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PONTRYAGIN DUALITY FOR PRODUCTS AND COPRODUCTS OF ABELIAN k-GROUPS

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ABSTRACT. The k-group dual $G^\wedge$ of a T_2 abelian k-group G is the group of all k-group morphisms from G into the circle group, provided with the k-refinement of the compact-open topology. We shall quite easily show that the dual of the coproduct of a collection of T_2 k-groups is the product of the respective duals. The fact that the dual of the product of the collection is the coproduct of the respective duals requires a more lengthy proof. Saying that G satisfies duality when $G \cong G^{\wedge\wedge}$ via the canonical map, as a simple corollary to these two results we get that the product and the coproduct of any collection of k-groups, each of which satisfies duality, also satisfy duality.

Introduction. The k-group dual $G^\wedge$ of a T_2 abelian k-group G is the group of all k-group morphisms from G into the circle group, provided with the k-refinement of the compact-open topology. We shall quite easily show that the dual of the coproduct of a collection of T_2 k-groups is the product of the respective duals. The fact that the dual of the product of the collection is the coproduct of the respective duals requires a more lengthy proof. Saying that G satisfies duality when $G \cong G^{\wedge\wedge}$ via the canonical map, as a simple corollary to these two results we get that the product and coproduct of any collection of k-groups, each of which satisfies duality, also satisfy duality.

This last result is analogous to one for abelian topological groups proved by Kaplan in [1]. One should note, however, the dissimilarity of the proofs required for the results in the case of topological groups and those for k-groups. Indeed, theorems for topological groups do not transfer well to k-groups; and the reason for this is that the multiplication is not necessarily continuous on the topological product for k-groups. Even when the same result is true for k-groups as for topological groups, a completely different proof is often required.

Preliminaries. We begin with some background material on T_2 k-spaces. For a Hausdorff (T_2) topological space X, let kX denote the space with the same underlying set as X and the topology consisting of all k-open sets, where U is k-open if $U \cap C$ is open in C for all compact C in X. The space kX is called the k-refinement of X; if $X \cong kX$

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via the identity, then X is called a k-space.

Examples of T_2 spaces which are also k-spaces include those that are locally compact, first countable, or k_ω-spaces. For information regarding T_2 k-spaces see [5]; for information concerning non-Hausdorff k-spaces as well see [3].

A T_2 k-group is a group with a T_2 k-topology such that inversion is continuous and such that multiplication is continuous on the k-product — instead of the topological product as is the case for topological groups. One notes that if H is a topological group, then kH is a k-group. For examples of k-groups which are not topological groups see [3]. Unlike topological groups, T_2 k-groups may not be regular, hence not completely regular. In what follows all groups are T_2 and abelian, and their identities will be denoted by 0.

If $\{G_i \mid i \in I\}$ is any collection of T_2 abelian k-groups, $\prod_{i \in I} G_i$ will denote the k-product of the collection, which is the product provided with the k-refinement of the usual product topology, a finer topology, in general, than the product topology. The coproduct of the collection will be denoted by $\coprod_{i \in I} G_i$. Algebraically the coproduct is just the direct sum of the collection, and with a bit of work one can verify that its topology is the k-refinement of the restriction to the direct sum of the box topology on the product. Since we shall not need this explicit description of the coproduct's topology, we leave the verification of this to the reader. One easily has that finite products and coproducts are the same, and we record this as lemma 1.

LEMMA 1. *For any finite* $F \subset I$, $\prod_{i \in F} G_i = \coprod_{i \in F} G_i$.

Since for each finite F, $\prod_{i \in F} G_i$ is embedded as a closed subgroup in $\prod_{i \in I} G_i$ with $g_i = 0$ if $i \notin F$, for notational convenience we shall consider it as a closed subgroup.

For T_2 k-spaces X and Y and $K_{c-0}(X, Y)$ denoting the set of continuous functions from X into Y, provided with the compact-open topology, let $\underline{K}(X, Y) = k[K_{c-0}(X, Y)]$. One recalls that subbasic open sets in the compact-open topology are of the form $W(C, U) = \{f \mid f(C) \subset U\}$ where C is compact in X and U is open in Y. From [3], for example, the function $\varphi : \underline{K}(X, \underline{K}(Y, Z)) \to \underline{K}(Y, \underline{K}(X, Z))$ given by $\varphi(f)(y)(x) = f(x)(y)$ is a topological isomorphism, natural in X and Y.

Letting $|G|$ denote the underlying k-space of the T_2 abelian k-group G, one checks that the inclusion $\underline{KG}(G, H) \to \underline{K}(|G|, |H|)$ is a closed embedding, where $KG(G, H)$ is the group (since H is abelian) of all k-group morphisms from G into H. A proof of this is given in [3], where it is also shown that the functor $\underline{K}(X, \_) : K \to K$ preserves closed embeddings.

