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BIFURCATION OF 2*m*th ORDER NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Eigenvalues of the linearized part of $2m$th order elliptic partial differential equations are shown to be bifurcation points.

Introduction. C. V. Coffman [5] and E. T. Dean and P. L. Chambré [6] among others (see for example [9]) investigated the bifurcation problem for elliptic partial differential equations of the form $Au = \lambda P(x)u + G(\lambda, u, x)$, restricting themselves to second order equations. Coffman showed that if $G(\lambda, u, x) = \lambda G(u, x)$ and either G_u is bounded or G is odd in u then every eigenvalue of the linearized part is a bifurcation point. Apparently his methods cannot be extended to the instance where G depends nonlinearly on λ. Under less severe conditions Dean and Chambré proved that the principle eigenvalue is a bifurcation point. In this paper we consider the equation where A is a linear partial differential operator of order $2m$, G is a nonlinear function of λ, and trade off Coffman's unduly restrictive assumptions for a greater degree of differentiability of the terms to show that bifurcation occurs at every eigenvalue.

Main Results. Let A be a formally selfadjoint elliptic [1, pp. 95–96, 45] linear partial differential operator of order $2m$ defined on a bounded domain Ω in $\mathbf{R}^n$ with sufficiently smooth boundary. Consider the boundary value problem

$$(1)\qquad \begin{aligned} &Au = \lambda P(x)u + G(\lambda, u, x) \qquad x \in \Omega \\ &D^\alpha u = 0 \qquad x \in \partial\Omega \qquad |\alpha| \leqq m - 1 \end{aligned}$$

where $\lambda \in \mathbf{R}$ and $P(x)$ and $G(\lambda, t, x)$ are real valued continuous functions on $\mathbf{R} \times \mathbf{R} \times \bar{\Omega}$. We are interested in the existence of solutions (λ, u) satisfying (1) for u small and λ near the eigenvalues of the linearized equation

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$$(2)\qquad \begin{aligned} &Au = \lambda P(x)u, && x \in \Omega, \\ &D^\alpha u = 0, \quad x \in \partial\Omega, && |\alpha| \leqq m-1. \end{aligned}$$

To be more precise we assume A is uniformly elliptic [1, p. 71] and can be expressed in divergence form

$$Au = \sum_{|\alpha|,|\beta| \leqq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u)$$

where α is the n-tuple of nonnegative integers: $\alpha = (\alpha_1, \cdots, \alpha_n)$, $D^\alpha = \prod_{j=1}^n (\partial/\partial x_j)^{\alpha_j}$ with the order of D^α defined by $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $a_{\alpha\beta}(x) = a_{\beta\alpha}(x) \in C^{2m}(\bar{\Omega})$ ($C^k(R)$ the space of bounded k times continuously differentiable functions defined on R). We further assume that $P(x) \in C^{3m}(\bar{\Omega})$, $|P(x)| \neq 0$ for $x \in \bar{\Omega}$, $G \in C^2(\mathbf{R} \times \mathbf{R} \times \bar{\Omega})$ and $G(\lambda, t, x) = o(t)$ uniformly for λ near eigenvalues of (2) and all x. An eigenvalue λ_0 of (2) is said to be a bifurcation point if every neighborhood of $(\lambda_0, 0)$ (in the $\mathbf{R} \times C(\Omega)$ topology) contains a nontrivial solution, that is a solution $(\lambda, u) \neq (\lambda, 0)$, of (1). With our assumptions we can prove

THEOREM. *Every eigenvalue of* (2) *is a bifurcation point of* (1).

PROOF. Let λ_0 be an eigenvalue of (2). To complete the proof we will show that A is a closed linear operator in a suitable Banach space and reduce the problem to one of finite dimension and apply M. S. Berger's bifurcation theorem [3] which we state in our context as follows. In a real Hilbert space H, let L be a compact selfadjoint map of $H \rightarrow H$ and let $T \in C^2$ be a gradient operator [10, p. 54] (for fixed λ) mapping a neighborhood of $(\lambda_0, 0) \in \mathbf{R} \times H$ into H such that $T(\lambda, 0) \equiv 0$ and $T_x(\lambda, 0) \equiv 0$, and suppose λ_0 is an eigenvalue of L. Then λ_0 is a bifurcation point of the equation $Lx = \lambda x + T(\lambda, x)$.

