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TWO GENERALIZATIONS OF THE GRONWALL INEQUALITY BY PRODUCT INTEGRATION

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ABSTRACT. Definitions and integrals are of the subdivision-refinement type, and functions are from R to R or $R \times R$ to R, where R denotes the set of real numbers. Further, c is a nonnegative constant, h is a bounded function from R to R, each of F and G is a nonnegative function from $R \times R$ to R, each of $\int_a^b F$ and $\int_a^b G$ exists and all other stated integrals are assumed to exist. Two integral inequalities are established. First, if

$$h(t) \leqq c + \int_a^t h(u)G(u,v) + \int_a^t \left[\int_a^u h(r)F(r,s) \right] G(u,v)$$

for $a \leqq t \leqq b$, then

$$h(t) \leqq c\left\{1 + \int_a^t [{}_a\Pi^u (1 + F + G)]\, G(u,v)\right\}$$

for $a \leqq t \leqq b$. Second, if

$$h(t) \leqq c + \int_a^t h(u)G(u,v) + \int_a^t \left[\int_a^u h(r)G(r,s) \right] F(u,v)$$

for $a \leqq t \leqq b$, then

$$h(t) \leqq c\left\{1 + \int_a^t G(u,v)\, [{}_v\Pi^t(1 + F + G)] \right\}$$

for $a \leqq t \leqq b$.

Introduction. In a recent paper, B. G. Pachpatte [14, Theorem 1, p. 758] obtains the following extension of the Gronwall inequality.

THEOREM. *If c is a nonnegative constant, each of f, g and h is a real-valued nonnegative continuous function and*

$$h(t) \leqq c + \int_a^t h(s)g(s)\,ds + \int_a^t \left[\int_a^s h(\tau)f(\tau)\,d\tau \right] g(s)\,ds$$

for $a \leqq t \leqq b$, then

$$h(t) \leqq c\left\{1 + \int_a^t \left[\exp \int_a^s f(\tau) + g(\tau)\,d\tau \right] g(s)\,ds \right\}$$

for $a \leqq t \leqq b$.

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In this paper, related results are obtained for integrals of interval functions. This development involves the use of product integrals instead of exponentials in obtaining the desired bound. Two integral inequalities are established. First, if c is a nonnegative constant, h is a function from R to R, each of F and G is a nonnegative function from $R \times R$ to R, each of $\int_a^b F$ and $\int_a^b G$ exists, all other stated integrals are assumed to exist and

$$h(t) \leqq c + \int_a^t h(u)G(u, v) + \int_a^t \left[\int_a^u h(r)F(r, s) \right] G(u, v)$$

for $a \leqq t \leqq b$, then

$$h(t) \leqq c\left\{1 + \int_a^t [{}_a\Pi^u (1 + F + G)]\, G(u, v)\right\}$$

for $a \leqq t \leqq b$. Second, if c, h, F and G are defined as before and

$$h(t) \leqq c + \int_a^t h(u)G(u, v) + \int_a^t \left[\int_a^u h(r)G(r, s) \right] F(u, v)$$

for $a \leqq t \leqq b$, then

$$h(t) \leqq c \left\{1 + \int_a^t G(u, v)\, [{}_v\Pi^t (1 + F + G)] \right\}$$

for $a \leqq t \leqq b$. When f and g are continuous functions,

$$G(u, v) = g(u)(v - u) \quad \text{and} \quad F(u, v) = f(u)(v - u),$$

the first result reduces to the previously stated theorem by B. G. Pachpatte.

Background. Definitions and integrals are of the subdivision-refinement type, and functions are from R to R or $R \times R$ to R, where R denotes the set of real numbers. Further, interval functions are assumed to be defined only for elements $\{x, y\}$ of $R \times R$ such that $x < y$. Lower case letters are used to denote functions defined on R, and upper case letters are used to denote functions defined on $R \times R$. If h and G are functions defined on R and $R \times R$, respectively, and $\{x_i\}_{i=0}^n$ is a subdivision of some interval $[a, b]$, then $h_i = h(x_i)$ for $i = 0, 1, \cdots, n$ and $G_i = G(x_{i-1}, x_i)$ for $i = 1, 2, \cdots, n$.

