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ON IDEALS HAVING ONLY SMALL PRIME FACTORS

D. G. HAZLEWOOD

1. **Introduction.** Let K be a fixed algebraic number field of degree n, with discriminant Δ and regulator R. Let r_1 and $2r_2$ denote the number of real and complex conjugates, respectively, ω the number of roots of unity, $r = r_1 + r_2 - 1$ the maximum number of independent nontrivial units,

$$d_k = \begin{cases} 1 & \text{if } 1 \leqq k \leqq r_1 \\ 2 & \text{if } r_1 + 1 \leqq k \leqq r_1 + r_2, \end{cases}$$

and

$$\lambda = \frac{2^{r_1+2r_2}\pi^{r_2} R}{\omega d_{r+1}|\Delta|^{1/2}}. \tag{1.1}$$

Let O denote the ring of integers in K, $\mathfrak{a}$ an integral ideal in O, $\mathfrak{p}$ a prime ideal in O, h the number of ideal classes, and $N\mathfrak{a}$ the norm of $\mathfrak{a}$. For real numbers $x \geqq 1$, $t \geqq 0$, and an ideal $\mathfrak{k}$ of O, $\mathfrak{k} \neq (0)$, we denote by $\psi(x^t, x; \mathfrak{k})$ the number of integral ideals $\mathfrak{a}$ of O with $N\mathfrak{a} \leqq x^t$, $(\mathfrak{a}, \mathfrak{k}) = (1)$, and if $\mathfrak{p}$ is a prime ideal dividing $\mathfrak{a}$, then $N\mathfrak{p} \leqq x$.

J. B. Friedlander [1] and J. R. Gillett [2] derived essentially the following estimate for $\psi(x^t, x; \mathfrak{k})$ with *t fixed* and $\mathfrak{k} = (1)$:

$$\psi(x^t, x; \mathfrak{k}) = h\lambda Z_1(t)x^t + O\left(\frac{x^t}{\log x}\right) \tag{1.2}$$

where $Z_1(t)$ is the well-known Dickman function satisfying the differential-difference equation

$$tZ_1'(t) = -Z_1(t-1) \tag{1.3}$$

with initial condition $Z_1(t) = 1$ for $0 \leqq t \leqq 1$ and the constant implied by the use of the O-notation depends not only on the field K, but also on the parameter t.

The object of this report is to establish an asymptotic estimate for $\psi(x^t, x; \mathfrak{k})$ generalizing (1.2) where the O-constant is independent of x, t, and $\mathfrak{k}$ and depends only on the field K unless otherwise indicated.

Also, as a consequence of the theory, we derive an asymptotic esti-

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mate for $\Phi(x^t, x; \mathfrak{k})$, the number of integral ideals $\mathfrak{a}$ in O with $N\mathfrak{a} \leqq x^t$, $(\mathfrak{a}, \mathfrak{k}) = (1)$, and if $\mathfrak{p}$ is a prime ideal dividing $\mathfrak{a}$, then $N\mathfrak{p} > x$.

Before stating the main theorem, we define the following functions. The function $q(\mathfrak{a})$ defined on the ideals of O is a generalization of the Möbius function given by

$$(1.4) \qquad q(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = (1) \\ 0 & \text{if } \mathfrak{p}^2/\mathfrak{a} \\ (-1)^s & \text{if } \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_s, \mathfrak{p}_i \neq \mathfrak{p}_j \text{ for } i \neq j. \end{cases}$$

For M a natural number with $0 \leqq m \leqq M$ and $r = 0$ or 1, the function $\xi_r(m; \mathfrak{k})$, derived in Section 4, is given by

$$(1.5) \quad \xi_r(m; \mathfrak{k}) = \sum_{\mathfrak{b}|\mathfrak{k}} \frac{q_r(\mathfrak{b})}{N\mathfrak{b}} \sum_{s=0}^{m} (-1)^s \binom{m}{s} (\log N\mathfrak{b})^{m-s} \left\{ \frac{(\log N\mathfrak{b})^{s+1}}{s+1} + s! C_s(k) \right\}$$

where

$$(1.6) \qquad q_r(\mathfrak{a}) = \begin{cases} q(\mathfrak{a}) & \text{if } r = 0 \\ |q(\mathfrak{a})| & \text{if } r = 1, \end{cases}$$

and

$$(1.7) \qquad C_s(k) = (-1)^s (h\lambda)^{-1} \left\{ 1 - \sum_{m=0}^{s} \frac{\Gamma_m(k)}{m!} \right\}$$

where $\Gamma_m(K)$ is a generalization of Euler's constant for the algebraic number field K defined by

$$(1.8) \qquad \Gamma_m(k) = \lim_{x \to \infty} \left\{ \sum_{N\mathfrak{a} \leqq x} \frac{(\log N\mathfrak{a})^m}{N\mathfrak{a}} - \frac{h\lambda (\log x)^{m+1}}{m+1} \right\}$$

As proved at the end of Section 4, we point out that

$$(1.9) \qquad \xi_r(m; \mathfrak{k}) = O_m(\log 2N\mathfrak{k} (\log \log 3N\mathfrak{k})^{m+1}).$$

Finally, we define $H_1(x; \mathfrak{k})$ by

$$(1.10) \qquad H_1(x; \mathfrak{k}) = (n\nu(N\mathfrak{k}) + 1)\exp(-C(\log x)^{1/2})$$

where $\nu(m)$ denotes the number of distinct prime factors of the rational integer m, n is the degree of K, and $C = a(4n^{1/2})^{-1}$ for an absolute constant $a > 0$.

