \frac{1}{2} (\partial_x \sigma)^2 + \frac{\lambda}{4} (\sigma^2 - f^2)^2 \right\} \tag{11}$$

and a piece coming from the fermion interacting with the σ meson. This latter piece is, from (8b), just

$$\mathscr{E}_\psi(x) = \omega_0 \psi_0{}^\dagger \psi_0. \tag{12}$$

Thus when one fermion is present, so that the normalization condition $\int \psi_0{}^\dagger \psi_0 \, dx = 1$ applies, we can write the action per unit time as

$$\begin{aligned} S/T = -\Bigg\{ \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_x \sigma)^2 \right.\\ \left. + \frac{\lambda}{4} (\sigma^2 - f^2)^2 \right] + \omega_0[\sigma] \Bigg\}. \end{aligned} \tag{13}$$

[To derive this result directly from the expression $S = \int dx\, dt \mathscr{L}$ is somewhat subtle because the linear nature of $\mathscr{L}$ as a function of ψ and $\bar{\psi}$—recall that ψ and $\bar{\psi}$ are treated as independent variables—means that when the Dirac equation is satisfied, say by a wave function $\tilde{\psi}$, one finds $\mathscr{L}(\tilde{\psi}) = 0$! To resolve this apparent paradox we must recall that the actual variation of S is *subject to the constraint* $\int dx \, \psi^\dagger \psi = 1$. Thus we need to introduce

$$\mathscr{L}_{\text{eff}} \equiv \mathscr{L} - \mu \psi^\dagger \psi$$

where μ is a Lagrange multiplier. It is then intuitively clear that one will obtain $\mu = \omega_0$ and thence (13).]

Here we have written $\omega_0[\sigma]$ to emphasize that ω_0 is functionally dependent on σ. It is clear that our arguments so far, although highly heuristic, are correct, since we can easily verify that the form of S in (13) still implies the field equations (8). Explicitly, using the result

$$\frac{\delta\omega_0[\sigma]}{\delta\sigma} = -\,g\bar{\psi}_0\psi_0 \tag{14}$$

familiar from inverse scattering theory, it is trivial to verify that $\delta S/\delta\sigma = 0$, with S given by (13), implies (8a). Further, to determine $\bar{\psi}_0\psi_0$, and hence to solve (8a), we must simultaneously solve (8b).

In (13) we have already partially succeeded in effecting the "change of variables" to scattering data. It remains to express the first term in (13)—the "meson" contribution—in terms of scattering data. Since the field σ plays the role of the "potential" in the Dirac equation, it is clear that the trace identities, which relate integrals over expressions involving the potential to the scattering data, may be relevant. To derive the trace identities for the Dirac equation we can apply the methods used by Zakharov and Faddeev [21] in the case of the Schrödinger equation. In Appendix A we present for completeness the details of this derivation. Here we mention only two mathematical points necessary to understand the results. First, the relativistic kinematics lead to two possible energy values for a plane wave of momentum $k : \omega(k)^{(\pm)} = \pm\,(k^2 + m^2)^{1/2}$. This implies, among other things, that the reflection coefficient—call it $s_{12}(k)$—has two Riemann sheets and thus should be written as $s_{12}^{(\pm)}(k)$. Second, as illustrated in detail in Appendix B, in the case when only a σ field is present—as in (8)—the positive and negative energies must come in "charge conjugate" pairs, $\omega_i^{(\pm)} = |\omega_i^{(-)}| \equiv \omega_i$. When both σ and π are present, this symmetry is *not* required. [This symmetry requirement is most easily seen by studying the explicit reconstructions in Appendix B. If one does not have $\omega_0^{(+)} = |\omega_0^{(-)}| \equiv \omega_0$, then one obtains both σ and π fields in the Dirac potential. When we refer to a "single discrete state" in the sigma only case, we mean a "charge conjugate" pair which satisfies this symmetry requirement.]

From Appendix A we see that the first two trace identities become, when only a sigma field is present, and keeping only the single discrete state ω_0,

$$\int_{-\infty}^{\infty} dx(\sigma^2 - f^2) = -\frac{4}{g^2}\,\kappa_0 - \frac{1}{2\pi g^2}\,F_0[s_{12}] \tag{15a}$$

and

$$\int_{-\infty}^{\infty} dx \left\{ \frac{1}{2}(\partial_x\sigma)^2 + \frac{g^2}{2}(\sigma^2 - f^2)^2 \right\} = \frac{8}{3g^2}\kappa_0^3 - \frac{1}{\pi g^2}F_1[s_{12}]. \tag{15b}$$