As our first simplification we set $|P(x)|^{-1/2}\, v = u$. Then (1) is equivalent to the problem

$$(3)\qquad \begin{aligned} &\tilde{A}v = \lambda\mu v + \tilde{G}(\lambda, v, x), && x \in \Omega, \\ &D^\alpha v = 0, \quad x \in \partial\Omega, && |\alpha| \leqq m-1, \end{aligned}$$

where $\tilde{A} = |P|^{-1/2} A |P|^{-1/2}$, $\tilde{G} = |P(x)|^{-1/2} G(\lambda, |P(x)|^{-1/2} v, x)$ and $\mu = P(x)/|P(x)|$ (that is ± 1). It is readily verified that

$$\tilde{A}v = \sum_{|\alpha|,|\beta| \leqq m} (-1)^{|\alpha|} D^\alpha(\tilde{a}_{\alpha\beta}(x) D^\beta v)$$

is a selfadjoint uniformly elliptic linear operator, $\tilde{a}_{\alpha\beta}(x) \in C^{2m}(\bar{\Omega})$ and λ_0 is an eigenvalue of the corresponding linearized part.

To find a suitable Banach space and domain for $\tilde{A}$ we let $C_0{}^\infty(\Omega)$ be the space of infinitely continuously differentiable functions with compact support in Ω and let $\mathring{W}_2^m(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{2,m}^2 = \sum_{|\alpha| \leqq m} \|D^\alpha v\|_{L^2(\Omega)}^2 .$$

By Gårding's inequality there exist constants γ, $\delta > 0$ such that for $v \in \mathring{W}_2^m(\Omega)$,

$$(4) \qquad \sum_{|\alpha|,|\beta| \leqq m} \int_\Omega \tilde{a}_{\alpha\beta}(x) D^\alpha v \cdot D^\beta v \, dx \geqq \gamma \|v\|_{2,m} - \delta \|v\|_{L^2(\Omega)}$$

[1, p. 78]. Thus for $f \in C(\bar{\Omega})$ there eixsts a unique generalized solution $v \in \mathring{W}_2^m(\Omega)$ such that $(\tilde{A} + \delta I)v = f$ [4, p. 199], [1, p. 102]. Moreover, by regularity (see [2, section V] and the references cited therein) we also have $v \in C^{2m-1}(\Omega) \cap C^{m-1}(\bar{\Omega})$, $D^\alpha v = 0$ for $x \in \partial\Omega$ and $|\alpha| \leqq m - 1$ and for $|\alpha| = 2m$, $D^\alpha v \in L^2(\Omega)$. If f is also Hölder continuous [2] then $v \in C^{2m}(\Omega)$. We therefore let C be the space of continuous functions on $\bar{\Omega}$ vanishing on the boundary, with the supremum norm and define the domain of $\tilde{A}$, $D(\tilde{A}) = (\tilde{A} + \delta I)^{-1}(C)$ and let $\tilde{A}$ be defined by $\tilde{A}u = f - \delta u$ for $u \in D(\tilde{A})$ where $u = (\tilde{A} + \delta I)^{-1} f$. Now $D(\tilde{A})$ is dense in C. Indeed, if $v \in C$, for any subdomain Ω' of Ω with closure contained in Ω we can define $v' = v$ for $x \in \bar{\Omega}'$ and $v' = 0$ otherwise. Let J_ϵ be a mollifier as defined in [1, p. 5]. Then $J_\epsilon v' \in C_0^\infty(\Omega)$ and $J_\epsilon v' \to v'$ uniformly in Ω'' as $\epsilon \to 0$ for any Ω'', *with* $\bar{\Omega}'' \subseteq \Omega'$ [1, p. 5]. Thus $C_0^\infty(\Omega)$ is dense in C and as $(\tilde{A} + \delta I)(C_0^\infty(\Omega)) \subset C$ it follows $C_0^\infty(\Omega) \subset D(\tilde{A})$ and so $D(\tilde{A})$ is dense in C.

