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ON UNCONDITIONAL SECTION BOUNDEDNESS IN SEQUENCE SPACES

J. J. SEMBER

1. **Introduction.** The concepts of section boundedness (AB), section convergence (AK) and functional section convergence (FAK) have been of great interest in summability and in the study of topological sequence spaces. More general notions of Cesàro-section boundedness and convergence and T-section boundedness and convergence have also been investigated and shown to be significant [2, 3]. The usual sections associated with a sequence x are the finite sequences at the front of x, that is, the sequences $P_n(x) = (x_1, x_2, \cdots, x_n, 0, 0, \cdots) = \sum_{k=1}^{n} x_k\delta^k$. The purpose of this work is to investigate the topological significance in a sequence space of the set of unrestricted sections of a sequence x, that is, the set of all sequences having the form $\sum_{k \in F} x_k\delta^k$, where F is some finite subset of the positive integers.

We show (Theorem 7) that a necessary and sufficient condition for an FK space E to be factorable in the form $E = cE$, where c denotes the space of convergent sequences, is that the unrestricted sections form a bounded set for every element of E.

Among the consequences of the results are an inclusion theorem due to Bennett, Kalton, Snyder and Wilansky ([1], [10]), an improvement of a result of Goes on solid FK spaces, and a factorization theorem for FK spaces related to the unconditional convergence of the series $\sum x_k\delta^k$.

2. **Preliminaries and Notation.** We let ω denote the linear space of all real or complex sequences, φ the subspace of sequences x for which $x_k \neq 0$ at most finitely often. The sequence δ^k has 1 in the k^{th} position and 0 for every other coordinate. If x is any sequence, $P_n(x) = \sum_{k=1}^{n} x_k\delta^k$ denotes the n^{th} section of x. A K-space is a linear space of sequences containing φ and having a locally convex Hausdorff topology with the property that the coordinate linear functionals $x \to x_k$ are continuous. An FK space is a complete metrizable K-space. If E is a K-space, the topological dual of E will be denoted by E'. A sequence x in a K-space E is said to have section boundedness (AB) in case $P(x) = \{P_n(x)\}$ is bounded in E, section convergence (AK) in case $P_n(x) \to x$ in E, weak section convergence (SAK) in case $P_n(x) \to x$ in the weak topology $\sigma(E, E')$, and functional section convergence

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(FAK) in case $\{P_n(x)\}$ is Cauchy in $\sigma(E, E')$ (equivalently, in case $\{f(P_n(x))\}$ converges for every $f \in E'$).

A series $\sum z_k$ in a topological vector space E is unconditionally convergent to $z \in E$ in case the net $(\sum_{k \in F} z_k)_{F \in \Phi}$ converges to z, where Φ is the collection of all finite subsets of the positive integers, directed by set inclusion. The series $\sum z_k$ is unconditionally Cauchy in case the net $(\sum_{k \in F} z_k)_{F \in \Phi}$ is a Cauchy net [8].

We let H denote the set of all sequences in φ consisting of 0's and 1's, thus $H = \{h = (h_k) \in \varphi : h_k = 0 \text{ or } h_k = 1, k = 1, 2, \cdots\}$. If $x \in \omega$, we let $H(x) = \{h(x) : h \in H\}$, where $h(x)$ denotes the coordinatewise product $h(x) = (h_k x_k)$. If E is a K-space, then $H(x) \subset E$ for any sequence x. We say that a sequence $x \in \omega$ has *unconditionally bounded sections* (UAB) in E in case $H(x)$ is a bounded subset of E, has *unconditional section convergence* (UAK) in case the net $H(x)$ converges to x in E, has *unconditional weak section convergence* $(USAK)$ in case the net $H(x)$ converges to x in $\sigma(E, E')$, and has *unconditional functional section convergence* $(UFAK)$ in case the net $H(x)$ is Cauchy in $\sigma(E, E')$.