Here κ_0, the bound state "momentum", is related to ω_0 by $\kappa_0 = (m^2 - \omega_0^2)^{1/2}$, where, as noted previously, $m = gf$. In (15) we have introduced the simplifying notation

$$F_n[s_{12}] \equiv \int_{-\infty}^{\infty} q^{2n}\, dq\{\ln - |s_{12}^{(+)}(q)|^2] + \ln[1 - |s_{12}^{(-)}(q)|^2]\}. \tag{16}$$

Comparing the expression involving the σ field in (15b) with that in (13), we see that if $\lambda = 2g^2$, we can write the full action per unit time completely in terms of the scattering data:

$$S/T = -\frac{8}{3g^2}\kappa_0^3 - F_1[s_{12}] - \omega_0. \tag{17}$$

[The restriction on the coupling constants, although representing a very important limitation on the method, does not lessen greatly the physical value of our results, since the structure of the theory is expected to be similar for other values of λ/g^2. Further, this restriction is hardly surprising, since $(g\sigma)$ is, loosely speaking, the "potential", and the nonlinearity of the trace identities will thus require some relation between g and λ for the change of variables to be allowed.] If we now vary S independently with respect to $s_{12}^{(\pm)}$ and ω_0—or, equivalently, κ_0—we find that

$$\frac{\delta S}{\delta s_{12}^{(\pm)}(q)} = 0 \tag{18}$$

implies that

$$s_{12}^{(\pm)}(q) = 0 \tag{19}$$

and that

$$\frac{\delta S}{\delta \kappa_0} = 0 \tag{20}$$

implies that

$$\kappa_0\omega_0 \equiv \kappa_0(m^2 - \kappa_0^2)^{1/2} = g^2/8. \tag{21}$$

[Note that the independence of $|s_{12}^{(\pm)}|$ and κ_0 is crucial for this manipulation to be allowed. In higher spatial dimensions, the problem of establishing a useful complete and independent "set" of scattering data is, as far as I know, unsolved.] Thus the σ "potential" in the Dirac equation is reflectionless and has one bound state; using Frolov's generalization [15] of the Gelfand-Levitan-Marchenko equation, it is straightforward to reconstruct σ and ψ. This is done explicitly in Appendix B, and the result is

$$\sigma(x) = f - (\kappa_0^2/g\omega_0)\,\mathrm{sech}[\kappa_0(x + x_0)]\,\mathrm{sech}[\kappa_0(x - x_0)] \tag{22a}$$

and

$$\psi(x, t) = \left(\frac{\kappa_0}{8}\right)^{1/2} e^{-i\omega_0 t} \begin{pmatrix} \mathrm{sech}[\kappa_0(x + x_0)] + \mathrm{sech}[\kappa_0(x - x_0)] \\ -\mathrm{sech}[\kappa_0(x + x_0)] + \mathrm{sech}[\kappa_0(x - x_0)] \end{pmatrix}. \tag{22b}$$

Here $\kappa_0 x_0 = \tanh^{-1}((m - \omega_0)/\kappa_0)$. The interested reader can verify that, when (21) is invoked, these forms of $\sigma(x)$ and $\psi(x, t)$ do satisfy the field equations in (8).

From the discussion it is clear that precisely the same approach can be used to solve the equations when both σ and π are present. The only difference is that, since the symmetry $\omega_0^{(+)} = |\omega_0^{(-)}|$ is *not* required, we keep only the single $\omega_0^{(+)} \equiv \omega_0$ contribution in the sum over bound states. Thus the action per unit time becomes

$$S/T = -\frac{4}{3g^2}\kappa_0^3 - \omega_0 - F_1[s_{12}]. \tag{23}$$

Minimization with respect to $s_{12}^{(\pm)}$ and κ_0 leads to the results

$$s_{12}^{(\pm)}(q) = 0 \tag{24a}$$

and

$$\kappa_0\omega_0 = g^2/4. \tag{24b}$$

Again the reconstruction is simple; referring to Appendix B we find

$$\sigma(x) = f - \frac{\kappa_0^2}{gm}[1 - \tanh \kappa_0 x] \tag{25a}$$

$$\pi(x) = -\frac{\omega_0\kappa_0}{gm}[1 - \tanh \kappa_0 x] \tag{25b}$$

and

$$\psi(x, t) = \left(\frac{\kappa_0(m - \omega_0)}{4m}\right)^{1/2} e^{-i\omega_0 t}\,\mathrm{sech}\,\kappa_0 x \begin{pmatrix} \dfrac{\kappa_0}{m - \omega_0} \\ 1 \end{pmatrix}. \tag{25c}$$

Again one can verify that the expressions in (25) do satisfy the field equations in (5), provided (24b) is invoked.