For any abelian k-group G, its k-group dual is $G^\wedge = \underline{KG}(G, S^1)$, where S^1 is the circle with its usual compact group topology. Note that $G^\wedge$ is T_2 since S^1 is T_2; in fact, since the continuous real-valued functions separate the points of S^1 the same is true for $G^\wedge$. Then, for any two abelian k-groups G and H, because of the remarks in the previous paragraph, the vertical arrows in the following diagram are closed embeddings. It follows, therefore, that the induced bijection φ' is, in fact, a topological isomorphism.

$$\begin{tikzcd}
\underline{KG}(H, G^\wedge) \arrow[r, dashed, "\varphi'"] \arrow[d] & \underline{KG}(G, H^\wedge) \arrow[d] \\
\underline{K}(|H|, |G^\wedge|) \arrow[d] & \underline{K}(|G|, |H^\wedge|) \arrow[d] \\
\underline{K}(|H|, \underline{K}(|G|, |S^1|)) \arrow[r, "\varphi"] & \underline{K}(|G|, \underline{K}(|H|, |S^1|))
\end{tikzcd}$$

Restating this categorically, with KG denoting the category of T_2 abelian k-groups, we have:

Proposition 2. *The functor* $\wedge : KG \to KG^{op}$ *is K-left adjoint to* $\wedge : KG^{op} \to KG$.

One notes for the unit η of this adjunction that each $\eta_G : G \to G^{\wedge\wedge}$ is given by $\eta_G(g)(\beta) = \beta(g)$ for each $g \in G$ and $\beta \in G^\wedge$. Each η_G is a k-group morphism. If, in fact, η_G is an isomorphism so that $G \cong G^{\wedge\wedge}$, we say that G satisfies k-group duality. This, of course, is in analogy to the case for topological groups, where H is said to satisfy Pontryagin duality if $H \cong H^{\vee\vee}$ via the canonical map, and $H^\vee = TG_{c-0}(H, S^1)$, the group of all topological group morphisms from H into S^1.

If G is locally compact, it is well known that G satisfies Pontryagin duality. Also, if G is locally compact, $G^\vee$ is locally compact and, therefore, $G^\vee = G^\wedge$, and it follows that G satisfies k-group duality as well.

The Main Theorem. In [1] Kaplan proved that if each member of a collection of T_2 topological groups satisfies Pontryagin duality, then their product also satisfies Pontryagin duality. Our objective is to prove a stronger result for k-group duality. Specifically, we shall prove:

Theorem 3. *For any collection* $\{G_i \mid i \in I\}$ *of* T_2 *abelian k-groups,* $[\coprod_{i\in I} G_i]^\wedge \cong \prod_{i\in I} G_i^\wedge$ *and* $[\prod_{i\in I} G_i]^\wedge \cong \coprod_{i\in I} G_i^\wedge$.

Before proving this, however, we note the following corollary, which is the k-group analog of Kaplan's result.

THEOREM 4. *If $\{G_i \mid i \in I\}$ is a collection of T_2 abelian k-groups, each satisfying k-group duality, then the product and the coproduct of the collection also satisfy k-group duality.*

PROOF. Using theorem 3 we have $[\prod_{i\in I} G_i]^{\wedge\wedge} \cong [\coprod_{i\in I} G_i^{\wedge}]^{\wedge} \cong \prod_{i\in I} G_i^{\wedge\wedge}$ which equals $\prod_{i\in I} G_i$ since $G_i \cong G_i^{\wedge\wedge}$ for each i. In a similar manner the coproduct is shown to satisfy duality.

We proceed now with the proof of theorem 3. Since $\wedge : KG \to KG^{op}$ is a left adjoint, it turns coproducts in KG into products in KG, and we record this as:

PROPOSITION 5. $[\coprod_{i\in I} G_i]^{\wedge} \cong \prod_{i\in I} G_i^{\wedge}$.

The proof of the second part of theorem 3 is somewhat more complicated. We begin with the following lemma.

