

A CRITERION OF OSCILLATION FOR GENERALIZED DIFFERENTIAL EQUATIONS*

WILLIAM T. REID

1. **Introduction.** For a second order ordinary linear differential equation

$$(1.1) \quad [r(t)u'(t)]' + q(t)u(t) = 0,$$

with real-valued coefficient functions r, q continuous and $r(t)$ positive on a non-compact interval $I = [a, \infty)$ the well-known Wintner-Leighton criterion, (see Wintner [12], and Leighton [4]), states that (1.1) is oscillatory on arbitrary subintervals $[c, \infty)$ of I whenever

$$(1.2a) \quad \int_a^\infty \frac{ds}{r(s)} = \lim_{t \rightarrow \infty} \int_a^t \frac{ds}{r(s)} = \infty,$$

and

$$(1.2b) \quad \int_a^\infty q(s) ds = \lim_{t \rightarrow \infty} \int_a^t q(s) ds = \infty.$$

The present paper presents an extension of this criterion for generalized matrix differential equations of the type previously considered by the author [6, 9]. In particular, this extended criterion implies for self-adjoint systems of difference equations a result, which in the case of a scalar self-adjoint difference equation yields a generalization of a theorem on oscillation established by McCarthy [5], and answers a question raised by that author.

Matrix notation is used throughout; in particular, matrices of one column are called vectors, and for a vector (y_α) , $(\alpha = 1, \dots, n)$, the norm $|y|$ is given by $(|y_1|^2 + \dots + |y_n|^2)^{1/2}$. The $n \times n$ identity matrix is denoted by E_n , or merely by E when there is no ambiguity, and 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is designated by M^* . The relations $M \cong N$, $(M > N)$, are used to signify that M and N are hermitian matrices of the same dimensions and $M - N$ is a non-negative, (positive), definite matrix. A matrix function is called continuous,

Received by the editors on October 1, 1976.

*This research was supported by the National Science Foundation under Grant MPS74-15648.

integrable, absolutely continuous, etc., when each element of the matrix possesses the specified property. If a matrix function $M(t)$ is locally absolutely continuous on an interval I , then $M'(t)$ signifies the matrix of derivatives at values where these derivatives exist, and zero elsewhere. Similarly, if $M(t)$ is Lebesgue integrable on a compact subinterval $[a, b]$ of I then $\int_a^b M(t) dt$ denotes the matrix of integrals of respective elements of $M(t)$. If $M(t)$ and $N(t)$ are matrix functions which have a common domain of existence, and $M(t)$ and $N(t)$ are equal almost everywhere (Lebesgue) on this domain, we write simply $M(t) = N(t)$.

2. The basic oscillation theorem. Consider a generalized vector differential system

$$(2.1) \quad -dv(t) - [dS(t)]u(t) = 0, \quad u'(t) - B(t)v(t) = 0,$$

in n -dimensional vector functions u, v with coefficient matrix functions satisfying the following hypothesis on a given non-compact interval $I = [a, b)$, where $-\infty < a < b \leq \infty$:

(\S) *B and S are hermitian $n \times n$ matrix functions, with B locally of class \mathcal{L}^∞ and S locally of bounded variation, while $B(t) \geq 0$ for $t \in I$.*

For basic properties of such systems, and the relation of such systems to ordinary differential systems to which (2.1) reduces when $S(t)$ is locally absolutely continuous, the reader is referred to [6] and [9] of the Bibliography. In particular, two values t_1 and t_2 of I are called (*mutually*) *conjugate* with respect to (2.1) if there exists a solution $(u; v)$ of this system with $u \neq 0$ on the subinterval with endpoints t_1 and t_2 , while $u(t_1) = 0 = u(t_2)$. Such a system is said to be *disconjugate* on a subinterval I_0 of I provided no two distinct points of I_0 are conjugate. On the non-compact interval $I = [a, b)$ the system is said to be *oscillatory near b* , (*oscillatory for large t* if $b = \infty$), in case (2.1) is not disconjugate on an arbitrary non-degenerate subinterval $[c, b)$ of I .