5. **Discussion.** Using the semi-classical techniques illustrated, for example, in the article by Hasslacher and Neveu [7] in these proceedings, one can study the implications of these classical solutions for the bound state spectrum of the sigma model quantum field theory [13], [14]. However, since these implications are not immediately relevant to our present considerations, we shall instead close by summarizing the three interesting mathematical problems suggested by our analysis.

(1) As emphasized in the introductory remarks, the one plus one dimensional sigma model must be regarded as only a possibly suggestive "toy" model in nuclear and particle physics. The real interest lies in the structure of the full three space-one time dimensional model. The first problem is thus the obvious and long-standing one: "Can one develop systematic, constructive methods—for example, a generalization of the inverse scattering method—for solving (possibly coupled) non-linear PDE's in more than one spatial dimension?" Notice that for a naive generalization of our method we would need not only the trace identities in higher dimensions, on which some progress has been made [23], but also knowledge of the independence and completeness of the "scattering data", which is in general apparently quite a difficult problem.

(2) A less general question, more immediately associated with our specific problem, is "Can one develop a general approach to non-linear PDE's in which the eigenfunctions of an "associated" linear equation appear as dependent variables in the non-linear equation itself?" Our analysis provides one simple example of a possible technique; perhaps a more thorough analysis could categorize the types of "coupled" equations which can be solved by, say, inverse scattering techniques.

(3) Finally, our present method can not be used to find σ fields with non-trivial time dependence because there is no immediately useful analog of the trace identities for time-dependent potentials. Thus a third problem is, "Can one find the analog of "trace identities" for time-dependent potentials?" Notice that the scattering data are much more complicated in this case, since the potential's time dependence implies that even though the potential is localized in space, the momentum of the scattered wave is not necessarily conserved. Hence, in physicist's terms, the "scattering matrix" will not be diagonal in momentum, i.e., $s_{12} = s_{12}(k, k')$. This problem does not seem completely intractable, however, since for purposes of the semi-classical analysis one need study only potentials periodic in time; it is then possible that

some of the exciting recent developments in studies of spatially periodic potentials can be adapted for this application [24], [25].

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Appendix A: Trace Identities for the Dirac Equation. To motivate our derivation of the trace identities for the Dirac equation let us recall the two essential ingredients used by Zakharov and Faddeev [21] in the case of the Schrödinger equation. First, one needs a dispersion relation relating the transmission coefficient—which we shall call $s_{11}(k)$—as a function of momentum k to an expression involving the modulus of the reflection coefficient—$s_{12}(k)$—and the bound state poles at $k = i\kappa_\ell$. Second, one introduces an auxiliary function—call it $\chi(x)$—constructed from a fundamental solution to the Schrödinger equation in a manner such that

$$\int_{-\infty}^{\infty} \chi(x)\, dx = \ln s_{11}(k). \tag{A.1}$$

From the Schrödinger equation one deduces the differential equation satisfied by χ. Then comparing the asymptotic expansion (in $1/k$) of the solution to this equation with that of the dispersion relation, one obtains the trace identities.

In the case of the Dirac equation, both steps involve some additional technical complications. First, the analytic structure of $s_{11}(k)$ as a function of k is two-sheeted [22], with a branch cut running from $k = +\,\mathrm{im}$ to $i\infty$ connecting the sheets. On the first sheet, the energy satisfies $E = +\,(k^2 + m^2)^{1/2}$, whereas on the second sheet, $E = -\,(k^2 + m^2)^{1/2}$. Since the branch point is second order, however, linear combinations of the function on the first and second sheets can be chosen so as to remove the integral along this cut from the dispersion relation. In particular, one can show that the following representation holds:

$$\begin{aligned} &\ln s_{11}^{(+)}(k) + \ln s_{11}^{(-)}(k) \\ &\quad = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dq\, \frac{\ln[1 - |s_{12}^{(+)}(k)|^2] + \ln[1 - |s_{12}^{(-)}(k)|^2]}{q - k} \\ &\qquad + \sum_{\ell=1}^{N_+} \ln\left(\frac{k + i\kappa_\ell^{(+)}}{k - i\kappa_\ell^{(+)}}\right) \\ &\qquad + \sum_{\ell=1}^{N_-} \ln\left(\frac{k + i\kappa_\ell^{(-)}}{k - i\kappa_\ell^{(-)}}\right). \end{aligned} \tag{A.2}$$