LEMMA 6. *For each finite $F = \{i_1, \cdots, i_n\} \subset I$, $[\prod_{i\in F} G_i]^{\wedge}$ is embedded as a closed subgroup in $[\prod_{i\in I} G_i]^{\wedge}$ via $(f_{i_1} + \cdots + f_{i_n})(x_i)_{i\in I} = f_{i_1}(x_{i_1}) + \cdots + f_{i_n}(x_{i_n})$.*

PROOF. Consider the following commuting diagram

$$
\begin{tikzcd}
 & \prod_{i\in I} G_i \arrow[dr, "q_F"] & \\
\prod_{i\in F} G_i \arrow[ur, "e_F"] \arrow[rr, "1"'] & & \prod_{i\in F} G_i
\end{tikzcd}
$$

where e_F is the canonical embedding and q_F is the canonical quotient. Applying the functor $\wedge$ to this we have

$$
\begin{tikzcd}
 & \left[\prod_{i\in I} G_i\right]^{\wedge} \arrow[dr, "e_F^{\wedge}"] & \\
\left[\prod_{i\in F} G_i\right]^{\wedge} \arrow[ur, "q_F^{\wedge}"] \arrow[rr, "1^{\wedge}"'] & & \left[\prod_{i\in F} G_i\right]^{\wedge}
\end{tikzcd}
$$

That $q_F{}^\wedge$ is a closed embedding and the map required by the lemma is now obvious.

In particular, each $G_i{}^\wedge$ is embedded into $[\prod_{i\in I} G_i]^\wedge$. Therefore, by the universal property of coproducts there is a unique k-group morphism $\psi : \coprod_{i\in I} G_i{}^\wedge \to [\prod_{i\in I} G_i]^\wedge$, and it is easy to see that $\psi(f_{i_1} + \cdots + f_{i_n})(g_i)_{i\in I} = f_{i_1}(g_{i_1}) + \cdots + f_{i_n}(g_{i_n})$. The remainder of the paper consists of showing that ψ is a k-group isomorphism. Since it is a k-group morphism, ψ is certainly continuous.

LEMMA 7. *ψ is one-to-one.*

PROOF. Let $\psi(f_{i_1} + \cdots + f_{i_n}) = 0$. Then $\psi(f_{i_1} + \cdots + f_{in})(g_i)_{i\in I} = 0$ for all $(g_i)_{i\in I}$; that is $f_{i_1}(g_i) + \cdots + f_{i_n}(g_{i_n}) = 0$. Define $(\bar{g}_i)_{i\in I}$ by $\bar{g}_i = 0$ except for $i = i_1$. Then $0 = f_{i_1}(\bar{g}_{i_1}) + \cdots + f_{i_n}(\bar{g}_{i_n}) = f_{i_1}(\bar{g}_{i_1})$ for all $\bar{g}_{i_1} \in G_{i_1}$. Thus $f_{i_1} = 0$; similarly $f_{ik} = 0$ for $k = 2, \cdots, n$.

Before showing that ψ is surjective we need the following lemma. There is an analogous result for topological groups, and it is instructive to compare the standard proof for topological groups with the proof presented here for k-groups.

LEMMA 8. *For any $f \in [\prod_{i\in I} G_i]^\wedge$ there exists a finite $F \subset I$ with $f(G_i) = 0$ if $i \notin F$.*

PROOF. Assume that this is false and that there is, therefore, an infinite $A \subset I$ with $f(G_i) \neq 0$ if $i \in A$. Let U be any open set in S^1 containing 0 which contains no non-trivial subgroups. Then, for each $i \in A$ there exists $g_i \in G_i$ with $f(g_i) \notin U$. Since f is continuous on the k-product, it is continuous on compact subsets of the topological product. In particular, it is continuous on the compact subset $\prod_{i\notin A} \{0\} \times \prod_{i\in A} \{0, g_i\}$, and there is, therefore, a finite $J \subset A$ with $(0)_{i\in I} \in \prod_{i\notin A} \{0\} \times \prod_{i\in J} \{0\} \times \prod_{i\in A\setminus J} \{0, g_i\} \subset f^{-1}(U)$. But this requires that $f(g_i) \in U$ for all $i \in A\setminus J$; therefore, $f(g_i) \in U$ for an infinite number of the i in A, and this is a contradiction to the fact that $f(g_i) \notin U$ when $i \in A$.

For each f in $[\prod_{i\in I} G_i]^\wedge$ let $\sup(f) = \{(g_i)_{i\in I} \mid f((g_i)_{i\in I}) \neq 0\}$. Lemma 8 says that $\sup(f) \subset \prod_{i\in F} G_i$ for some finite $F \subset I$.

LEMMA 9. *ψ is surjective.*

PROOF. For $f \in [\prod_{i\in I} G_i]^\wedge$ with $\sup(f) \subset \prod_{i\in F} G_i$ and $F = \{i_1, \cdots, i_n\}$, let $f^* = f \mid G_{i_1} + \cdots + f \mid G_{i_n}$. Then $f^* \in \coprod_{i\in I} G_i{}^\wedge$, and clearly $\psi(f^*) = f$.