We let

$$E_{UAB} = \{x \in \omega : x \text{ has } UAB\}$$
$$E_{UFAK} = \{x \in \omega : x \text{ has } UFAK\}$$
$$E_{UAK} = \{x \in E : x \text{ has } UAK\}$$
$$E_{USAK} = \{x \in E : x \text{ has } USAK\}.$$

Since $P(x) \subset H(x)$, it is clear that $E_{UT} \subset E_T$ for $T = AB, FAK, AK, SAK$. We also define

$$C_\phi{}^+ = \{t \in \varphi : 0 \leqq t_k \leqq 1, k = 1, 2, \cdots\}$$
$$C_\phi = \{t \in \varphi : |t_k| \leqq 1, k = 1, 2, \cdots\}$$
$$C_0 = \{t \in c_0 : |t_k| \leqq 1, k = 1, 2, \cdots\}$$
$$C_\lambda = \{t \in c : |t_k| \leqq 1, k = 1, 2, \cdots\}$$

and, for any sequence x, $C_\phi{}^+(x)$, $C_\phi(x)$, $C_0(x)$ and $C_\lambda(x)$ will denote the corresponding sets of coordinatewise products. Thus C_0 is the unit ball in the space of null sequences and, for example,

$$C_0(x) = \{t(x) = (t_k x_k) : t \in C_0\}.$$

3. **Unconditional Section Boundedness.** The properties of section boundedness (AB) and functional section convergence (FAK) are, in general, different. Our first result gives a characterization of uncon-

ditional section boundedness in terms of the dual space and shows that the properties UAB and $UFAK$ are the same.

THEOREM 1. *If E is a K-space, then*

$$E_{UAB} = \{x \in \omega : \sum |x_k|\,|f(\delta^k)| < +\infty \;\; \forall f \in E'\} = E_{UFAK}.$$

PROOF. By the Banach-Mackey Theorem, $H(x)$ is bounded if and only if it is weakly bounded. Thus $x \in E_{UAB}$ if and only if, for each $f \in E'$, $\{\sum_{k\in F} x_k f(\delta^k) : F \in \Phi\}$ is a bounded set of real numbers. By Riemann's Theorem on the rearrangement of series in a finite-dimensional space, this last condition is equivalent to the absolute convergence of $\sum x_k f(\delta^k)$. Clearly $E_{UFAK} \subset E_{UAB}$ and, if $\sum x_k f(\delta^k)$ is absolutely convergent for every $f \in E'$, then $H(x)$ is weakly Cauchy.

In particular, we have

COROLLARY 1.1. *Let x be any sequence with unconditionally bounded sections in E. Then x has FAK in E.*

We observe next that if a sequence with unconditionally bounded sections has section convergence in the weak topology, then this convergence is unconditional. More precisely,

THEOREM 2. *If E is a K-space, then*

$$E_{UAB} \cap E_{SAK} = E_{USAK}$$

PROOF. If $H(x)$ is bounded and $\sum x_k \delta^k = x$ weakly, then for $\epsilon > 0$ and $f \in E'$ we can choose N sufficiently large so that $\sum_{k=N+1}^{\infty} |x_k f(\delta^k)| < \epsilon/2$ and $|f(x) - \sum_{k=1}^{N} x_k f(\delta^k)| < \epsilon/2$. Then if F is any finite set of positive integers containing $\{1, 2, \cdots, N\}$, we have $|f(\sum_{k\in F} x_k \delta^k) - f(x)| < \epsilon$ and it follows that $H(x)$ converges to x in the weak topology.