Here the superscripts $(\pm)$ refer to quantities on the first $(E = +(k^2 + m^2)^{1/2} > 0)$ and second $(E = -(k^2 + m^2)^{1/2} < 0)$ Riemann sheets and the sums are over the $N^+(N^-)$ bound state poles located at $0 \leqq \kappa_\ell \leqq m$ on the first (second) Riemann sheet. To clarify the form of (A.2) we observe that for a function analytic in the upper half plane and suitably behaved at $|z| \to \infty$, there exists an integral representation for the function in terms of its real part on the real axis:

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{du \, \mathrm{Re} f(u)}{u - z}, \quad u \in \mathrm{R}, \ \ \mathrm{Im}\, z > 0.$$

Since the $\mathrm{Re} \ln s_{11}(k) = \ln|s_{11}(k)|$, and since unitarity implies [22], for real k, $|s_{11}(k)|^2 = 1 - |s_{12}(k)|^2$, if we momentarily ignore the singularities of $s_{11}(k)$ in the UHP, we see that using (A.3) with $f(z) = \ln s_{11}(k)$ yields the first term in (A.2). To include the effects of the singularities in $s_{11}(k)$ for $\mathrm{Im}\, k > 0$, we note that because the function is two sheeted, the integral along the branch cut $im \leqq k < i\infty$ enters with opposite signs for $\ln s_{11}^{(+)}$ and $\ln s_{11}^{(-)}$ and thus cancels in the sum. Further, the explicit form of the contribution from the bound state poles follows by observing that for real k, unitarity requires their contributions to the real part of (A.2) to vanish, and hence the expressions in the logarithms must be unimodular for real k, that is, of the form $(k + i\kappa)/(k - i\kappa)$. For large k the asymptotic expansion of (A.2) in powers of $(1/k)$ is

$$\ln s_{11}^{(+)}(k) + \ln s_{11}^{(-)}(k) = \sum_{n=1}^{\infty} \frac{c_n}{k^n} \tag{A.4}$$

where $c_{2n} = 0$

$$\begin{aligned} c_{2n+1} = &- \frac{1}{2\pi i} \int_{-\infty}^{\infty} q^{2n} dq \\ &\left\{ \ln[1 - |s_{12}^{(+)}(q)|^2] + \ln[1 - |s_{12}^{(-)}(q)^2|] \right\} \\ &+ \frac{2}{2n+1} \left(\sum_{\ell=1}^{N_+} (i\kappa_\ell^{(+)})^{2n+1} \right. \\ &\left. + \sum_{\ell=1}^{N_-} (i\kappa_\ell^{(-)})^{2n+1} \right). \end{aligned} \tag{A.5}$$

To obtain the second expression for $\ln s_{11}(k)$, we resort directly to the Dirac equation. We need study only the equation in which both σ and π are present, since the results for σ only can also be obtained from this case. One minor technical problem is *a posteriori* evident from the form of the σ and π fields in equation (25): namely, although the "potential" in the Dirac equation in the form of (5)—which potential is determined by $\sigma(x) - f$ and $\pi(x)$—vanishes as $x \to +\infty$, it does not vanish as $x \to -\infty$. To avoid problems arising from the apparent non-localization of the potential, we can define new dependent variables (ρ, θ) by the transformation

$$\sigma + i\pi = \rho e^{i\theta/f} \tag{A.6}$$

and by introducing

$$\psi' = e^{i\gamma_5\theta/2f}\psi. \tag{A.7}$$

For time-independent σ and π the Dirac equation for $\tilde{\psi}'$ defined by $\psi'(x, t) = e^{-i\omega t}\tilde{\psi}'(x)$ then becomes

$$\left[\omega\gamma_0 + i\gamma_1 \frac{d}{dx} - m - g\hat{\rho}(x) - \gamma_5\gamma_1 s(x) \right] \tilde{\psi}'(x) = 0 \tag{A.8}$$

where $m = gf$, $\hat{\rho}(x) = \rho(x) - f$, and $s(x) \equiv (1/2f)\, d\theta/dx$. For localized, "solitary wave" solutions, $\hat{\rho}(x)$ and $s(x)$ must both approach zero as $x \to +\infty$. Hence, in these variables the "potential" is localized, and we expect no technical problems.

