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REPRESENTATION OF LINEAR FUNCTIONALS IN A BANACH SPACE

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ABSTRACT. In this paper we prove a Riesz representation theorem for linear functionals in a reflexive Banach space.

1. **Introduction.** As an aid to the extension of operator theory from the Hilbert space setting to a general Banach space, G. Lumer [4] introduced the notion of a semi-inner-product on a normed linear space X as a complex function $[\cdot, \cdot]$ on $X \times X$ which is linear in the first argument only, strictly positive, and satisfies a Schwarz inequality $|[x, y]|^2 \leqq [x, x][y, y]$. The form $[\cdot, \cdot]$ induces a norm in the natural way of putting $[x, x]^{1/2} = \|x\|$. Lumer showed that every normed linear space has a semi-inner-product which is compatible with the norm in this fashion.

J. R. Giles [2] later showed that the axioms of a semi-inner-product can be extended to include conjugate homogeneity in the second component without any loss of generality with respect to applications in normed linear spaces. In view of this refinement Giles was able to prove that in a smooth semi-inner-product space which is uniformly convex, a Riesz Representation theorem holds. That is to say that if $x^* \in X^*$, there is a unique $y \in X$ such that $x^*(x) = [x, y]$ for all $x \in X$. The purpose of this paper is to extend the representation theorem of Giles to Reflexive Banach Spaces.

2. DEFINITION 1. Let X be a normed linear space. A function $[\cdot, \cdot] : X \times X \to C$ is a *semi-inner-product* on X (s.i.p.) if and only if it satisfies the following:

(a) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$ for all $\alpha, \beta \in C$,

(b) $[x, x] = \|x\|^2 > 0$ if $x \neq 0$,

(c) $|[x, y]|^2 \leqq [x, x][y, y]$,

(d) $[x, \beta y] = \bar{\beta}[x, y]$.

DEFINITION 2. A B-space X is *uniformly convex* if and only if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ so that if $\|x\| = \|y\| = 1$ and $\|x - y\| > \epsilon$ then $\|(x + y)/2\| < 1 - \delta$.

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DEFINITION 3. A B-space X is *strictly convex* if and only if whenever $\|x\| + \|y\| = \|x + y\|$ where $x, y \neq 0$ then $x = \lambda y$ for some $\lambda > 0$.

DEFINITION 4. A B-space X is *reflexive* if and only if the mapping $J : X \to X^{**}$ given by $(Jx)(x^*) = x^*(x)$ is surjective.

It is well known that uniformly convex implies both strictly convex and reflexive. However, neither converse holds.

DEFINITION 5. A B-space X is *smooth* if and only if for each $x \in X$ with $\|x\| = 1$ there is a unique $x^* \in X^*$ such that $x^*(x) = \|x^*\|$ (The Hahn-Banach theorem guarantees the existence of at least one such functional).

DEFINITION 6. Following James [3], an element $x \in X$ is orthogonal to $y \in X$ if and only if for each $\lambda \in C$ $\|x + \lambda y\| \geqq \|x\|$. This will be written as $x \perp y$. If for each $x \in M, y \in N$, $M, N \subseteq X$, we have $x \perp y$ we write $M \perp N$.

DEFINITION 7. For $x, y \in X$ we say that x is *normal* to y with respect to or relative to the s.i.p. $[\,\cdot, \cdot\,]$ if and only if $[y, x] = 0$. If M and N are subsets of X, we say that M is normal to N if and only if $[y, x] = 0$ for all $y \in N, x \in M$.

3. THEOREM 1. *Let M and N be subspaces of a normed linear space X. A necessary and sufficient condition for $M \perp N$ is that there exists a* s.i.p. $[\,\cdot, \cdot\,]$ *relative to which M is normal to N.*

PROOF. Suppose that M is normal to N with respect to $[\,\cdot, \cdot\,]$. If $x \in M$ and $y \in N$ we have $\|x + y\| \, \|x\| \geqq [x + y, x] = \|x\|^2$, from which it follows that $M \perp N$.

Let us now suppose that $M \perp N$, so that $M \cap N = \{0\}$. Hence, for each $x \in M$ we may define a linear functional f_x on $s_x = \operatorname{span}\{x, N\} = \{\alpha x + n \mid \alpha \in C, n \in N\}$ as follows:

$$f_x(\alpha x + n) = \alpha \|x\|^2.$$

Now f_x is clearly linear and in view of

$$|f_x(\alpha x + n)| = |\alpha| \, \|x\|^2 \leqq \|x\| \, |\alpha| \, \|x + n/\alpha\| = \|x\| \, \|\alpha x + n\|,$$

f_x is bounded by $\|f_x\| \leqq \|x\|$. By observing that $f_x(x/\|x\|) = \|x\|$ we obtain $\|f_x\| = \|x\|$. f_x also satisfies $f_x(x) = \|x\|^2$ and $f_x(n) = 0$ $\forall n \in N$. For $z \notin M$ we may define $f_z(\alpha z) = \alpha \|z\|^2$ on the span of $\{z\}$. Clearly for these $z \in X$, $\|f_z\| = \|z\|$ and $f_z(z) = \|z\|^2$. Thus for each $x \in X$ we obtain a bounded linear functional f_x which may be extended to the entire space by the Hahn-Banach theorem. We therefore consider f_x