The statement that $\int_a^b G$ exists means there exists a number L such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\left| L - \sum_{i=1}^{n} G_i \right| < \epsilon.$$

Similarly, ${}_a\Pi^b\,(1 + G)$ exists if there exists a number L such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\left| L - \prod_{i=1}^{n} (1 + G_i) \right| < \epsilon.$$

Also, G has bounded variation on $[a, b]$ if there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\sum_{i=1}^{n} |G_i| < B.$$

If G has bounded variation on $[a, b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y\,(1 + G)$ exists for $a \leqq x \leqq y \leqq b$.

For convenience in notation, we adopt the conventions that

$$\prod_{i=p+1}^{p} (1 + G_i) = 1 \quad \text{and} \quad \sum_{i=p+1}^{p} G_i = 0.$$

These conventions simplify the representation of certain expressions that occur later in the paper.

Right and left integrals arise in this paper. These are denoted by $\int_a^b G(u, v)h(v)$ and $\int_a^b h(u)G(u, v)$, respectively. Suppose $\{x_i\}_{i=0}^n$ denotes a subdivision of some interval $[a, b]$. Then, the preceding right and left integrals have approximating sums of the form

$$\sum_{i=1}^{n} G_i h_i \quad \text{and} \quad \sum_{i=1}^{n} h_{i-1} G_i,$$

respectively. Through the paper, several different functions are involved in right or left integrals. For example, integrals of the form

$$\int_a^b \left[\int_a^u h(r)F(r, s) \right] G(u, v)$$

and

$$\int_a^b G(u, v)\,[{}_v\Pi^b\,(1 + F + G)]$$

arise. Here, the approximating sums are of the form

$$\sum_{i=1}^{n} \left[\int_a^{x_{i-1}} h(r)F(r,s) \right] G_i$$

and

$$\sum_{i=1}^{n} G_i [{}_{x_i}\Pi^b (1 + F + G)],$$

respectively. Representations involving right and left integrals are necessary due to possible discontinuities of the functions involved.

If $\int_a^b G$ exists, then $\int_a^b |G(u, v) - \int_u^v G|$ exists and is zero. This result is due to A. Kolmogoroff [11, p. 669]. Further, a proof of it is also given by W. D. L. Appling [1, Theorem 1, p. 155]. This result is of use in switching between difference inequalities and integral inequalities. It is used extensively in the development of the results of this paper.

Additional background on product integration can be obtained in papers by P. R. Masani [13], J. S. MacNerney [12], B. W. Helton [5], J. C. Helton [8, 9] and J. C. Helton and S. Stuckwisch [10]. An extensive survey of Gronwall inequalities is provided by P. R. Beesack [2]. Further, recent papers by J. Chandra and B. A. Fleishman [3] and B. W. Helton [7] also provide background on this inequality. For monotone maps on partially ordered sets, the reader is referred to a paper by A. Tarski [15].

Statement and Proof of Results. The first integral inequality is now established. One lemma is needed.

LEMMA 1. *Suppose c is a positive constant, h is a bounded function from R to R, each of F and G is a nonnegative function from $R \times R$ to R, each of $\int_a^b F$ and $\int_a^b G$ exists, each of*

$$\int_a^b h(u)G(u,v) \quad \textit{and} \quad \int_a^b \left[\int_a^u h(r)F(r,s) \right] G(u,v)$$

exists and

$$h(t) \leqq c + \int_a^t h(u)G(u,v) + \int_a^t \left[\int_a^u h(r)F(r,s) \right] G(u,v)$$

for $a \leqq t \leqq b$. Then, if $a < t \leqq b$ and $\{x_i\}_{i=0}^n$ is a subdivision of $[a, t]$, the following inequality holds:

$$h(t) \leqq c \left\{1 + \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j)\right] G_i \right\}$$

$$+ \sum_{i=1}^{n} (c_i + d_i)H(i, n),$$

where $H(n, n) = 1$ *and*

$$c_i = \int_{x_{i-1}}^{x_i} h(u)G(u, v) - h_{i-1} G_i \quad \textit{for } i = 1, 2, \cdots, n,$$

$$d_i = \int_{x_{i-1}}^{x_i} \left[\int_a^u h(r)F(r, s)\right] G(u, v)$$

$$- \left[\sum_{j=1}^{i-1} h_{j-1}F_j\right] G_i \quad \textit{for } i = 1, 2, \cdots, n,$$

and

$$H(i, n) = H(i, n-1) + H(i, n-1)G_n$$

$$+ \left[\sum_{j=i}^{n-2} H(i, j)F_{j+1} \right] G_n \quad \textit{for } i = 1, 2, \cdots, n-1.$$