Using our standard representation for the γ matrices, $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_1$, $\gamma_5 = \gamma_0\gamma_1 = -\sigma_2$, we can write (A.8) in terms of the two components of $\tilde{\psi}'$ as

$$\frac{d\psi_2}{dx} + a_-(x)\psi_1(x) = 0 \tag{A.9a}$$

and

$$\frac{d\psi_1}{dx} + a_+(x)\psi_2(x) = 0 \tag{A.9b}$$

where $\tilde{\psi}' \equiv \binom{\psi_1}{\psi_2}$ and

$$a_\pm(x) \equiv m + g\hat{\rho}(x) \pm (\omega + s(x)) \tag{A.10}$$

Consider the specific soluton $f \equiv \binom{f_1}{f_2}$ to (A.9) which satisfies

$$\ln f_1(x, k) \underset{x\to+\infty}{\simeq} ikx + c(k),$$

where $c(k)$ is constant depending on the normalization of f. Frolov's results [22] establish that, for $\operatorname{Im} k > 0$

$$\ln f_1(x, k) \underset{x\to-\infty}{\simeq} ikx - \ln s_{11}(k) + c(k).$$

Thus if one introduces

$$\chi(x) \equiv \frac{d}{dk}(\ln f_1(x, k)) - ik, \tag{A.11}$$

it follows that

(A.12) $$\int_{-\infty}^{\infty} \chi(x)\, dx = \ln s_{11}(k).$$

Further, from the Dirac equation in the form (A.9) one can derive an equation for χ. Differentiating (A.9b) with respect to x and substituting from (A.9a) we obtain

(A.13) $$\frac{d^2 f_1}{dx^2} - \frac{1}{a_+}\left(\frac{da_+}{dx}\right)\frac{df_1}{dx} - a_+ a_- f_1 = 0.$$

Using (A.11) this can be written, after some algebra, in terms of χ as

(A.14) $$\frac{d\chi}{dx} + \chi^2 + \alpha(x)\chi + \beta(x) = 0$$

where

$$\alpha(x) = 2ik - [\omega + m + g\hat{\rho} + s]^{-1}\left(g\frac{d\hat{\rho}}{dx} + \frac{ds}{dx}\right)$$

and

$$\beta(x) = 2\omega s + s^2 - 2gf\hat{\rho} - g^2\hat{\rho}^2 - ik\left(g\frac{d\hat{\rho}}{dx} + \frac{ds}{dx}\right)[\omega + m + g\hat{\rho} + s]^{-1}.$$

Up to now we have not specified which Riemann sheet we are considering. Indeed it is clear that if we consider $\chi^{(\pm)}$, defined by (A.11) in terms of $f_1^{(\pm)}$, both satisfy (A.14) with the only difference being that for $\chi^{(+)}$, $\omega = +(k^2 + m^2)^{1/2}$, whereas for $\chi^{(-)}$, $\omega = -(k^2 + m^2)^{1/2}$. This change of sign leads to important cancellations in the sum, $\chi^{(+)} + \chi^{(-)}$, whose integral satisfies

(A.15) $$\int_{-\infty}^{\infty} (\chi^{(+)}(x) + \chi^{(-)}(x))\, dx = \ln s_{11}^{(+)}(k) + \ln s_{11}^{(-)}(k).$$

The remaining calculations are straightforward but tedious. Expanding

$$\chi^{(\pm)}(x) = \sum_{n=0}^{\infty} \frac{\chi_n^{(\pm)}(x)}{(2ik)^n}$$

and equating powers of $(1/k)$ in (A.14) leads eventually to the following expressions for the $\chi_n^{(+)} + \chi_n^{(-)}$:

(A.16) $$\chi_0^{(+)} + \chi_0^{(-)} = 0$$

(A.16b) $$\chi_1^{(+)} + \chi_1^{(-)} = 2g^2(\rho^2 - f^2)$$

(A.16c) $$\chi_2^{(+)} + \chi_2^{(-)} = 4g \frac{d}{dx}(\rho s)$$

and

(A.16d) $$\chi_3^{(+)} + \chi_3^{(-)} = -2g^4(\rho^2 - f^2)^2 - 2g^2\left(\frac{d\rho}{dx}\right)^2 - 8g^2\rho^2 s^2$$

where in the last equation we have dropped certain total derivative terms which vanish in (A.15). Note that $\chi_2^{(+)} + \chi_2^{(-)}$ is a total derivative and thus there is no $0(1/k^2)$ term in (A.15). This is of course necessary for consistency with (A.4), since $c_{2n} = 0$.