THEOREM 1. *If $\mathfrak{k}$ is an arbitrary integral ideal of O, $\mathfrak{k} \neq (0)$, $x \geqq 1$, $t \geqq 0$ are real numbers, and M is an even integer, then*

$$
\begin{aligned}
\psi(x^t, x; \mathfrak{k}) &= h\lambda x^t \Bigg\{ \sum_{\mathfrak{d}|\mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} Z_1(t) \\
&\quad - \sum_{m=0}^{M-1} \frac{(-1)^m Z_1^{(m+1)}(t)}{m!\ (\log x)^{m+1}} \xi_0(m; \mathfrak{k}) \Bigg\} \\
&\quad + O_{M,\epsilon}\Bigg(x^t \Bigg\{ t^{A_1} H_1(x; \mathfrak{k}) (\log x)^{A_2} \\
&\quad + 2^{n\nu(N\mathfrak{k})} x^{-2\epsilon/(n+1)} (1 + Z_1(t)) + \xi_1(M; \mathfrak{k}) \frac{t Z_1^{(M)}(t)}{(\log x)^{M+1}} \Bigg\}\Bigg)
\end{aligned} \tag{1.11}
$$

uniformly in x, t, and $\mathfrak{k}$ for t outside the intervals $(\gamma, \gamma + \epsilon)$ where $\gamma = 1, 2, \cdots, M + 1$, ϵ is an arbitrary positive real number, n is the degree of K, and A_1 and A_2 are absolute constants.

We remark that this asymptotic formula is valid only for $t \leqq (\log x)^{1/2}$ due to the behavior of $Z_1(t)$. We will consider other ranges for t in a later work.

An immediate corollary to Theorem 1 gives a better view of the leading term.

COROLLARY. *If $0 \leqq t \leqq (\log x)^{1/2}$, then*

$$
\begin{aligned}
\psi(x^t, x; \mathfrak{k}) &= h\lambda x^t \sum_{\mathfrak{d}|\mathfrak{k}} \frac{q(\mathfrak{d})}{n\mathfrak{d}} Z_1(t) \\
&\quad + O_\epsilon\Bigg(x^t \Bigg\{ t^{A_1} H_1(x; \mathfrak{k}) (\log x)^{A_2} \\
&\quad + 2^{n\nu(N\mathfrak{k})} x^{-2\epsilon/(n+1)} (1 + z_1(t) + \xi_1(0; \mathfrak{k}) \frac{t Z_1(t)}{\log x} \Bigg\}\Bigg)
\end{aligned} \tag{1.12}
$$

uniformly in x, t, and $\mathfrak{k}$ for t outside the interval $(1, 1 + \epsilon)$ *for arbitrary* $\epsilon > 0$.

The particular interest of (1.12) is that if $2 < t$, then ϵ can be chosen larger than 1 so that if $\nu(N\mathfrak{k}) \ll (2/n(n + 1)) \log x$, the last term of the O-term of (1.12) is dominant to yield

$$
\begin{aligned}
\psi(x^t, x; \mathfrak{k}) &= h\lambda x^t \sum_{\mathfrak{d}|\mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} Z_1(t) \\
&\quad + O_\epsilon\left(x^t \log 2N\mathfrak{k} \log\log 3N\mathfrak{k} \frac{t Z_1(t)}{\log x}\right)
\end{aligned} \tag{1.13}
$$

Specifically, if $\mathfrak{k} = (1)$ and $2 < t \leqq (\log x)^{1/2}$, then

$$\psi(x^t, x; \mathfrak{k}) = h\lambda x^t Z_1(t) + O_\epsilon\left(x^t \frac{tZ_1(t)}{\log x}\right) \tag{1.14}$$

to improve (1.2).

For the function $\Phi(x^t, x; \mathfrak{k})$, we obtain the following asymptotic estimate using Lemma 3.2.

THEOREM 2. *If $\mathfrak{k}$ is an integral ideal of O, $\mathfrak{k} \neq (0)$, $x \geqq 1$, $t \geqq 0$, then*

$$\phi(x^t, x; \mathfrak{k}) = \int_1^t Z_2'(u)x^u \, du + O(x^t t^{A_1} H_1(x; \mathfrak{k})(\log x)^{A_2}) \tag{1.15}$$

uniformly in x, t, and $\mathfrak{k}$ for absolute constants A_1 and A_2 where $Z_2(t)$ *is de Bruijn's function satisfying the equation*

$$tZ_2'(t) = Z_2(t-1) \tag{1.16}$$

with initial condition $Z_2(t) = 1$ for $0 \leqq t \leqq 1$.