In view of the assumption that $B(t) \geq 0$ and B is locally of class \mathcal{L}^∞ on I , the smallest eigenvalue $\lambda_{\text{Min}}[B(t)]$ of $B(t)$ is a nonnegative, real-valued function that is locally of class \mathcal{L}^∞ on I , (see, for example, [10; Theorem 3.1]). The basic oscillation result to be established is as follows.

THEOREM 2.1. *If hypothesis (\S) holds, then (2.1) is oscillatory near b whenever the following two conditions are satisfied:*

- (2.2) (i) $\lambda(t) = \lambda_{\text{Min}}[B(t)]$ is such that $\int_a^b \lambda(s) ds = \lim_{t \rightarrow b} \int_a^t \lambda(s) ds = \infty$;
- (ii) there exists an n -dimensional vector ξ such that $|\xi| = 1$, and $\xi^*S(t)\xi \rightarrow \infty$ as $t \rightarrow b$.

For r, q continuous real-valued functions on $[a, \infty)$ with r positive, the result of the above theorem for $n = 1$, $B(t) = 1/r(t)$, and $S(t) = \int_a^t q(s) ds$, reduces to the Wintner-Leighton criterion for the scalar ordinary differential equation (1.1).

If hypothesis (S) holds, and there is a subinterval (c, b) on which (2.1) is disconjugate, then there exists an hermitian matrix function $N(t)$ and an hermitian constant matrix χ such that for $s \in (c, b)$,

$$(2.3) \quad N(t) - \int_s^t N(r)B(r)N(r) dr = S(t) + \chi, t \in (c, b).$$

This result is of the form of conclusion (iii) of Theorem 5.1 of [9], with the substitutions $N(t) = -W(t)$, $S(t) = -M(t)$, $\chi = -\Psi$, but is not a direct consequence of the cited result of [9] as the latter deals with a compact interval. The stated result may be proved by the same method as that presented for conclusion (iii) of Theorem 5.1 of [9], however, with the modification that one now refers to Theorem 5.3 of the author's earlier paper [8], rather than to Theorem 5.1 of [8]. In this connection, it is to be emphasized that the results of § 5 of [8] do not involve any assumption of normality on subintervals, so that for the existence of $N(t)$ and χ satisfying (2.3) no such assumption is needed.

Now if $|\xi| = 1$, and $\xi^*S(t)\xi \rightarrow \infty$ as $t \rightarrow b$, then also $\xi^*[S(t) + \chi]\xi \rightarrow \infty$ as $t \rightarrow b$, and as $B(t) \geq 0$ on I equation (2.3) implies that $\xi^*N(t)\xi \rightarrow \infty$ as $t \rightarrow b$. Also, since $N(t)$ and $B(t)$ are hermitian we have $\xi^*N(r)B(r)N(r)\xi \geq \lambda(r)\xi^*N^2(r)\xi$. Moreover, by the Schwarz inequality, it follows that $\xi^*N^2(r)\xi = [\xi^*\xi][\xi^*N^2(r)\xi] \geq |\xi^*N(r)\xi|^2$. Consequently, whenever conditions (2.2) are satisfied, and $N(t), \chi$ are as in (2.3), there exists a value $s \in (c, b)$ such that $\theta(t) = \xi^*N(t)\xi$ satisfies the conditions

- (2.4) (i) $\theta(t) > 0$ for $t \in [s, b)$,
- (ii) $\theta(t) - \int_s^t \lambda(r)\theta^2(r) dr \rightarrow \infty$ as $t \rightarrow b$.