Comparing the expressions in (A.16) with those in (A.5), we arrive at the first two non-trivial trace identities:

(A.17a) $$\begin{aligned}\int_{-\infty}^{\infty} dx(\sigma^2 + \pi^2 - f^2) &= \int_{-\infty}^{\infty} dx(\rho^2 - f^2) \\ &= -\frac{1}{2\pi g^2}\int_{-\infty}^{\infty} dq\{\ln[1 - |s_{12}^{(+)}(q)|^2] \\ &\quad + \ln[1 - |s_{12}^{(-)}(q)|^2]\} \\ &\quad - \frac{2}{g^2}\left(\sum_{\ell=1}^{N_+} \kappa_\ell^{(+)} + \sum_{\ell=1}^{N_-} \kappa_\ell^{(-)}\right)\end{aligned}$$

(A.17b) $$\begin{aligned}\int_{-\infty}^{\infty} &\left[\frac{1}{2}\left(\left(\frac{d\sigma}{dx}\right)^2 + \left(\frac{d\pi}{dx}\right)^2\right) + \frac{g^2}{2}(\sigma^2 + \pi^2 - f^2)^2\right] dx \\ &= \int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\left(\frac{d\rho}{dx}\right)^2 + 2\rho^2 s^2\right) + \frac{g^2}{2}(\rho^2 - f^2)^2\right] dx \\ &= -\frac{1}{\pi g^2}\int_{-\infty}^{\infty} q^2 dq\{\ln[1 - |s_{12}^{(+)}(q)|^2] \\ &\quad + \ln[1 - |s_{12}^{(-)}(q)|^2]\} \\ &\quad + \frac{4}{3g^2}\left\{\sum_{\ell=1}^{N_+} (\kappa_\ell^{(+)})^3 + \sum_{\ell=1}^{N_-} (\kappa_\ell^{(-)})^3\right\}.\end{aligned}$$

These are the forms used in the text to express the action entirely in terms of the scattering data.

One final point is appropriate. In the text, equations (A.17) were used with the contribution of only one discrete state—consistent with the symmetry requirements mentioned above—included. It is relatively easy to see, at least in the time-independent case studied explicitly, that this is sufficient for determining the classical field configurations which lead to bound states in the quantum field theory. Consider keeping two distinct discrete states. Since their contributions to the action are additive and since, for time-independent, the energy of the full field configuration is just $E = -S/T$, the energy of any possible field configuration found by keeping two discrete states would simply be the exact sum of the two separate energies. Since a bound state must have *lower* energy than its separated constituents, keeping two discrete states in (A.17) can not lead to any new bound states in the quantum field theory.

Appendix B: Reconstruction of the Meson Fields and Fermion Bound State Wave Functions. To reconstruct the meson fields and Dirac wave functions we use the techniques developed by Frolov [22], who has established that a matrix generalization of the Gelfand-Levitan-Marchenko formalism is applicable to the inverse problem for the Dirac equation. The procedure can be simply summarized. From the scattering data one forms a matrix kernel, $F(x, y) = F_S(x, y) + F_{BS}(x, y)$; for reflectionless potentials—$s_{12}(k) = 0$—the scattering contribution, F_S, vanishes and only the bound state contribution, F_{BS}, remains. In our standard representation for the Dirac γ-matrices,

$$F_{BS}(x, y) = \sum_{\ell=1}^{N_+ N_-} c_\ell e^{-\kappa_\ell (x+y)} M_\ell \tag{B.1}$$

where the matrices M_ℓ are given by

$$M_\ell = \begin{pmatrix} \dfrac{m + (m^2 - \kappa_\ell^2)^{1/2}}{m - (m^2 - \kappa_\ell^2)^{1/2}} & \dfrac{m + (m^2 - \kappa_\ell^2)^{/2}}{\kappa_\ell} \\ \dfrac{m + (m^2 - \kappa_\ell^2)^{1/2}}{\kappa_\ell} & 1 \end{pmatrix} \tag{B.2a}$$

for $\omega_\ell > 0$ and by

$$M = \begin{pmatrix} 1 & \dfrac{m + (m^2 - \kappa_\ell^2)^{1/2}}{\kappa_\ell} \\ \dfrac{m + (m^2 - \kappa_\ell^2)^{1/2}}{\kappa_\ell} & \dfrac{m + (m^2 - \kappa_\ell^2)^{1/2}}{m - (m^2 - \kappa_\ell^2)^{1/2}} \end{pmatrix} \tag{B.2b}$$

for $\omega_\ell < 0$.