2. The General Question. After the manner of B. V. Levin and A. S. Fainleib [6] and [3], [4], we let $x \geqq 1$ and fix

$$0 = B_0 < B_1 < \cdots < B_{k-1} < B_k = +\infty \tag{2.1}$$

for some natural number k. We say that an ideal $\mathfrak{a}$ belongs to $\mathfrak{M}_m$ for $1 \leqq m \leqq k$ if either $\mathfrak{a} = (1)$ or if all the prime ideal factors of $\mathfrak{a}$ have norms greater than $x^{B_{m-1}}$ but not exceeding x^{B_m}. Thus any integral ideal $\mathfrak{a}$ can be uniquely expressed in the form

$$\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_k, \quad \mathfrak{a}_m \in \mathfrak{M}_m, \quad 1 \leqq m \leqq k. \tag{2.2}$$

We let f_m, $1 \leqq m \leqq k$, denote completely multiplicative functions. Then for $t \geqq 0$, we define

$$m_f(x^t) = \sum_{N\mathfrak{a} \leqq x^t} f(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_k}} f_1(N\mathfrak{a}_1) \cdots f_k(N\mathfrak{a}_k). \tag{2.3}$$

If $k = 2$, $B_1 = 1$, and

$$f_1(N\mathfrak{a}) = \begin{cases} 1 & \text{if } N\mathfrak{a} = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

$$f_2(N\mathfrak{a}) = \begin{cases} 1 & \text{if } (\mathfrak{a}, \mathfrak{k}) = (1) \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

then $m_f(x^t) = \psi(x^t, x; \mathfrak{k})$.

Of course, the object is now to estimate the sum $m_f(x^t)$. To do this, we define for each function f_m, the function λ_{f_m} by the following rule:

$$f_m(N\mathfrak{a}) \log N\mathfrak{a} = \sum_{\mathfrak{b} \mid \mathfrak{a}} f_m(N\mathfrak{b})\lambda_{f_m}\left(N \frac{\mathfrak{a}}{\mathfrak{b}}\right). \tag{2.6}$$

Since the functions f_m are completely multiplicative, λ_{f_m} can be characterized as follows:

$$\lambda_{f_m}(N\mathfrak{a}) = \begin{cases} \log N\mathfrak{a} f(N\mathfrak{a}) & \text{if } \mathfrak{a} = \mathfrak{p}^r \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Obviously, there must be some restriction on the functions f_m in order to estimate $m_f(x^t)$. We shall study the behavior of $m_f(x^t)$ for two classes of functions f_m. For $x \geqq 0$, $y \geqq 0$ the first class is determined by the conditional existence of the following functions:

$$L_{f_m}(x, y) = \sum_{\substack{N\mathfrak{p}^r \leqq x \\ N\mathfrak{p} \leqq y}} \lambda_{f_m}(N\mathfrak{p}^r) = \sum_{\substack{N\mathfrak{p}^r \leqq x \\ N\mathfrak{p} \leqq y}} \log N\mathfrak{p} f_m(N\mathfrak{p}^r) \tag{2.8}$$

and

$$\prod_{f_m}(x) = \prod_{N\mathfrak{a} \leqq x} \left(1 + \sum_{r=1}^{\infty} |f_m(N\mathfrak{p}^r)|\right). \tag{2.9}$$

The alternate class of functions will be determined by conditions on the functions:

$$\begin{aligned} L^*_{f_m}(x,y) &= \sum_{\substack{N\mathfrak{p}^r \leqq x \\ N\mathfrak{p} \leqq y}} \lambda_{f_m}(N\mathfrak{p}^r) N\mathfrak{p}^{-r} \\ &= \sum_{\substack{N\mathfrak{p}^r \leqq x \\ N\mathfrak{p} \leqq y}} \log N\mathfrak{p} f_m(N\mathfrak{p}^r) N\mathfrak{p}^{-r} \end{aligned} \tag{2.10}$$

and

$$\prod^*_{f_m}(x) \prod_{N\mathfrak{p}^r \leqq x} = \left(1 + \sum_{r=1}^{\infty} |f_m(N\mathfrak{p}^r)| N\mathfrak{p}^{-r}\right). \tag{2.11}$$

Now we define a class of functions Ω as those functions f_m, $1 \leqq m \leqq k$ satisfying the following requirements:

$$L_{f_m}(x, y) = \tau_m \log(\min(x, y)) + D_m + h_m(x, y) \tag{2.12}$$

where τ_m is a complex number, D_m is an absolute constant, and $h_m(x, y) = O(H(x) + H(y))$, $H(x)$ is a nonincreasing, nonnegative function; and

$$\prod_{f_m}(x) = O(\log^{A_m} x) \tag{2.13}$$

where A_m is an absolute constant.

Similarly, we define the class of functions Ω^* with equivalent conditions on $L^*_{f_m}(x, y)$ and $\prod^*_{f_m}(x)$.

The condition (2.13) will be necessary only if the functions f_m have negative values.

We are now ready to state the basic general result necessary to estimate $m_f(x^t)$. The proof is omitted since it is similar to the proof of Lemma 4 of [4].