For any sequence x an element in the convex hull of $H(x)$ has the form

$$y = \sum_{j=1}^{k} \mu_j h^j(x),$$

where $\mu_j \geqq 0$ for each $j = 1, 2, \cdots, k$ and $\sum_{j=1}^{k} \mu_j = 1$. Letting $F = F_1 \cup F_2 \cup \cdots \cup F_k$, where $F_j = \{n : h_n^j = 1\}$, and $\lambda_n = \sum_{n\in F_j} \mu_j$, we can write

$$y = \sum_{n\in F} \lambda_n x_n \delta^n,$$

where $0 \leqq \lambda_n \leqq 1$ for each n. Thus $y \in C_\phi{}^+(x)$. Conversely if $y = \sum_{n \in F} \lambda_n x_n \delta^n$ with $0 \leqq \lambda_n \leqq 1$ and F finite, then the λ_n can be rearranged so that

$$0 \leqq \lambda_{n(1)} \leqq \lambda_{n(2)} \leqq \cdots \leqq \lambda_{n(k)} \leqq 1$$

and we can write

$$y = \lambda_{n(1)} \sum_{i=1}^{k} x_{n(i)}\delta^{n(i)} + \sum_{j=2}^{k} (\lambda_{n(j)} - \lambda_{n(j-1)}) \sum_{i=j}^{k} x_{n(i)}\delta^{n(i)}$$

$$+ (1 - \lambda_{n(k)}) \sum_{k \in \phi} x_k \delta^k \qquad (\emptyset = \text{the empty set})$$

which is in the convex hull of $H(x)$. This argument is given in [8] for series in a general topological vector space.

We thus have

LEMMA 1. *If E is a K-space and x is any sequence, then the convex hull of $H(x)$ is $C_\phi{}^+(x)$.*

As in the first part of the argument above, we see that every element in the absolutely convex hull of $H(x)$ has the form

$$y = \sum_{n \in F} \lambda_n x_n \delta^n,$$

where F is some finite subset of the positive integers and $|\lambda_n| \leqq 1$ for each n. Thus if $\Gamma(H(x)$ denotes the absolutely convex hull of $H(x)$, we have

LEMMA 2. *If E is a K-space and x is any sequence, then*

$$\Gamma(H(x)) \subseteq C_\phi(x) \subseteq C_\phi^+(x) - C_\phi^+(x) + iC_\phi^+(x) - iC_\phi^+(x) \subseteq 4\Gamma(H(x)).$$

Since a K-space is locally convex, we then have

THEOREM 3. *If E is a K-space and x is any sequence, the following are equivalent:*

(i) *x has unconditionally bounded sections in E*

(ii) *$C_\phi{}^+(x)$ is bounded in E*

(iii) *$C_\phi(x)$ is bounded in E.*

4. Unconditional Section Boundedness and Sequential Completeness. The set H with coordinatewise multiplication is a semigroup of continuous linear operators on any K-space E. The same is true of $C_\phi{}^+$ and C_ϕ. We show that the "completion" of H, from this operator point of view, is the unit ball C_0 of sequences converging to zero. This result parallels [5], Proposition 7, where it is shown that the "completion" of the set P of usual section operators is the unit ball B_0 of the null sequences of bounded variation.

THEOREM 4. *Let E be a sequentially complete K-space and let x be any sequence with unconditionally bounded sections in E. Then $C_0(x) \subseteq E$ and $C_0(x)$ is bounded in E. Furthermore every element of $C_0(x)$ has unconditional section convergence.*

PROOF. Suppose $H(x)$ is bounded in E, let p be any continuous seminorm on E, and let $\eta \in C_0$. It follows from Theorem 3 that $C_\phi(x)$ is bounded, hence there exists $M > 0$ so that $p(t(x)) < M$ for each $t \in C_\phi$. We observe that, if $m > n$,

$$\begin{aligned} P_m(\eta x) - P_n(\eta x) &= (0, 0, \cdots, 0, \eta_{n+1}x_{n+1}, \eta_{n+2}x_{n+2}, \cdots, \eta_m x_m, 0, 0, \cdots) \\ &= (\max_{n+1 \leqq k \leqq m} |\eta_k|)\, \eta'(x), \end{aligned}$$

where $\eta' \in C_\phi$. Therefore

$$p(P_m(\eta x) - P_n(\eta x)) = (\max_{n+1 \leqq k \leqq m} |\eta_k|) p(\eta'(x)) < (\max_{n+1 \leqq k \leqq m} |\eta_k|) M$$

and, since $\eta \in C_0$, it follows that $\{P_n(\eta x)\}$ is a Cauchy sequence in E. Since the coordinate linear functionals are continuous, $\{P_n(\eta x)\}$ must converge to ηx. Thus $C_0(x) \subseteq E$.