The c_ℓ are normalization constants, and the sum is over all positive and negative energy bound states. From the kernel F one constructs the transformation operator, $K(x, y)$ by solving the equation

$$K(x, y) + F(x, y) + \int_x^\infty K(x, t)F(t, y)\, dt = 0. \tag{B.3}$$

Then the "potential"

$$\begin{aligned} V(x) &\equiv \gamma_0 g\hat{\sigma} + i\gamma_0\gamma_5 \pi \\ &= \begin{pmatrix} g\hat{\sigma} & -g\pi \\ -g\pi & -g\hat{\sigma} \end{pmatrix} \end{aligned} \tag{B.4}$$

is given by the commutator

$$V(x) = [-i\gamma_5, K(x, x)]. \tag{B.5}$$

Further, the Dirac wave functions are given by

$$f(x, k) = e(x, k) + \int_x^\infty K(x, y)e(y, k)\, dy \tag{B.6}$$

where $e(x, k)$ is a plane wave solution to the free Dirac equation

$$e(x, k) = \begin{pmatrix} \dfrac{ik}{(k^2 + m^2)^{1/2} - m} \\ 1 \end{pmatrix} e^{+ikx}, \tag{B.7a}$$

for $\omega > 0$ and

$$e(x, k) = \begin{pmatrix} 1 \\ \dfrac{ik}{(k^2 + m^2)^{1/2} - m} \end{pmatrix} e^{+ikx}, \quad \omega < 0. \tag{B.7b}$$

To proceed we must distinguish the σ only and the $\sigma + \pi$ cases. For the first case, the requirement of charge symmetry for the σ only potential is satisfied by having one positive and one negative energy bound state with $\omega_0^{(+)} = |\omega_0^{(-)}| = \omega_0$ and with identical normalizations. Thus, with $\omega_0 \equiv (m^2 - \kappa_0^2)^{1/2}$

$$\begin{aligned} F(x, y) &= c_0 e^{-\kappa_0(x+y)} \begin{pmatrix} \dfrac{2m}{m - \omega_0} & \dfrac{2(m + \omega_0)}{\kappa_0} \\ \dfrac{2(m + \omega_0)}{\kappa_0} & \dfrac{2m}{m - \omega_0} \end{pmatrix} \\ &\equiv c_0 e^{-\kappa_0(x+y)} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \end{aligned} \tag{B.8}$$

The assumption $K(x, y) = \hat{K}(x)e^{-\kappa_0 y}$ separates the integral equation (B.3), leaving an algebraic matrix equation whose explicit solution is

$$\hat{K}(x) = \frac{-c_0 e^{-\kappa_0 x}}{[1 + 2\alpha z + z^2(\alpha^2 - \beta^2)]} \tag{B.9}$$
$$\begin{pmatrix} \alpha + z(\alpha^2 - \beta^2) & \beta \\ \beta & \alpha + z(\alpha^2 - \beta^2) \end{pmatrix} \tag{B.9}$$

where $z \equiv c_e{}^{-2\kappa_0 x}/2\kappa_0$.

Thus using (B.5) we find

$$V(x) \equiv \begin{pmatrix} g\hat{\sigma} & 0 \\ 0 & -g\hat{\sigma} \end{pmatrix} \tag{B.10}$$

where

$$g\hat{\sigma} = \frac{-4\beta\kappa_0 z}{[1 + (\alpha + \beta)z][1 + (\alpha - \beta)z]}. \tag{B.11}$$

After some algebra, (B.11) can be cast into the more familiar form quoted (for $\delta = 0$) in the text:

$$g\hat{\sigma} = \frac{-\kappa_0{}^2}{\cdot\ \omega_0} \operatorname{sech}(\kappa_0(x + x_0) + \delta) \operatorname{sech}(\kappa_0(x - x_0) + \delta) \tag{B.12}$$

where $\kappa_0 x_0 = \tanh^{-1}((m - \omega_0)/\kappa_0)$ and $\tanh\delta = (\kappa_0(m - \omega_0) - c_0\omega_0)/(\kappa_0(m - \omega) + c_0\omega_0$.

Reconstructing the positive energy bound state wave function from (B.6) we obtain

$$\psi_0(x) = \frac{e^{-\kappa_0 x}}{1 + 2\alpha z + z^2(\alpha^2 - \beta^2)} \begin{pmatrix} \beta z(\alpha/2 - 1) + \beta/2 \\ 1 + z(\alpha - \beta^2/2) \end{pmatrix}. \tag{B.13}$$

After a substantial amount of algebra, we can write this in the form

$$\psi_0(x) = \left(\frac{\kappa_0}{8}\right)^{1/2} \begin{pmatrix} \operatorname{sech}(\kappa_0(x + x_0) + \delta) + \operatorname{sech}(\kappa_0(x - x_0) + \delta) \\ -\operatorname{sech}(\kappa_0(x + x_0) + \delta) + \operatorname{sech}(\kappa_0(x - x_0) + \delta) \end{pmatrix} \tag{B.14}$$

where we have normalized ψ_0 in (B.14) to 1 and where the parameters are all as previously defined. Of course, the full time-dependent bound state wave function is

$$\psi_0(x, t) = e^{-i\omega_0 t}\psi_0(x).$$

When both σ and π are present, we need consider only a single discrete state of positive energy, $\omega_0^{(+)} \equiv \omega_0$. Thus

$$F(x, y) = c_0 e^{-\kappa_0(x,y)} \begin{pmatrix} \dfrac{m+\omega_0}{m-\omega_0} & \dfrac{m+\omega_0}{\kappa_0} \\ \dfrac{m+\omega_0}{\kappa_0} & 1 \end{pmatrix}. \tag{B.15}$$