FUNDAMENTAL LEMMA. *Suppose the completely multiplicative functions* f_m, $1 \leqq m \leqq k$, *satisfy* (2.12) *and* (2.13). *Then* $m_f(x^t)$ *as defined by* (2.3) *satisfies the following equation:*

$$\begin{aligned} tm_f(x^t) - \int_0^t m_f(x^u)\,du &= \sum_{m=1}^{k} \tau_m \int_{t-B_m}^{t-B_{m-1}} m_f(x^u)\,du \\ &\quad + \frac{D_1}{\log x} m_f(x^t) \\ &\quad + \frac{1}{\log x} \sum_{N\mathfrak{a} \leqq x^t} f(N\mathfrak{a}) h_1\left(\frac{x^t}{N\mathfrak{a}}, x^{B_1}\right) \\ &\quad + \frac{1}{\log x} \sum_{m=2}^{k} \sum_{N\mathfrak{a} \leqq x^{t-B_{m-1}}} f(N\mathfrak{a}) \left\{ h_m\left(\frac{x^t}{N\mathfrak{a}}, x^{B_m}\right) \right. \\ &\quad \left. - h_m\left(\frac{x^t}{N\mathfrak{a}}, x^{B_{m-1}}\right)\right\}. \end{aligned} \tag{2.14}$$

To conclude this section on the general question, we shall also state a result that is proved in Levin and Fainleib [6]:

(Lemma 1.2.1 of [6]) Let $R(t, x)$ be a complex valued function of real variables t and x, integrable with respect to t; let a and $b_1, \cdots, b_m$ be complex numbers, $C_1 \geqq 0$, and $0 \leqq B_0 < B_1 < \cdots < B_m < +\infty$. Suppose further that $R(t, x) = 0$ for $t \leqq 0$ and that

$$\begin{aligned} tR(t, x) - (a + 1)\int_0^t R(u, x)\,du + \sum_{s=1}^{m} b_s \int_{t-B_s}^{t-B_{s-1}} R(u, x)\,du \\ = O(t^{C_1}) \end{aligned} \tag{2.15}$$

uniformly in x. If

$$\int_0^{-n} |R(u, x)|\,du = O(1) \tag{2.16}$$

uniformly in x, where η is a positive constant, then there exists a constant $C_2 > 0$ such that for all $t \geqq \eta$

$$R(t, x) = O(t^{C_2}) \tag{2.17}$$

uniformly in x.

3. **The General Case with $k = 2$.** For all our further considerations, we *fix* $k = 2$ and $B_1 = 1$. Further we let g be a completely multiplicative function, $\mathfrak{f} \neq (0)$ an ideal of O, and define the completely multiplicative function G by the following rule:

$$G(N\mathfrak{a}) = \begin{cases} g(N\mathfrak{a}) & \text{if } (\mathfrak{a}, \mathfrak{f}) = (1) \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We shall now prove our first asymptotic estimate for the special case of $m_f(x^t)$ defined in Section 2.

LEMMA 3.1. *Let G be a function defined by* (3.1) *where g is in Ω with $H(x) = \exp(-A(\log x)^a)$, $A > 0$, $a > 0$. If $x \geqq 1$ and $t \geqq 0$, then*

$$\sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} > x}} G(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} > x \\ (\mathfrak{a}, \mathfrak{f}) = (1)}} g(N\mathfrak{a}) = Z(t) + O(t^{A_1} H(x; \mathfrak{f})(\log x)^{A_2}) \tag{3.2}$$

uniformly in x, t, and $\mathfrak{f}$ where A_1 and A_2 are absolute constants,

$$H(x; \mathfrak{f}) = (n\nu(N\mathfrak{f}) + 1)\exp(-A/2(\log x)^a), \tag{3.3}$$

and $Z(t)$ satisfies the equation

$$tZ'(t) = \tau Z(t-1) \tag{3.4}$$

with initial condition $Z(t) = 1$ for $0 \leqq t \leqq 1$.

PROOF. Let f_1 be defined by (2.4) and $f_2 = G$. It is a straight forward argument similar to the proof of Lemma 2 of [4] that the conditions of the Fundamental Lemma are satisfied with

$$m_f(x^t) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} > x}} G(N\mathfrak{a}),$$

i.e.,

$$L_{f_1}(x, y) = 1$$

and

$$L_{f_2}(x, y) = \tau \log \min(x, y) + D(\mathfrak{f}) + h(x, y; \mathfrak{f}) \tag{3.5}$$

where

$$D(\mathfrak{k}) = D - \sum_{\mathfrak{p}|\mathfrak{k}} \sum_{r=1}^{\infty} \lambda_g(N\mathfrak{p}^r) \tag{3.6}$$

and

$$h(x, y; \mathfrak{k}) = h(x, y) + \sum_{\substack{\mathfrak{p}|\mathfrak{k} \\ N\mathfrak{p}^r > x}} \lambda_g(N\mathfrak{p}^r) + \sum_{\substack{\mathfrak{p}|\mathfrak{k} \\ N\mathfrak{p} > y}} \lambda_g(N\mathfrak{p}^r) - \sum_{\substack{\mathfrak{p}|\mathfrak{k} \\ N\mathfrak{p}^r > x \\ N\mathfrak{p} > y}} \lambda_g(N\mathfrak{p}^r). \tag{3.7}$$

In particular,

$$h(x, y; \mathfrak{k}) = O((n\nu(N\mathfrak{k}) + 1)\exp(-A/2(\log \min(x, y))^a)). \tag{3.8}$$

Hence

$$\begin{aligned} tm_f(x^t) - \int_0^t m_f(x^u)\,du = {} & \tau \int_0^{t-1} m_f(x^u)\,du \\ & + \frac{1}{\log x} \sum_{N\mathfrak{a} \leqq x^{t-1}} f(N\mathfrak{a}) \left\{ h\left(x^t, \frac{x^t}{N\mathfrak{a}}; \mathfrak{k}\right) \right. \\ & \left. - h\left(\frac{x^t}{N\mathfrak{a}}, x; \mathfrak{k}\right) \right\} \end{aligned} \tag{3.9}$$

since $\tau_1 = D_1 = 0$ and $\tau_2 = \tau$, $D_2 = D(\mathfrak{k})$.