to be defined throughout X. Now, let $\mathcal{V}$ be a well ordering of X, and let x be the initial element of $\mathcal{V}$. Define the functional Φ_x to be f_x; and if $z = \lambda x$ define $\Phi_z = \bar{\lambda}\Phi_x$. Similarly for x', the initial element of $\mathcal{V}$ not in the span of x, define $\Phi_{x'} = f_{x'}$ and $\Phi_{\lambda x'} = \bar{\lambda}\Phi_{x'}$. Continuing in this fashion we may, by transfinite induction, define Φ_z for each $z \in X$. Since each $z \in X$ has a unique initial generation ω relative to the order $\mathcal{V}$ (i.e., ω is the least element of $\mathcal{V}$ for which $z = \lambda\omega$) the indexing of the functionals Φ_z is clearly well-defined. We may now set $[x, z] = \Phi_z(x)$, and we need only verify that (a)–(b) of Definition 1 hold, since clearly for $x \in M, y \in N$ we have $[y, x] = \Phi_x(y) = 0$. The condition (a) is immediate from the linearity of Φ_z. For condition (b) suppose that ω is the initial generator of $x \in X$, say $x = \lambda\omega$, then $[x, x] = \Phi_{\lambda\omega}(\lambda\omega) = |\lambda|^2 f_\omega(\omega) = \|\lambda\omega\|^2 = \|x\|^2 > 0$. Similarly, for condition (c), if $x = \lambda\omega$ and $y = \mu\nu$ for both ω and ν initial in $\mathcal{V}$, then we have

$$\begin{aligned}|[x, y]|^2 = |\Phi_y(x)|^2 = |\Phi_{\mu\nu}(\lambda\omega)|^2 = |\mu|^2|\lambda|^2|\Phi_\nu(\omega)|^2 \\ \leqq |\mu|^2|\lambda|^2\|f_\nu\|^2\|\omega\|^2 = \|\mu\nu\|^2\|\lambda\omega\|^2 = [y, y]\ [x, x].\end{aligned}$$

Finally for part (d) $[x, \beta y] = [x, (\beta\mu)\nu] = \overline{\beta\mu}\Phi_\nu(x) = \bar{\beta}\Phi_{\mu\nu}(x) = \bar{\beta}[x, y]$. This concludes the proof.

Remark. We may observe that there exist subspaces M and N, both with dimension larger than one, that satisfy the hypothesis of this theorem. For example if the Banach space has a monotone base $\{x_i\}$ then for every n, span $\{x_1, \cdots x_n\}$ is orthogonal to its algebraic complement.

James [3], in his 1947 paper introducing the notion of orthogonality given in Definition 6, observed that in order for $x \in X$ to be orthogonal to the null-space $N(f)$ of a functional f it is necessary and sufficient that $|f(x)| = \|f\|\ \|x\|$. The sufficiency is easily seen by the following: For $n \in N(f)$,

$$\|f\|\ \|x + n\| \geqq |f(x + n)| = |f(x)| = \|f\|\ \|x\|.$$

S. Mazur [5] shows that in a reflexive B-space X, for any functional f there is an $x \in X$ so that $|f(x)| = \|f\|\ \|x\|$. Consequently if X is reflexive and $f \in X^*$ there is an $x \in X$ so that $x \perp N(f)$. In view of this we prove.

Theorem 2. *Let X be a Banach space. Then a necessary and sufficient condition for X to be reflexive is that for every $f \in X^*$, there exists an* s.i.p. $[\ \cdot, \cdot\ ]$ *and an element $y \in X$ so that $f(x) = [x, y]$ for all $x \in X$.*

PROOF. (Necessity) If $N(f) = X$, any s.i.p. will suffice with $y = 0$. If $N = N(f) \neq X$, since X is reflexive, there is an $x_0 \in X$ with $x_0 \perp N$. The orthogonality relation is homogeneous, thus if M is the span of $\{x\}$ we have $M \perp N$. By Theorem 1 there is an s.i.p. $[\ \cdot, \cdot\ ]$ with respect to which M is normal to N. For $x \in X$ consider the element $z \in X$ given by $z = f(x)x_0 - f(x_0)x$. Clearly $z \in N$ so $0 = [z, x_0] = f(x)\|x_0\|^2 - f(x_0)[x, x_0]$. Consequently we have $f(x) = [x, \overline{(f(x_0)}/\|x_0\|^2)x_0] = [x, y]$. For sufficiency we need only observe that every functional assumes its norm on the unit sphere and hence by James [6], X is reflexive.

THEOREM 3. *In the event that the normed linear space X is strictly convex the y found in Theorem 2 is unique with respect to* $[\ \cdot, \cdot\ ]$.

PROOF. The proof can be found in [2].

Since any separable B-space can be renormed so as to be strictly convex [1] we may in this setting assume the representing element to be unique.

In Giles' Theorem [2] the space is assumed to be both uniformly convex and smooth. In the case that the B-space is smooth then there is a unique semi-inner-product so that since uniformly convex B-spaces are reflexive Giles' theorem is a consequence of Theorem 2.

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