However, these conditions are incompatible with the assumption that $\int_s^t \lambda(r) dr \rightarrow \infty$ as $t \rightarrow b$. Indeed, if $M > 0$ and $\tau \in (s, b)$ is such that $\theta(t) - \int_s^t \lambda(r)\theta^2(r) dr \geq M$ for $t \in [\tau, b)$, then

$$\lambda(t)\theta^2(t) \left[M + \int_\tau^t \lambda(r)\theta^2(r) dr \right]^{-2} \geq \lambda(t), \text{ for } t \in [\tau, b),$$

and integration yields the inequality $M^{-1} \geq \int_{\tau}^t \lambda(r) dr$, for $t \in [\tau, b)$, contrary to (2.2i). Consequently, whenever (\mathcal{S}) and conditions (2.2i, ii) hold the system (2.1) fails to be disconjugate on subintervals of I of the form (c, b) ; that is, (2.1) is oscillatory near b .

If (2.1) is identically normal, and $S(t)$ is locally absolutely continuous with derivative $S'(t) = C(t)$ almost everywhere on I , then in case $C(t) \geq 0$ on a subinterval (c, b) of I it follows from Theorem 3.3 of [7] that the conclusion of Theorem 2.1 remains valid when (2.2i) is replaced by the weaker condition

$$(2.5_0) \quad \xi^* \left[\int_a^t B(s) ds \right] \xi \rightarrow \infty \text{ as } t \rightarrow b, \text{ for arbitrary non-zero vectors } \xi,$$

which, in view of the non-negative definite character of $B(t)$, is equivalent to the condition

$$(2.5) \quad \lambda_{\text{Min}} \left[\int_a^t B(s) ds \right] \rightarrow \infty \text{ as } t \rightarrow b.$$

For a discussion of various related criteria, the reader is referred to the papers [1, 2] of Ahlbrandt. It is to be remarked, however, that it is not known whether or not the result of Theorem 2.1 remains valid whenever (2.2i) is replaced by (2.5) and no additional conditions are imposed.

3. Criteria for related systems. Now consider a self-adjoint generalized differential system of the form

$$(3.1) \quad \begin{aligned} & -dv_1(t) + [C_1(t)u_1(t) \\ & - A_1^*(t)v_1(t)] dt - [dS_1(t)]u_1(t) = 0, \\ & u_1'(t) - A_1(t)u_1(t) - B_1(t)v_1(t) = 0, \end{aligned}$$

wherein the $n \times n$ matrix functions A_1, B_1, C_1, S_1 satisfy on $I = [a, b)$ the following hypothesis.

(\mathcal{S}_1) A_1, B_1, C_1 are locally of class \mathcal{L}^∞ , with B_1, C_1 hermitian and $B_1(t) \geq 0$ almost everywhere on I , while S_1 is locally of bounded variation on this interval.

A system (3.1) is reducible to the form (2.1) under various substitutions. In particular, if $Y_1(t)$ is a fundamental matrix solution of the ordinary differential equation $Y_1'(t) - A_1(t)Y_1(t) = 0$, then under the substitution

$$(3.2) \quad u_1(t) = Y_1(t)u(t), \quad v_1(t) = Y_1^{*-1}(t)v(t),$$

the system (3.1) is reduced to (2.1) with

$$B(t) = Y_1^{-1}(t)B_1(t)Y_1^{*-1}(t), \tag{3.3}$$

$$S(t) = \int_{t_0}^t Y_1^*(s)[dS_1(s)]Y_1(s) - \int_{t_0}^t Y_1^*(s)C(s)Y_1(s) ds,$$

(see, for example, [9, § 2]).

For a system (3.1) with

$$S_1(t) = M(t) + M_1(t), \tag{3.4}$$

where M and M_1 are hermitian matrix functions that are locally of bounded variation on I , and the system

$$\begin{aligned} & - dv_1(t) + [C_1(t)u_1(t) \\ & - A_1^*(t)v_1(t)] dt - [dM_1(t)]u_1(t) = 0, \\ & u_1'(t) - A_1(t)u_1(t) - B_1(t)v_1(t) = 0, \end{aligned} \tag{3.5}$$

is disconjugate on a subinterval (c, b) of I , let $(U_1(t); V_1(t))$ be a conjoined basis of (3.5) with $U_1(t)$ non-singular on (c, b) . Under the substitution

$$(3.6) \quad u_1(t) = U_1(t)u(t), \quad v_1(t) = V_1(t)u(t) + U_1^{*-1}(t)v(t),$$

the system (3.1) reduces on (c, b) to the form (2.1) with

$$(3.7) \quad B(t) = U_1^{-1}(t)B_1(t)U_1^{*-1}(t), \quad S(t) = \int_{t_0}^t U_1^*(s)[dM(s)]U_1(s).$$