Now G satisfies (2.13) so that

$$\sum_{N\mathfrak{a} \leqq x^t} |G(N\mathfrak{a})| = O(t^A \log^A x).$$

Thus (3.9) becomes

$$\begin{aligned} tm_f(x^t) - \int_0^t m_f(x^u)\,du - \tau \int_0^{t-1} m_f(x^u)\,du \\ = O(t^A H(x; \mathfrak{k})(\log x)^{A-1}) \end{aligned} \tag{3.10}$$

uniformly in x, t, and $\mathfrak{k}$.

Now we let $R(t, x; \mathfrak{k})$ be a function such that

$$m_f(x^t) = Z(t) + R(t, x; \mathfrak{k})H(x; \mathfrak{k})(\log x)^{A-1} \tag{3.11}$$

and substitute into (3.10) to get

$$\begin{aligned} tZ(t) - \int_0^t Z(u)\,du - \tau \int_0^{t-1} Z(u)\,du + tR(t, x; \mathfrak{k})H(x; \mathfrak{k})(\log x)^{A-1} \\ - \int_0^t R(u, x; \mathfrak{k})H(x; \mathfrak{k})(\log x)^{A-1}\,du \end{aligned}$$

$$- \tau \int_0^{t-1} R(u, x; \mathfrak{k}) H(x; \mathfrak{k}) (\log x)^{A-1}\, du$$
$$= O(t^A H(x; \mathfrak{k})(\log x)^{A-1}).$$

Hence

$$tR(t, x; \mathfrak{k}) - \int_0^t R(u, x; \mathfrak{k})\, du - \tau \int_0^{t-1} R(u, x; \mathfrak{k})\, du = O(t^A) \tag{3.12}$$

uniformly in x, t, and $\mathfrak{k}$.

We also note that if $t = 1$, then $\int_0^1 |R(u, x; \mathfrak{k})|\, du = O(1)$. Thus, using the Levin and Fainleib result at the end of Section 2, there exists a constant $A_1 > 0$ such that $R(t, x; \mathfrak{k}) = O(t^{A_1})$ uniformly in x, t, and $\mathfrak{k}$, so that (3.11) implies (3.2) to prove Lemma 3.1.

Using Abel's summation on (3.2) we can prove the following lemma where g is in Ω^*. In particular, if $g(N\mathfrak{a}) = 1$, we shall see in Section 4 that $\tau = 1$ and $H(x) = \exp(-a/(2n^{1/2})(\log x)^{1/2})$ so that (3.13) implies (1.15) to prove Theorem 2.

Lemma 3.2. *Let G be a function defined by* (3.1) *where g is in Ω^* with $H(x) = \exp(-A(\log x)^a)$, $A > 0$, $a > 0$. If $x \geqq 1$ and $t \geqq 0$, then*

$$\sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} > x}} G(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} > x \\ (\mathfrak{a},\mathfrak{k})=(1)}} g(N\mathfrak{a})$$
$$= \int_1^t Z'(u) x^u\, du + O(x^t t^{A_1} H(x; \mathfrak{k})(\log x)^{A_2}) \tag{3.13}$$

uniformly in x, t, and $\mathfrak{k}$.

Now we let

$$S(x^t; \mathfrak{k}) = \sum_{N\mathfrak{a} \leqq x^t} G(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ (\mathfrak{a},\mathfrak{k})=(1)}} g(N\mathfrak{a}) \tag{3.14}$$

and let $f_1 = G$ and f_2 be defined by (2.4). Then

$$m_f(x^t) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} > x}} G(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x \\ (\mathfrak{a},\mathfrak{k})=(1)}} g(N\mathfrak{a}). \tag{3.15}$$

The object of the next lemma is to write (3.15) in terms of (3.14) so that we will need only a good estimate for (3.14) to get one for (3.15).

LEMMA 3.3. *Let G be a function defined by* (3.1) *where g is in Ω with $H(x) = \exp(-A(\log x)^a)$, $A > 0$, $a > 0$. If $x \geqq 1$ and $t \geqq 0$, then*

$$\sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x}} G(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x \\ (\mathfrak{a},\mathfrak{f})=(1)}} g(N\mathfrak{a})$$

$$(3.16) \qquad = S(x^t; \mathfrak{f}) + \int_0^t Z'(t-u)S(x^u; \mathfrak{f})\, du + O(t^{A_3}H(x; \mathfrak{f})(\log x)^{A_4})$$

uniformly in x, t, and $\mathfrak{f}$ where $Z(t)$ satisfies the equation

$$(3.17) \qquad tZ'(t) = -\tau Z(t-1)$$

with initial condition $Z(t) = 1$ for $0 \leqq t \leqq 1$ and A_3 and A_4 are absolute constants.