Proof. This lemma is established by induction. We initially note that $h(a) \leqq c$. Our induction argument begins with $n = 1$. Suppose $a < t \leqq b$ and $\{x_i\}_{i=0}^{1}$ is a subdivision of $[a, t]$. Then,

$$h(t) \leqq c + \int_a^t h(u)G(u, v)$$

$$+ \int_a^t \left[\int_a^u h(r)F(r, s)\right] G(u, v)$$

$$= c + h_0G_1 + c_1 + d_1$$

$$\leqq c + c\, G_1 + c_1 + d_1$$

$$= c \left\{1 + \sum_{i=1}^{1} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j)\right] G_i \right\}$$

$$+ \sum_{i=1}^{1} (c_i + d_i)H(i, 1).$$

Therefore, the result is true for $n = 1$.

The result is now assumed to be true for all positive integers less than or equal to n. That is, if $a < t \leqq b$, $1 \leqq m \leqq n$ and $\{x_i\}_{i=0}^m$ is a subdivision of $[a, t]$, then

$$
\begin{aligned}
h(t) &\leqq c + \int_a^t h(u)G(u, v) \\
&\quad + \int_a^t \left[\int_a^u h(r)F(r, s) \right] G(u, v) \\
&\leqq c \left\{ 1 + \sum_{i=1}^m \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] G_i \right\} \\
&\quad + \sum_{i=1}^m (c_i + d_i) H(i, m).
\end{aligned} \tag{1}
$$

The desired inequality is next established for $n + 1$.

Suppose $a < t \leqq b$ and $\{x_i\}_{i=0}^{n+1}$ is a subdivision of $[a, t]$. In order to simplify the proof of the final result, several intermediate results are established.

If $\{A_i\}_{i=1}^m$ and $\{B_i\}_{i=1}^m$ are nonnegative sequences, then

$$
\prod_{j=1}^m (1 + A_j + B_j) = 1 + \sum_{i=1}^m \left[\prod_{j=1}^{i-1} (1 + A_j + B_j) \right] [A_i + B_i], \tag{2}
$$

and

$$
1 + \sum_{i=1}^m \left[\prod_{j=1}^{i-1} (1 + A_j + B_j) \right] B_i \leqq \prod_{j=1}^m (1 + A_j + B_j). \tag{3}
$$

These relations can be established by induction. By employing the two preceding relations, we have that

$$
\begin{aligned}
& c \left\{ 1 + \sum_{i=1}^n \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] G_i \right\} G_{n+1} \\
&\quad + \sum_{i=1}^n c \left\{ 1 + \sum_{j=1}^{i-1} \left[\prod_{k=1}^{j-1} (1 + F_k + G_k) \right] G_j \right\} F_i G_{n+1} \\
&\leqq c \left\{ 1 + \sum_{i=1}^n \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] G_i \right\} G_{n+1}
\end{aligned} \tag{4}
$$

$$+ \sum_{i=1}^{n} c \left\{ \prod_{j=1}^{i-1} (1 + F_j + G_j) \right\} F_i G_{n+1}$$

[From (3)]

$$= c \left\{ 1 + \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] [F_i + G_i] \right\} G_{n+1}$$

$$= c \left[\prod_{j=1}^{n} (1 + F_j + G_j) \right] G_{n+1}.$$

[From (2)]

This relation is used in the next paragraph.