For hermitian ordinary differential systems appearing as the accessory system for variational problems of Lagrange or Bolza type this transformation is essentially the classical Clebsch transformation, (see [11, Lemma 4.2 and Corollary of Chapter VII], and also [9, Lemma 5.1]).

Moreover, in view of the equivalence of (3.5) to an ordinary differential system as in [9, § 2], and the results on principal solutions of self-adjoint ordinary differential systems presented in [11, § 3 of Chapter VII], it follows that if (3.5) is identically normal on I then $B(t)$ defined by (3.7) is such that for $c < s < t < b$ the hermitian matrix function $\Theta(t, s | U_1) = \int_s^t B(r) dr$ is positive definite and $\Theta^{-1}(t, s | U_1) \rightarrow 0$ as $t \rightarrow b$, which is equivalent to (2.5₀) or (2.5).

4. Self-adjoint systems of difference equations. Now for a given non-compact interval $I = [a, b)$ let $\{t_j\}$, $(j = 0, 1, \dots)$, be a sequence of values satisfying

$$(4.1) \quad a = t_0 < t_1 < \dots < t_j < t_{j+1} < \dots, \lim_{j \rightarrow \infty} t_j = b,$$

and let $R_j, S_j, (j = 0, 1, \dots)$, be hermitian $n \times n$ matrices with each R_j positive definite. Consider a system (2.1) wherein for $j = 0, 1, \dots$ we have

$$(4.2) \quad B(t) \equiv R_j^{-1}, S(t) \equiv S_j, \text{ for } t \in (t_j, t_{j+1}).$$

For definiteness, one may also suppose

$$S(t_{j+1}) = \frac{1}{2} [S_j + S_{j+1}], (j = 0, 1, \dots).$$

As presented in [6, § 6], if $(u; v)$ is a solution of (2.1) on I then on this interval $u(t)$ is a continuous vector function that is linear on each subinterval $[t_j, t_{j+1}]$, and for

$$(4.3) \quad K_{j+1} = S_{j+1} - S_j, \Delta u(t_j) = u(t_{j+1}) - u(t_j), \\ \Delta t_j = t_{j+1} - t_j$$

the sequence $\{u(t_j)\}$ satisfies the self-adjoint system of difference equations

$$(4.4) \quad R_{j+1} \frac{\Delta u(t_{j+1})}{\Delta t_{j+1}} - R_j \frac{\Delta u(t_j)}{\Delta t_j} + K_{j+1} u(t_{j+1}) = 0, (j = 0, 1, \dots).$$

Conversely, if on I the vector function $u(t)$ is continuous, linear on each subinterval $[t_j, t_{j+1}]$, and satisfies (4.4), then a solution of (2.1) is given by $u = u(t), v = v(t)$ with

$$v(t_0) = R_0 \frac{\Delta u(t_0)}{\Delta t_0} - [S(t_0) - S_0] u(t_0), \\ v(t) = R_j \frac{\Delta u(t_j)}{\Delta t_j} \text{ for } t \in (t_j, t_{j+1}), \\ v(t_{j+1}) = R_j \frac{\Delta u(t_j)}{\Delta t_j} - [S(t_{j+1}) - S_j] u(t_{j+1}), (j = 0, 1, \dots).$$

For a direct treatment of real self-adjoint systems (4.4), with the derivation of certain central oscillation and comparison theorems, the reader is referred to Harris [3].

Since for $t \in (t_j, t_{j+1})$ we have $\lambda_{\text{Min}}[B(t)] = 1/\lambda_{\text{Max}}[R_j]$, a direct application of Theorem 2.1 yields the following result.