PROOF. Now recall from (3.15) that

$$m_f(x^t) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{a}=\mathfrak{a}_1 \cdot \mathfrak{a}_2}} f_1(N\mathfrak{a}_1)f_2(N\mathfrak{a}_2) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x \\ (\mathfrak{a},\mathfrak{f})=(1)}} g(N\mathfrak{a}).$$

We define functions $\hat{f}_1$ and $\hat{f}_2$ by the relations

$$(3.18) \qquad \sum_{\mathfrak{d}|\mathfrak{f}} f_m(N\mathfrak{d})\hat{f}_m(N\mathfrak{a}/\mathfrak{d}) = f_1(N\mathfrak{a}),\ m = 1, 2.$$

It is easy to see that (3.18) implies that $\hat{f}_1$ is defined by (2.4) and $\hat{f}_2 = f_1$. Hence by Lemma 3.1

$$(3.19) \qquad m_f(x^t) = \hat{Z}(t) + O(t^{A_1}H(x; \mathfrak{f})(\log x)^{A_2})$$

where

$$(3.20) \qquad t\hat{Z}'(t) = \tau\hat{Z}(t-1)$$

with initial condition $\hat{Z}(t) = 1$ for $0 \leqq t \leqq 1$.

Now using essentially the same argument as used in the proof of Theorem 1 of [3] and the fact that

$$(3.21) \qquad \int_0^t Z'(t-u)\hat{Z}'(u)\, du + Z'(t) + \hat{Z}'(t) = 0,$$

we prove that

$$S(x^t; \mathfrak{f}) = m_f(x^t) - \int_0^t Z'(t-u)S(x^u; \mathfrak{f}) + O(t^{A_3}H(x; \mathfrak{f})(\log x)^{A_4})$$

which is (3.16) to prove Lemma 3.3.

Again using Abel's summation, we prove Lemma 3.4 where g is in Ω^*. This functional equation (3.22) will be the initial step toward proving Theorem 1.

LEMMA 3.4. *Let G be a function defined by* (3.1) *where g is in Ω^* with $H(x) = \exp(-A(\log x)^\alpha)$, $A > 0$, $a > 0$. If $x \geqq 1$ and $t \geqq 0$, then*

$$\sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x}} G(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x \\ (\mathfrak{a},\mathfrak{k})=(1)}} g(N\mathfrak{a})$$

$$(3.22) \qquad = S(x^t; \mathfrak{k}) + \int_0^t x^{t-u} Z'(t-u) S(x^u; \mathfrak{k})\, du + O(x^t t^{A_3} H(x; \mathfrak{k})(\log x)^{A_4})$$

uniformly in x, t, and $\mathfrak{k}$ where $Z(t)$ satisfies (3.17), *and A_3 and A_4 are absolute constants.*

4. **The Proof of Theorem 1.** If we define the function $g = 1$ in (3.1), then

$$(4.1) \qquad \psi(x^t, x; \mathfrak{k}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x}} G(N\mathfrak{a}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ \mathfrak{p}|\mathfrak{a} \Rightarrow N\mathfrak{p} \leqq x \\ (\mathfrak{a},\mathfrak{k})=(1)}} 1.$$

From Theorem 190 of Landau [5],

$$(4.2) \qquad \sum_{N\mathfrak{a} \leqq x} \log N\mathfrak{p} = x + O\left(x \exp\left(-a/n^{1/2}(\log x)^{1/2}\right)\right)$$

where $a > 0$ is an absolute constant and n is the degree of K. Thus it is easy to see that

$$(4.3) \qquad \sum_{N\mathfrak{a} \leqq x} \frac{\log N\mathfrak{p}}{N\mathfrak{p}} = \log x + D + O\left(\exp\left(-a/2n^{1/2}(\log x)^{1/2}\right)\right)$$

where D is an absolute constant.

Hence with $g = 1$

$$(4.4) \qquad L_g^*(x, y) = \log(\min(x, y)) + D_1 + h_1(x, y)$$

where D_1 is an absolute constant and

$$(4.5) \qquad h_1(x, y) = O(H_1(x) + H_1(y))$$

where

$$(4.6) \qquad H_1(x) = \exp\left(-a/(2n^{1/2})(\log x)^{1/2}\right).$$

Further, we note that

$$\prod_g{}^*(x) = \prod_{N\mathfrak{p} \leqq x} \left(1 + \sum_{r=1}^{\infty} N\mathfrak{p}^{-r}\right) = O(\log x). \tag{4.7}$$

Therefore the conditions of Lemma 3.4 are satisfied with $g = 1$ so that

$$\begin{aligned} \psi(x^t, x; \mathfrak{k}) = S_1(x^t; \mathfrak{k}) + \int_0^t x^{t-u} Z_1{}'(t-u) S_1(x^u; \mathfrak{k})\, du \\ + O(x^t t^{A_1} H_1(x; \mathfrak{k})(\log x)^{A_2}) \end{aligned} \tag{4.8}$$

uniformly in x, t, and $\mathfrak{k}$ where A_1 and A_2 are absolute constants, $H_1(x; \mathfrak{k})$ is given by (1.10), $Z_1(t)$ by (1.3), and

$$S_1(x^t; \mathfrak{k}) = \sum_{\substack{N\mathfrak{a} \leqq x^t \\ (\mathfrak{a}, \mathfrak{k}) = (1)}} 1. \tag{4.9}$$

As stated previously, a good estimate for $S_1(x^t; \mathfrak{k})$ will yield a good estimate for $\psi(x^t, x; \mathfrak{k})$. For the estimate for $S_1(x^t; \mathfrak{k})$ we define the following functions:

$$S_1(x) = \sum_{N\mathfrak{a} \leqq x} 1 \tag{4.10}$$

and

$$R_1(x) = (h\lambda x)^{-1} \{h\lambda x - S_1(x)\} \tag{4.11}$$

where h is the number of ideal classes of K and λ is the constant given by (1.1).