By using (1) and the relation from the preceding paragraph, we have that

$$\int_{x_n}^{x_{n+1}} h(u)G(u,v) + \int_{x_n}^{x_{n+1}} \left[\int_a^u h(r)F(r,s) \right] G(u,v)$$

$$= h_n G_{n+1} + c_{n+1} + \left[\sum_{i=1}^{n} h_{i-1} F_i \right] G_{n+1} + d_{n+1}$$

[Definitions of c_{n+1} and d_{n+1}]

$$\leqq \left(c \left\{ 1 + \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] G_i \right\} + \sum_{i=1}^{n} (c_i + d_i) H(i,n) \right) G_{n+1} + c_{n+1}$$

(5)

$$+ \left(\sum_{i=1}^{n} \left[c \left\{ 1 + \sum_{j=1}^{i-1} \left[\prod_{k=1}^{j-1} (1 + F_k + G_k) \right] G_j \right\} + \sum_{j=1}^{i-1} (c_j + d_j) H(j, i-1) \right] F_i \right) G_{n+1} + d_{n+1}$$

[From (1)]

$$\leqq c \left[\prod_{j=1}^{n} (1 + F_j + G_j) \right] G_{n+1}$$

$$+ \left[\sum_{i=1}^{n} (c_i + d_i)H(i, n) \right] G_{n+1} + c_{n+1}$$

$$+ \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} (c_j + d_j)H(j, i-1) \right] F_i G_{n+1} + d_{n+1}.$$

[From (4)]

It can be established by induction that

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} (c_j + d_j)H(j, i-1) \right] F_i = \sum_{i=1}^{n-1} (c_i + d_i) \sum_{j=i}^{n-1} H(i, j)F_{j+1}. \tag{6}$$

Now, by using the preceding relation, we have that

$$\begin{aligned} &\sum_{i=1}^{n} (c_i + d_i)H(i, n) \\ &\quad + \left[\sum_{i=1}^{n} (c_i + d_i)H(i, n) \right] G_{n+1} + c_{n+1} \\ &\quad + \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} (c_j + d_j)H(j, i-1) \right] F_i G_{n+1} + d_{n+1} \\ &= \sum_{i=1}^{n} (c_i + d_i)H(i, n) \\ &\quad + \left[\sum_{i=1}^{n} (c_i + d_i)H(i, n) \right] G_{n+1} + c_{n+1} \\ &\quad + \left[\sum_{i=1}^{n-1} (c_i + d_i) \sum_{j=i}^{n-1} H(i, j)F_{j+1} \right] G_{n+1} + d_{n+1} \end{aligned} \tag{7}$$

[From (6)]

$$\begin{aligned} &= \sum_{i=1}^{n} [c_i + d_i] \Big[H(i, n) + H(i, n)\, G_{n+1} \\ &\quad + \sum_{j=i}^{n-1} H(i, j)F_{j+1}G_{n+1} \Big] + [c_{n+1} + d_{n+1}] \end{aligned}$$

$$= \sum_{i=1}^{n+1} (c_i + d_i)H(i, n + 1).$$

[Definition of $H(i, n + 1)$]

We are now prepared to complete the argument for $n + 1$. By using the results of the two preceding paragraphs, we have that

$$h(t) \leqq c + \int_a^{x_{n+1}} h(u)G(u, v)$$

$$+ \int_a^{x_{n+1}} \left[\int_a^u h(r)F(r, s) \right] G(u, v)$$

$$\leqq c \left\{1 + \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j)\right] G_i\right\}$$

$$+ \sum_{i=1}^{n} (c_i + d_i)H(i, n) + \int_{x_n}^{x_{n+1}} h(u)G(u, v)$$

$$+ \int_{x_n}^{x_{n+1}} \left[\int_a^u h(r)F(r, s)\right] G(u, v)$$

[From (1)]

$$\leqq c \left\{1 + \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j)\right] G_i \right\}$$

$$+ \sum_{i=1}^{n} (c_i + d_i)H(i, n)$$

$$+ c \left[\prod_{j=1}^{n} (1 + F_j + G_j)\right] G_{n+1}$$

$$+ \left[\sum_{i=1}^{n} (c_i + d_i)H(i, n)\right] G_{n+1} + c_{n+1}$$

$$+ \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} (c_j + d_j)H(j, i - 1)\right] F_i G_{n+1} + d_{n+1}$$

[From (5)]

$$= c \left\{1 + \sum_{i=1}^{n+1} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j)\right] G_i \right\}$$

$$+ \sum_{i=1}^{n+1} (c_i + d_i)H(i, n+1).$$

[From (7)]

Therefore, the result is true for $n + 1$. Hence, the proof of the lemma is completed.