THEOREM 4.1. *If $I = [a, b)$, and $t_j, R_j, S_j, (j = 0, 1, \dots)$, are as specified above, then (4.4) is oscillatory near b when the following conditions are satisfied:*

$$(i) \sum_{j=0}^{\infty} (\Delta t_j) / \lambda_{\text{Max}} [R_j] = \infty,$$

$$(ii) \text{ there exists a vector } \xi \text{ with } |\xi| = 1 \text{ and } \sum_{j=0}^{\infty} \xi^* K_{j+1} \xi \\ = \lim_{m \rightarrow \infty} \sum_{j=1}^m \xi^* K_{j+1} \xi = \infty.$$

For $n = 1$, $a = 0$, $b = \infty$, $t_j = j$, ($j = 0, 1, \dots$), and r_j, k_j real values with each r_j positive, this criterion states that the real self-adjoint difference equation

$$(4.5) \quad r_{j+1} \Delta u(t_{j+1}) - r_j \Delta u(t_j) + k_{j+1} u(t_{j+1}) = 0, \quad (j = 0, 1, \dots),$$

is oscillatory near ∞ whenever

$$(4.6) \quad (i) \sum_{j=0}^{\infty} 1/r_j = \infty, \quad (ii) \sum_{j=0}^{\infty} k_{j+1} = \infty.$$

This condition generalizes the result of McCarthy [5], whose criterion for the oscillation of (4.5) near infinity involved (4.6ii) and the boundedness of the positive sequence $\{r_j\}$, and who stated [5, p. 204] that he did not know whether the boundedness of $\{r_j\}$ could be replaced by the divergence of the series $\sum_{j=0}^{\infty} 1/r_j$.

Added in proof. Subsequent to the submission of the present paper the author received from Don B. Hilton and Roger T. Lewis a copy of a manuscript entitled *Spectral Analysis of Order Difference Equations*, wherein they also obtain the above stated generalization of the result of McCarthy [5].

BIBLIOGRAPHY

1. C. D. Ahlbrandt, *Disconjugacy criteria for self-adjoint differential systems*, (Dissertation, University of Oklahoma, 1968), *Journal of Differential Equations* **6** (1969), 271-295. MR 39 #3555.
2. ———, *The question of equivalence of principal and co-principal solutions of self-adjoint differential systems*, *Illinois Journal of Mathematics* **20** (1976), 72-81. MR 53 #940.
3. V. C. Harris, *A system of difference equations and an associated boundary value problem*, (Dissertation, Northwestern University, 1950).
4. W. Leighton, *On self-adjoint differential equations of the second order*, *Journal of the London Mathematical Society* **27** (1952), 37-47. MR 13-745.
5. P. J. McCarthy, *Note on the oscillation of solutions of second order difference equations*, *Portugaliae Mathematica* **18** (1959), 203-205. MR 23 #A1953.
6. W. T. Reid, *Generalized linear differential equations*, *Journal of Mathematics and Mechanics* **8** (1959), 705-726. MR 21 #5777.
7. ———, *Oscillation criteria for self-adjoint differential systems*, *Transactions of the American Mathematical Society* **101** (1961), 91-106. MR 24 #A3349.
8. ———, *Principal solutions of nonoscillatory linear differential systems*, *Journal of Mathematical Analysis and Applications* **9** (1964), 397-423. MR 29 #6110.

9. ———, *Generalized linear differential systems and related Riccati matrix integral equations*, Illinois Journal of Mathematics **10** (1966), 701-722. **MR 37 #1682**.

10. ———, *Some elementary properties of proper values and proper vectors of matrix functions*, SIAM Journal of Applied Mathematics **18** (1970), 259-266. **MR 41 #8623**.

11. ———, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1971. **MR 42 #7963**.

12. A. Wintner, *A criterion of oscillatory stability*, Quarterly of Applied Mathematics **7** (1949), 115-117. **MR 10-456**.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA
73019

Present address: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT
AUSTIN, AUSTIN, TEXAS 78712