From Theorem 210 of Landau [5],

$$R_1(x) = O(x^{-2/(n+1)}) \tag{4.12}$$

where n is the degree of K.

Using the function q given by (1.4), we see that

$$\begin{aligned} S_1(x^t; \mathfrak{k}) &= \sum_{\mathfrak{d} | \mathfrak{k}} q(\mathfrak{d}) S_1(x^t / N\mathfrak{d}) \\ &= h\lambda x^t \left\{ \sum_{\mathfrak{d} | \mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} - \sum_{\mathfrak{d} | \mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} R_1(x^t / N\mathfrak{d}) \right\}. \end{aligned}$$

We define

$$R_1(x^t; \mathfrak{k}) = \sum_{\mathfrak{d} | \mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} R_1(x^t / N\mathfrak{d}) \tag{4.13}$$

so that

$$S_1(x^t; \mathfrak{k}) = h\lambda x^t \left\{ \sum_{\mathfrak{d}|\mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} - R_1(x^t; \mathfrak{k}) \right\}. \tag{4.14}$$

Substituting (4.14) in (4.8) we then use basically the same argument beginning with (7.7) of [3] to show that

$$\begin{aligned} \psi(x^t, x; \mathfrak{k}) = h\lambda x^t \quad & \left\{ \sum_{\mathfrak{d}|\mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} Z_1(t) \right. \\ & \left. - \sum_{m=0}^{M-1} \frac{(-1)^m}{m!} \frac{Z_1{}^{(m+1)}(t)}{(\log x)^{m+1}} \int_1^\infty \frac{(\log u)^m R_1(u; \mathfrak{k})}{u} du \right\} \\ & + O_{M,\epsilon}\Bigg(x^t \Bigg\{ t^{A_1} H_1(x; \mathfrak{k})(\log x)^{A_2} + 2^{n\nu(N\mathfrak{k})} x^{-2\epsilon/(n+1)}(1 + Z_1(t)) \\ & + \frac{tZ_1{}^{(M)}(t)}{(\log x)^{M+1}} \int_1^\infty \frac{(\log u)^M |R_1(u; \mathfrak{k})|}{u} du \Bigg\}\Bigg). \end{aligned} \tag{4.15}$$

To conclude the proof of Theorem 1 we must show

$$\xi_0(m; \mathfrak{k}) = \int_1^\infty \frac{(\log u)^m R_1(u; \mathfrak{k})}{u} du \tag{4.16}$$

which in turn implies that

$$\xi_1(M; \mathfrak{k}) = \int_1^\infty \frac{(\log u)^M |R_1(u; \mathfrak{k})|}{u} du. \tag{4.17}$$

To accomplish this, we use the following argument. Using (4.13) we see that

$$\int_1^\infty \frac{(\log u)^m R_1(u; \mathfrak{k})}{u} du = \sum_{\mathfrak{d}|\mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} \int_1^\infty \frac{(\log u)^m R_1(u/N\mathfrak{d})}{u} du \tag{4.18}$$

and changing the variable of integration the right hand side of (4.18) is equal to

$$\sum_{\mathfrak{d}|\mathfrak{k}} \frac{q(\mathfrak{d})}{N\mathfrak{d}} \sum_{s=0}^{m} \binom{m}{s} (\log N\mathfrak{d})^{m-s} \int_{1/N\mathfrak{d}}^\infty \frac{(\log u)^s R_1(u)}{u} du. \tag{4.19}$$

Breaking the integral in (4.19) into two parts we have

$$\int_{1/N\mathfrak{d}}^\infty \frac{(\log u)^s R_1(u)}{u} du = \frac{(-1)^s(\log N\mathfrak{d})^{s+1}}{s+1} + \int_1^\infty \frac{(\log u)^s R_1(u)}{u} du. \tag{4.20}$$

By Abel's summation for s a nonnegative integer,

$$\sum_{N\mathfrak{a}\leqq x} \frac{(\log N\mathfrak{a})^s}{N\mathfrak{a}} = \frac{h\lambda(\log x)^{s+1}}{s+1} - h\lambda(\log x)^s R_1(x)$$

$$+ sh\lambda \int_1^x \frac{(\log u)^{s-1}R_1(u)}{u}\, du - h\lambda \int_1^x \frac{(\log u)^s R_1(u)}{u}\, du$$

and using (4.12) we have for an arbitrary constant $\epsilon > 0$

$$h\lambda \int_x^\infty \frac{(\log u)^s R_1(u)}{x}\, du = O(x^{-\epsilon}),$$

$$sh\lambda \int_x^\infty \frac{(\log u)^s R_1(u)}{u}\, du = O_s(x^{-\epsilon}),$$

and

$$h\lambda(\log x)^s R_1(x) = O(x^{-\epsilon}).$$

Hence for s fixed, we see that

$$\lim_{x\to\infty} \left\{ \sum_{N\mathfrak{a}\leqq x} \frac{(\log N\mathfrak{a})^s}{N\mathfrak{a}} - \frac{h\lambda(\log x)^{s+1}}{s+1} \right\}$$
$$= sh\lambda \int_1^\infty \frac{(\log u)^{s-1}R_1(u)}{u}\, du - h\lambda \int_1^\infty \frac{(\log u)^s R_1(u)}{u}\, du, \tag{4.21}$$

but from this and (1.8) we see that

$$\Gamma_s(K) = sh\lambda \int_1^\infty \frac{(\log u)^{s-1}R_1(u)}{u}\, du$$
$$- h\lambda \int_1^\infty \frac{(\log u)^s R_1(u)}{u}\, du. \tag{4.22}$$