Now we show that $\tilde{A}$ is closed on $D(\tilde{A})$ in C. As $\tilde{A}$ defined on $D(\tilde{A})$ is symmetric in $L^2(\Omega)$ a Hilbert space, $\tilde{A}$ has a minimal closed extension in $L^2(\Omega)$, also denoted $\tilde{A}$, with domain $D_L(\tilde{A})$ [8, p. 56]. Thus since $k\|v\|_C \geqq \|v\|_{L^2(\Omega)}$ it follows that $\tilde{A}$ has a minimal closed extension in C with domain $D_C(\tilde{A})$. However $D(\tilde{A}) = D_C(\tilde{A})$. Indeed, suppose $v \in D_C(\tilde{A})$ and let $f = \tilde{A}v$. Then as $D_C(\tilde{A}) \subset D_L(\tilde{A})$ there exists a $\{v_i\} \subset D(\tilde{A})$ such that $v_i \to v$ in $L^2(\Omega)$ and $\tilde{A}v_i \to \tilde{A}v$ in $L^2(\Omega)$. By (4) and the Cauchy-Schwarz inequality

$$\begin{aligned} \gamma\|v_i - v_j\|_{2,m} &\leqq \sum_{|\alpha|,|\beta| \leqq m} \int_\Omega \tilde{a}_{\alpha\beta}(x) D^\alpha(v_i - v_j) \\ &\qquad \cdot D^\beta(v_i - v_j)\, dx + \delta\|v_i - v_j\|_{L^2(\Omega)}^2 \\ &\leqq \int_\Omega (\tilde{A} + \delta I)(v_i - v_j) \cdot (v_i - v_j)\, dx \\ &\leqq \|(\tilde{A} + \delta I)(v_i - v_j)\|_{L^2(\Omega)} \|v_i - v_j\|_{L^2(\Omega)}. \end{aligned}$$

Hence $\{v_i\}$ converges in $\mathring{W}_2{}^m(\Omega)$ and $v \in \mathring{W}_2{}^m(\Omega)$. As the bilinear form associated with $\tilde{A}$ is bounded in $\mathring{W}_2{}^m(\Omega)$ and $f + \delta v \in C$ it follows that v is a generalized solution of $(\tilde{A} + \delta I)v = g = f + \delta v$. But then $v \in D(\tilde{A})$. Therefore $\tilde{A}$ is closed on $D(\tilde{A})$ in C.

By the "Fredholm alternative" for uniformly elliptic operators [4, p. 199], [1, p. 102] it follows the null space of $\tilde{A} - \lambda_0 \mu I$, $N(\tilde{A} - \lambda_0 \mu I)$, is finite dimensional and by regularity contained in C. In addition $(\tilde{A} - \lambda_0 \mu I)\phi = f$ for $f \in C$ if and only if $\int_\Omega f \cdot \phi \, dx = 0$ for all $\phi \in N(\tilde{A} - \lambda_0 \mu I)$. Thus as $C = N(\tilde{A} - \lambda_0 \mu I) \oplus N(\tilde{A} - \lambda_0 \mu I)^\perp$(where

$$N(\tilde{A} - \lambda_0 \mu I)^\perp = \{\psi \in C \mid \int_\Omega \psi \cdot \phi \, dx = 0 \text{ for } \phi \in N(\tilde{A} - \lambda_0 \mu I)\})$$

it follows that $C = N(\tilde{A} - \lambda_0 \mu I) \oplus R(\tilde{A} - \lambda_0 \mu I)$ ($R(\cdot)$ denotes the range of $\tilde{A} - \lambda_0 \mu I$).

Therefore $v \in C$ is uniquely of the form $v = \phi + \psi$, $\phi \in N(\tilde{A} - \lambda_0 \mu I)$ and $\psi \in R(\tilde{A} - \lambda_0 \mu I)$ and $\tilde{G} = G_N(\lambda, \phi + \psi, x) + G_R(\lambda, \phi + \psi, x)$ where $G_N \in N(\tilde{A} - \lambda_0 \mu I)$ and $G_R \in R(\tilde{A} - \lambda_0 \mu I)$. Clearly $\tilde{A} - \lambda \mu I : N(\tilde{A} - \lambda_0 \mu I) \to N(\tilde{A} - \lambda_0 \mu I)$ for all λ and as the resolvent of a closed map is open, $\tilde{A} - \lambda \mu I$ is a one-one map, with uniformly bounded inverse, of $D(\tilde{A}) \cap R(\tilde{A} - \lambda_0 \mu I)$ onto $R(\tilde{A} - \lambda_0 \mu I)$ for all λ near λ_0. Thus finding solutions of (3) is equivalent to solving in $\mathbf{R} \times N(\tilde{A} - \lambda_0 \mu I) \times (D(\tilde{A}) \cap R(\tilde{A} - \lambda_0 \mu I))$ the system