To see that ψ^{-1} is continuous it is sufficient to show that it is continuous on compact subsets since $[\prod_{i\in I} G_i]^\wedge$ is a T_2 k-space. The following lemma is the key step in accomplishing this.

LEMMA 10. *For any compact C in $[\prod_{i\in I} G_i]^\wedge$, $\bigcup \{\sup(c) \mid c \in C\} \subset \prod_{i\in F} G_i$ for some finite $F \subset I$.*

PROOF. Assume that the lemma is false. We, then, can choose $c_0 \in C$ and $F_0 \subset I$ with $\sup(c_0) \subset \prod_{i\in F_0} G_i$; next we can choose $c_1 \in C$ and $F_1 \subset I$ with $\sup(c_1) \not\subset \prod_{i\in F_0} G_i$, $\sup(c_1) \subset \prod_{i\in F_1} G_i$, and $F_0 \subsetneqq F_1$. And in general we have $c_n \in C$ with $\sup(c_n) \not\subset \prod_{i\in F_{n-1}} G_i$, $\sup(c_n) \subset \prod_{i\in F_n} G_i$, and $F_{n-1} \subsetneqq F_n$. Thus for each non-negative integer n, there exists $i_n \in F_n \backslash F_{n-1}$ with $c_n(G_{i_n}) \neq 0$. Now let U be any neighborhood of 0 in S^1 containing no non-trivial subgroups. Then for each n there is a $g_{i_n} \in G_{i_n}$ with $c_n(g_{i_n}) \notin U$. Since C is compact the c_n have an accumulation point c. Let F be a finite subset of I with $\sup(c) \subset \prod_{i\in F} G_i$. Now, define $K = \prod_{i_n \in A} \{g_{i_n}, 0\} \times \prod_{i\notin A} \{0\}$, where $A = \{i \in I \mid i = i_n$ for some n and $i \notin F\}$. Clearly $c(k) = 0$ for all $k \in K$; thus, C is in the open set $W(K, U)$. However, $c_n \notin W(C, U)$ for all n except possibly some finite number of them for which $i_n \in F$. But this is impossible if the c_n accumulate at c; therefore, the lemma is true.

LEMMA 11. *ψ^{-1} is continuous.*

PROOF. Let C be a compact subset of $[\prod_{i\in I} G_i]^\wedge$ with F as in lemma 10, and consider the following commuting diagram where all maps are the canonical ones, and in particular $q_F{}^\wedge$ is the closed embedding of lemma 6.

$$\begin{tikzcd}
\coprod_{i\in I} G_i^\wedge \arrow[rrr, shift left, "\psi"] & & & {[\prod_{i\in I} G_i]^\wedge} \arrow[lll, shift left, "\psi^{-1}"] \\
\coprod_{i\in F} G_i^\wedge \arrow[u] \arrow[r, phantom, "="] & \prod_{i\in F} G_i^\wedge \arrow[r, phantom, "\cong"] & {[\coprod_{i\in F} G_i]^\wedge} \arrow[r, phantom, "="] & {[\prod_{i\in F} G_i]^\wedge} \arrow[u, "q_F{}^\wedge"']
\end{tikzcd}$$

Since $q_F{}^\wedge$ is a closed embedding, it is clear that ψ^{-1} restricted to $q_F{}^\wedge([\prod_{i\in F} G_i]^\wedge)$ is continuous and is, therefore, certainly continuous on C. Thus ψ^{-1} is continuous on all compact subsets, hence continuous.

We have now shown that ψ is a k-group isomorphism, and the proof of theorem 3 is complete.

Throughout the paper, the only property of S^1 that was of importance was that it had no small subgroups. Thus any other T_2 abelian k-group with this property would also yield a duality theory in which the dual of the product was the coproduct of the respective duals and vice versa.

If one takes an uncountable collection of copies of the reals $\{R_i \mid i \in I\}$, then $\prod_{i\in I} R_i$ satisfies k-group duality since each $R_i \cong R_i^{\wedge\wedge}$. Furthermore, this is an example of a k-group satisfying k-group duality which is not a topological group (the k-product $\prod_{i \in I} G_i$ is not even regular when I is uncountable). This k-product is, of course, the k-refinement of a topological group which satisfies Pontryagin duality, and this prompts one to ask the following: If H satisfies Pontryagin duality as a topological group, does kH satisfy k-group duality?

We should also remark that in [4] Noble gives a different—and inequivalent—definition of a k-group. In particular, all his k-groups are topological groups and need not be k-spaces. He then proves that every closed subgroup of a countable product of locally compact groups satisfies Pontryagin duality, and this is an extension of the principal theorem in Kaplan's second paper on duality [2].

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