THEOREM 1. *If c is a positive constant, h is a bounded function from R to R, each of F and G is a nonnegative function from $R \times R$ to R, each of $\int_a^b F$ and $\int_a^b G$ exists, each of*

$$\int_a^b h(u)G(u, v) \text{ and } \int_a^b \left[\int_a^u h(r)F(r, s)\right] G(u, v)$$

exists and

$$h(t) \leqq c + \int_a^t h(u)G(u, v) + \int_a^t \left[\int_a^u h(r)F(r, s)\right] G(u, v)$$

for $a \leqq t \leqq b$, then

$$h(t) \leqq c\left\{1 + \int_a^t [{}_a\Pi^u(1 + F + G)]\, G(u, v)\right\}$$

for $a \leqq t \leqq b$.

PROOF. Suppose the conclusion is false. Then, there exists a number t, $a < t \leqq b$, such that

$$h(t) > c\left\{1 + \int_a^t [{}_a\Pi^u(1 + F + G)]\, G(u, v)\right\}.$$

Let d denote the positive number such that

$$d = h(t) - c\left\{1 + \int_a^t [{}_a\Pi^u(1 + F + G)]\, G(u, v)\right\}.$$

We note that the existence of

$${}_a\Pi^u(1 + F + G) \text{ and } \int_a^t [{}_a\Pi^u(1 + F + G)]\, G(u, v)$$

can be established from the existence of $\int_a^b F$ and $\int_a^b G$.

It follows from the existence of the integrals involved that there exists a subdivision D_1 of $[a, t]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

$$\left| \int_a^t \left[{}_a\Pi^u(1 + F + G)\right] G(u, v) - \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] G_i \right| < d(2c)^{-1}.$$

Let β represent a nonnegative function of bounded variation from $R \times R$ to R such that, if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, t]$ and $1 \leqq i \leqq n$, then

$$2(|F_i| + |G_i|) < \beta_i.$$

There exist a subdivision D_2 of $[a, t]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then

$$\prod_{i=1}^{n} (1 + \beta_i) < B.$$

Since $\int_a^t h(u)G(u, v)$ exists, it follows that

$$\int_a^t \left| \int_x^y h(u)G(u, v) - h(x)G(x, y) \right|$$

exists and is zero. Hence, there exists a subdivision D_3 of $[a, t]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then

$$\sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} h(u)G(u, v) - h_{i-1}G_i \right| < d(4B)^{-1}.$$

Since $\int_a^t [\int_a^u h(r)F(r, s)] G(u, v)$ exists, it follows that

$$\int_a^t \left| \int_x^y \left[\int_a^u h(r)F(r, s) \right] G(u, v) - \left[\int_a^x h(r)F(r, s) \right] G(x, y) \right|$$

exists and is zero. From this, the existence of $\int_a^b h(r)F(r, s)$ and the bounded variation of G, it follows that there exists a subdivision D_4 of $[a, t]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_4, then

$$\sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} \left[\int_a^u h(r)F(r, s) \right] G(u, v) - \left[\sum_{j=1}^{i-1} h_{j-1}F_j \right] G_i \right| < d(4B)^{-1}.$$

Let D denote the subdivision $\bigcup_{i=1}^4 D_i$ of $[a, t]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. It follows from Lemma 1 that

$$h(t) \leqq c \left\{ 1 + \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] G_i \right\}$$

$$+ \sum_{i=1}^{n} (c_i + d_i) H(i, n),$$

where c_i, d_i and $H(i, n)$ are defined in Lemma 1. It follows from the manner in which the $H(i, n)$ are defined that

$$H(i, n) \leqq \prod_{j=1}^{n} (1 + \beta_j) < B.$$

Thus,

$$\begin{aligned} h(t) \leqq\ & c \left\{1 + \int_a^t [{}_a\Pi^u (1 + F + G)]\, G(u, v) \right\} \\ & + c \left| \sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j + G_j) \right] G_i \right. \\ & \left. - \int_a^t [{}_a\Pi^u (1 + F + G)]\, G(u, v) \right| \\ & + \sum_{i=1}^{n} |c_i|\, |H(i, n)| + \sum_{i=1}^{n} |d_i|\, |H(i, n)| \\ <\ & [h(t) - d] + c[d(2c)^{-1}] \\ & + B \sum_{i=1}^{n} |c_i| + B \sum_{i=1}^{n} |d_i| \\ <\ & [h(t) - d] + d/2 + B[d(4B)^{-1}] + B[d(4B)^{-1}] \\ =\ & h(t). \end{aligned}$$

This is a contradiction. Therefore, the desired inequality is established. This completes the proof of Theorem 1.