If we extend the definition (1.7) to $C_{-1}(K) = -1$, we can se from (4.22) that

$$\frac{(-1)^s}{s!} \int_1^\infty \frac{(\log u)^s R_1(u)}{u}\, du = (-1)^s (h\lambda)^{-1} \left\{1 - \sum_{m=0}^{s} \frac{\Gamma_m(K)}{m!}\right\} \tag{4.23}$$

so that $C_s(K)$ as defined by (1.7) is equal to

$$\frac{(-1)^s}{s!} \int_1^\infty \frac{(\log u)^s R_1(u)}{u}\, du. \tag{4.24}$$

Using (4.24) and (4.20) in (4.18) we have (4.16).
Finally we shall prove (1.9) that

$$\xi_r(m;\mathfrak{k}) = O_m(\log 2N\mathfrak{k}\log\log 3N\mathfrak{k})^{m+1}).$$

To do this we define the function

$$h_r(z) = \sum_{\mathfrak{p}|\mathfrak{k}} (-\log N\mathfrak{p}) \frac{q_r(\mathfrak{p})}{(N\mathfrak{p}^Z + q_r(\mathfrak{p}))} \tag{4.25}$$

for any complex number z, $r = 0$ or 1, and q_r defined by (1.6). Then for any natural number m, there exists integers a_{mj}, $1 \leqq j \leqq m+1$ with $a_{m1} = 1$ such that

$$h_r^{(m)}(z) = \sum_{\mathfrak{p}|\mathfrak{k}} (-\log N\mathfrak{p})^{m+1} q_r(\mathfrak{p}) \sum_{j=1}^{m+1} \frac{a_{mj}}{(N\mathfrak{p}^z + q_r(\mathfrak{p}))^j} \tag{4.26}$$

where $h_r^{(m)}(z)$ denotes the m-th derivative of $h_r(z)$ with respect to z. This is seen by a straightforward argument using induction on m.
Now we consider the function

$$g_r(z) = \sum_{\mathfrak{d}|\mathfrak{k}} q_r(\mathfrak{d})N\mathfrak{d}^{-z} = \prod_{\mathfrak{p}|\mathfrak{k}} (1 + q_r(\mathfrak{d})N\mathfrak{d}^{-z}). \tag{4.27}$$

Taking the logarithmic derivative

$$g_r'(z) = h_r(z)g_r(z) \tag{4.28}$$

with

$$g_r'(z) = \sum_{\mathfrak{d}|\mathfrak{k}} q_r(\mathfrak{d})N\mathfrak{d}^{-z}(-\log N\mathfrak{d}). \tag{4.29}$$

Using Leibnitz's rule we have

$$\begin{aligned} &\sum_{\mathfrak{d}|\mathfrak{k}} \frac{q_r(\mathfrak{d})}{N\mathfrak{d}}(\log N\mathfrak{d})^m \\ &= \sum_{s=0}^{m-1} \binom{m-1}{s}\left(\sum_{\mathfrak{d}|\mathfrak{k}} \frac{q_r(\mathfrak{d})}{N\mathfrak{d}}(\log N\mathfrak{d})^s \right)(-1)^{m-s}h_r^{(m-s-1)}(1) \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} h_r^{(s)}(1) &= O\left(\sum_{\mathfrak{p}|\mathfrak{k}} \frac{(\log N\mathfrak{p})^{s+1}}{N\mathfrak{p}} \right) \\ &= O_s\left((\log\log 3N\mathfrak{k})^{s+1} \right). \end{aligned} \tag{4.31}$$

Hence from (4.28), (4.30), and (4.31) we see that

$$\sum_{\mathfrak{d}|\mathfrak{k}} \frac{q_r(\mathfrak{d})}{N\mathfrak{d}}(\log N\mathfrak{d})^m = O_m(g_r(1)(\log\log 3N\mathfrak{k})^{m+1}) \tag{4.32}$$

where

$$g_r(1) = O(\log 2N\mathfrak{k}). \tag{4.33}$$

Therefore writing $\xi_r(m; \mathfrak{k})$ as

$$\frac{1}{m+1} \sum_{\mathfrak{d} \mid \mathfrak{k}} \frac{q_r(\mathfrak{d})}{N\mathfrak{d}} (\log N\mathfrak{d})^{m+1} + \sum_{s=0}^{m} \frac{m!}{(m-s)!} C_s(k) \sum_{\mathfrak{d} \mid \mathfrak{k}} \frac{q_r(\mathfrak{d})}{N\mathfrak{d}} (\log N\mathfrak{d})^{m-s} \tag{4.34}$$

we see that $\xi_r(m; \mathfrak{k})$ is $O_m(\log 2N\mathfrak{k}(\log\log 3N\mathfrak{k})^{m+1})$ to prove (1.9).

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Southwest Texas State University, San Marcos, Texas 78666.