Solving (B.3) as before yields

$$K(x, y) = \frac{-c_0 e^{-\kappa_0(x+y)}}{1 + z\left(\dfrac{\beta^2}{4} + 1\right)} \begin{pmatrix} \dfrac{\beta^2}{4} & \dfrac{\beta}{2} \\ \dfrac{\beta}{2} & 1 \end{pmatrix} \tag{B.16}$$

where $\beta = 2(m + \omega_0)/\kappa_0$ and $z = c_0 e^{-2\kappa_0 x}/2\kappa_0$.

Using (B.5) we find

$$g\hat{\sigma} = \frac{-2\beta\kappa_0 z}{1 + \left(\dfrac{\beta^2}{4} + 1\right) z} \tag{B.17a}$$

$$g\pi = \frac{-\left(\dfrac{\beta^2}{4} - 1\right) 2\kappa_0 z}{1 + \left(\dfrac{\beta^2}{4} + 1\right) z} \tag{B.17b}$$

or,

$$g\hat{\sigma} = \frac{-\kappa_0^2}{gf}(1 - \tanh(\kappa_0 x + \gamma)) \tag{B.18a}$$

$$g\pi = \frac{-\omega_0\kappa_0}{gf}(1 - \tanh(\kappa_0 x + \gamma)) \tag{B.18b}$$

where

$$\tanh\gamma = \frac{\dfrac{\kappa_0^3}{m(m+\omega_0)} - c_0}{\dfrac{\kappa_0^3}{m(m+\omega_0)} + c_0}. \tag{B.19}$$

Using (B.6) with (B.16) and simplifying, we find the normalized bound state wave function to be

$$\text{(B.20)} \quad \psi_0 = \left(\frac{\kappa_0(m - \omega_0)}{4m} \right)^{1/2} \operatorname{sech}(\kappa_0 x + \gamma) \begin{pmatrix} \dfrac{\kappa_0}{m - \omega_0} \\ 1 \end{pmatrix} .$$

Finally, we note that there exists an additional solution of physical interest in the σ only case. This solution is seen by noting that the *single* discrete state at $\omega_0 = 0$ automatically satisfied the symmetry requirement $\omega_0^{(+)} = |\omega_0^{(-)}|$ and hence could lead to a σ only solution. Further, as we can see by the following explicit reconstruction, this *does* lead to an exact solution of the field equations.

For $\omega_0 = 0$ the matrix F is trivial,

$$\text{(B.21)} \qquad F(x, y) = c_0 e^{-m(x+y)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

where $m = \kappa_0 = gf$. Solving (B.3) as before we find

$$\text{(B.22)} \qquad K(x, y) = \frac{-c_0 e^{-m(x+y)}}{1 + 2z} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

where $z = c_0 e^{-2mx}/2m$.

From (B.16) using (B.10) and simplifying we find

$$\text{(B.23)} \qquad \sigma = \hat{\sigma} + f = f \tanh(gfx + \alpha)$$

where $\tanh \alpha = (gf - c_0)/(gf + c_0)$.

The Dirac wave function, reconstructed from (B.6) using $K(x, y)$ as in (B.16) is

$$\text{(B.24)} \qquad \psi_0 = \left(\frac{gf}{4} \right)^{1/2} \operatorname{sech}(gfx + \alpha) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where we have normalized ψ_0 to 1.

This solution has the unique feature that the contribution of the eigenfunction, ψ_0, to the non-linear equation is identically zero: that is, $\bar{\psi}_0 \psi_0 = 0$. Further, although our inverse method does not indicate it, because of this lack of "feedback", this solution exists for arbitrary λ/g^2; explicitly, one has

$$\text{(B.25a)} \qquad \sigma(x) = f \tanh((\lambda/2)^{1/2} fx)$$

and

$$\text{(B.25b)} \qquad \psi_0(x, t) = N(\cosh(\lambda/2)^{1/2} fx)^{-g/(\lambda/2)^{1/2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

The existence of this solution for values of λ/g^2 other than that for which our method can be applied indicates an important limitation of our approach but may prove helpful in understanding the cases in which it can be used.

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