The second integral inequality is now established. One lemma is needed.

LEMMA 2. *Suppose c is a positive constant, h is a bounded function from R to R, each of F and G is a nonnegative function from $R \times R$ to R, each of $\int_a^b F$ and $\int_a^b G$ exists, each of*

$$\int_a^b h(u)G(u, v) \quad \text{and} \quad \int_a^b \left[\int_a^u h(r)G(r, s) \right] F(u, v)$$

exists and

$$h(t) \leqq c + \int_a^t h(u)G(u,v)$$

$$+ \int_a^t \left[\int_a^u h(r)G(r,s) \right] F(u,v)$$

for $a \leqq t \leqq b$. *Then, if* $a < t \leqq b$ *and* $\{x_i\}_{i=0}^n$ *is a subdivision of* $[a,t]$, *the following inequality holds:*

$$h(t) \leqq c \left\{ 1 + \sum_{i=1}^n G_i \left[\prod_{j=i+1}^n (1 + F_j + G_j) \right] \right\}$$

$$+ \sum_{i=1}^n (c_i + d_i) H(i,n),$$

where $H(n,n) = 1$ *and*

$$c_i = \int_{x_{i-1}}^{x_i} h(u)G(u,v) - h_{i-1}G_i \qquad \text{for } i = 1, 2, \cdots, n$$

$$d_i = \int_{x_{i-1}}^{x_i} \left[\int_a^u h(r)G(r,s) \right] F(u,v)$$

$$- \left[\sum_{j=1}^{i-1} h_{j-1}G_j \right] F_i \qquad \text{for } i = 1, 2, \cdots, n$$

and

$$H(i,n) = H(i,n-1) + H(i,n-1)G_n$$

$$+ \left[\sum_{j=i}^{n-2} H(i,j)\, G_{j+1} \right] F_n \qquad \text{for } i = 1, 2, \cdots, n-1.$$

Proof. This lemma can be established by a proof similar to the proof used to establish Lemma 1.

Theorem 2. *If* c *is a positive constant,* h *is a bounded function from* R *to* R, *each of* F *and* G *is a nonnegative function from* $R \times R$ *to* R, *each of* $\int_a^b F$ *and* $\int_a^b G$ *exists, each of*

$$\int_a^b h(u)G(u,v) \quad \text{and} \int_a^b \left[\int_a^u h(r)G(r,s) \right] F(u,v)$$

exists and

$$h(t) \leqq c + \int_a^t h(u)G(u, v) + \int_a^t \left[\int_a^u h(r)G(r, s) \right] F(u, v)$$

for $a \leqq t \leqq b$, *then*

$$h(t) \leqq c \left\{ 1 + \int_a^t G(u, v) \left[{}_v\Pi^t (1 + F + G) \right] \right\}$$

for $a \leqq t \leqq b$.

PROOF. This theorem can be established by a proof similar to the proof used to establish Theorem 1 by using Lemma 2 in place of Lemma 1.

W. P. Davis and J. A. Chatfield [4, Theorem 3, p. 744] establish that, if $\int_a^b G$ exists and $\int_a^b G^2$ exists and is zero, then

$${}_a\Pi^b (1 + G) = \exp \int_a^b G.$$

Thus, if the additional restriction that each of $\int_a^b F^2$ and $\int_a^b G^2$ exists and is zero is added to Theorems 1 and 2, then their conclusions are

$$h(t) \leqq c \left\{ 1 + \int_a^t \left[\exp \int_a^u (F + G) \right] G(u, v) \right\}$$

and

$$h(t) \leqq c \left\{ 1 + \int_a^t G(u, v) \left[\exp \int_v^t (F + G) \right] \right\},$$

respectively. When given in these forms, Theorems 1 and 2 more closely resemble the Gronwall inequality.