$$\psi = (\tilde{A} - \lambda \mu I)^{-1} G_R(\lambda, \phi + \psi, x)$$

$$\tilde{A}\phi = \lambda \mu \phi + G_N(\lambda, \phi + \psi, x).$$

By an application of the implicit function theorem [7, p. 265] there exists a unique twice continuously differentiable function $\psi = \psi(\lambda, \phi)$ such that

$$\psi(\lambda, \phi) \equiv (\tilde{A} - \lambda \mu I)^{-1} G_R(\lambda, \phi + \psi(\lambda, \phi), x)$$

for (λ, ϕ) near $(\lambda_0, 0)$.

Moreover, by regularity for each fixed λ and ϕ, $\psi \in C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ in x and $D^\alpha \psi = 0$ on $\partial\Omega$. Thus our problem is reduced to solving the finite dimensional equation

$$\tilde{A}\phi - \lambda \mu \phi - G_N(\lambda, \phi + \psi(\lambda, \phi), x) = 0 \tag{5}$$

An argument similar to that of [11, Theorem 3] will show that (5) is a gradient operator equation (for fixed λ) with potential

$$P(\lambda, \phi) = (1/2) \sum_{|\alpha|,|\beta| \leq m} \int_\Omega \tilde{a}_{\alpha\beta}(x) D^\alpha(\phi + \psi(\lambda, \phi))$$

$$\cdot D^\beta(\phi + \psi(\lambda, \phi))\, dx - (1/2)\lambda\mu \int_\Omega (\phi + \psi(\lambda, \phi))^2\, dx$$

$$- \int_\Omega G(\lambda, \phi + \psi(\lambda, \phi), x)\, dx$$

where $G(\lambda, t, x) = \int_0^t \tilde{G}(\lambda, s, x)\, ds$.

Indeed a simple computation and integration by parts yields for $\phi, f \in N(\tilde{A} - \lambda_0 \mu I)$

$$\lim_{t \to 0} t^{-1}(P(\lambda, \phi + tf) - P(\lambda, \phi))$$

$$= \int_\Omega [(\tilde{A} - \lambda\mu I)(\phi + \psi(\lambda, \phi)) - \tilde{G}(\lambda, \phi$$

$$+ \psi(\lambda, \phi))]\ (f + \psi_\phi(\lambda, \phi)(f))\, dx$$

$$= \int_\Omega [(\tilde{A} - \lambda\mu I)(\phi + \psi(\lambda, \phi)) - \tilde{G}(\lambda, \phi$$

$$+ \psi(\lambda, \phi))] f\, dx$$

$$+ \int_\Omega [(\tilde{A} - \lambda\mu I)(\phi + \psi(\lambda, \phi)) - \tilde{G}(\lambda, \phi$$

$$+ \psi(\lambda, \phi))]\ \psi_\phi(\lambda, \phi)(f)\, dx.$$

Now the second term in the last expression is zero. Clearly $\psi_\phi(\lambda, \phi)(f) \in R(\tilde{A} - \lambda_0 \mu I)$ since $\psi(\lambda, \phi) \in R(\tilde{A} - \lambda_0 \mu I)$ for all $\phi \in N(\tilde{A} - \lambda_0 \mu I)$. Thus this integral reduces to

$$\int_\Omega [(\tilde{A} - \lambda\mu I)(\psi\lambda, \phi) - G_R(\lambda, \phi + \psi(\lambda, \phi))]\ \psi_\phi(\lambda, \phi)(f)\, dx.$$

But this must be zero by the definition of $\psi(\lambda, \phi)$. Thus by orthogonality the last expression reduces to

$$\int_\Omega [(\tilde{A} - \lambda\mu I)\phi - G_N(\lambda, \phi + \psi(\lambda, \phi))] f\, dx.$$

Consequently (5) is a gradient operator equation and the theorem follows from Berger's bifurcation theorem.

Remark. In the proof of our theorem we never used the fact that G is continuous for all (λ, t, x). Thus it would have sufficed to assume that G is twice continuously differentiable for λ near λ_0 and t near 0 for all $x \in \bar{\Omega}$.

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