An Application. We now give an application of Theorem 1 to the determination of a bound for the solution of an integral equation. In the following, suppose each of f and g is a function from R to R, each of F and G is a function from $R \times R$ to R, f is quasi-continuous on $[a, b]$, each of $\int_a^b F$ and $\int_a^b G$ exists and each of F and G has bounded variation on $[a, b]$. We consider the integral equation

$$h(t) = f(t) + \int_a^t h(u)G(u, v) + \int_a^t \left[\int_a^u h(r)F(r, s) \right] G(u, v)$$

for $a \leqq t \leqq b$. The existence of a bounded solution for this integral equation can be established by constructing an appropriate sequence of Picard iterates. Further, the uniqueness of this solution can be established by using Theorem 1.

In the following, we show that in some instances the inequality given in Theorem 1 produces a better bound for the solution of the preceding integral equation than does the Gronwall inequality. We use the following form of the Gronwall inequality.

THEOREM 3. *If c is a positive constant, h is a bounded function from R to R, G is a nonnegative function from $R \times R$ to R, $\int_a^b G$ exists and*

$$h(t) \leqq c + \int_a^t h(u)G(u, v)$$

for $a \leqq t \leqq b$, then

$$h(t) \leqq c\, {}_a\Pi^t (1 + G)$$

for $a \leqq t \leqq b$ [6, Theorem 4, p. 495].

Initially, a bound for $h(t)$ is constructed by using Theorem 1. Since f is quasi-continuous on $[a,b]$, f is bounded on $[a, b]$. Let c denote a bound for f on $[a, b]$. Then,

$$|h(t)| \leqq c + \int_a^t |h(u)| \cdot |G(u, v)|$$

$$+ \int_a^t \left[\int_a^u |h(r)| \cdot |F(r, s)| \right] |G(u, v)|,$$

and hence by applying Theorem 1, we have that

$$\begin{aligned} |h(t)| &\leqq c \left\{ 1 + \int_a^t [{}_a\Pi^u (1 + |F| \right. \\ &\qquad \left. \cdot\, |G(u, v)| \right\} \\ &\leqq c \left\{ 1 + \int_a^t [{}_a\Pi^u (1 + |F| + |G|)] \right. \\ &\qquad \left. \cdot\, [|G(u, v)| + |F(u, v)|] \right\} \\ &= c\, {}_a\Pi^t (1 + |F| + |G|). \end{aligned}$$

Thus, $c_a\Pi^t (1 + |F| + |G|)$ is a bound for $h(t)$ for $a \leqq t \leqq b$.

Now, a bound for $h(t)$ is constructed by using the Gronwall inequality given in Theorem 3. To do this, we first note that

$$\int_a^t \left[\int_a^u h(r)F(r, s) \right] G(u, v) = \int_a^t h(u)F(u, v) \int_v^t G$$

for $a \leqq t \leqq b$. This relation can be established by using the identity

$$\sum_{i=1}^{n}\left[\sum_{j=1}^{i-1} h_{j-1}F_j\right] G_i = \sum_{i=1}^{n-1} h_{i-1}F_i \left[\sum_{j=i+1}^{n} G_j\right]$$

for subdivisions $\{x_i\}_{i=0}^{n}$ of $[a, t]$. Now, by using the preceding integral identity, we have that

$$\begin{aligned}|h(t)| &\leqq c + \int_a^t |h(u)| \cdot |G(u,v)| \\ &\quad + \int_a^t |h(u)| \cdot |F(u,v)| \cdot \left| \int_v^t G \right| \\ &= c + \int_a^t |h(u)| \Big[|G(u,v)| \\ &\quad + |F(u,v)| \cdot \left| \int_v^t G \right| \Big] \\ &\leqq c + \int_a^t |h(u)|[|G(u,v)| + B|F(u,v)|],\end{aligned}$$

where B denotes the least number such that $|\int_u^v G| \leqq B$ for $a \leqq u < v \leqq b$. Thus, by applying the form of the Gronwall inequality given in Theorem 3 to the preceding integral inequality, we have that

$$|h(t)| \leqq c\, {}_a\Pi^t\,[1 + (|G| + B|F|)]$$

for $a \leqq t \leqq b$. Whenever B satisfies the inequality $B > 1$, this bound for $h(t)$ is not as good as the bound for $h(t)$ obtained previously by using Theorem 1.

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