We have shown that every element of $C_0(x)$ is in the closure of the bounded set $C_\phi(x)$. Since E is locally convex it follows that $C_0(x)$ is bounded. Now let p be a continuous seminorm on E, $\eta \in C_0$, and $\epsilon > 0$. Since $P_n(\eta x) \to \eta x$ we can find N sufficiently large so that, for $n \geqq N$,

$$p(P_n(\eta x) - \eta x) < \epsilon/2$$

and

$$|\eta_n| < \epsilon/2M,$$

where $M > 0$ is such that $p(\tau(x)) < M$ for each $\tau \in C_\phi$. Then if F is any finite set of positive integers with $F \supset \{1, 2, \cdots, N\}$, we have

$$p\left(\sum_{k\in F}(\eta x)_k\delta^k-\eta x\right)\leqq p\left(\sum_{k\in F\setminus\{1,2,\cdots,N\}}\eta_k x_k\delta^k\right)$$
$$+\ p(P_N(\eta x)-\eta x)$$
$$\leqq\left(\max_{k\in F\setminus\{1,2,\cdots,N\}}|\eta_k|\right)p(\tau(x))+\epsilon/2\quad(\text{where }\tau\in C_\phi)$$
$$<(\epsilon/2M)M+\epsilon/2=\epsilon.$$

and therefore $\eta x \in E_{UAK}$.

A special case of Theorem 4 yields the inclusion result for c_0 given in [1] (Proposition 5, p. 565) and [10] (Corollary 5, p. 598).

COROLLARY 4.1. *Let E be a sequentially complete K-space. If* $\sum |f(\delta^k)| < +\infty$ *for each* $f \in E'$, *then* $c_0 \subseteq E$.

PROOF. The condition $\sum |f(\delta^k)| < +\infty$ is clearly the same, from Theorem 1, as $H(1)$ bounded in E, where $1 = (1, 1, \cdots, 1, \cdots)$.

COROLLARY 4.2. *An FK space E contains* c_0 *if and only if* $\sum |f(\delta^k)| < +\infty$ *for each* $f \in E'$.

PROOF. The sufficiency of the condition is given by Corollary 4.1. The necessity follows from the fact that the relative topology of E on c_0 is weaker than the usual sup norm topology ([11], p. 203, Coro. 1) and $H(1)$ is bounded in c_0 with its usual topology.

Theorem 4 asserts that $c_0(E) \subseteq E_{UAK}$ in a sequentially complete K-space in which every element has unconditionally bounded sections. If E is an FK space, we have

THEOREM 5. *Let E be an FK space in which every element has unconditionally bounded sections. Then* $C_0(E) = E_{UAK}$.

PROOF. The inclusion $C_0(E) \subseteq E_{UAK}$ follows from Theorem 4. Garling has shown ([5], p. 1006, Lemma 1) that, in an FK space, the inclusion $E_{AK} \subseteq B_0(E)$ holds, where B_0 is the unit ball of null sequences of bounded variation. Since we always have $E_{UAK} \subseteq E_{AK}$ and $B_0(E) \subseteq C_0(E)$, the result follows.

THEOREM 6. *Let E be a sequentially complete K-space and let x be a sequence in E with unconditionally bounded sections in E. Then* $C_\lambda(x) \subseteq E \cap E_{UAB}$.

PROOF. If $\tau \in C_\lambda$, we can write $\tau = L \cdot 1 + \eta$, where $|L| \leqq 1$ and $\eta \in C_0$. Thus $\tau(x) = Lx + \eta(x)$. But $\eta(x) \in E_{UAB}$ by Theorem 4 and $x \in E_{UAB}$ by hypothesis, and the result follows.

Since it is always true that $E \subset C_\lambda(E)$, we then have

COROLLARY 6.1. *If E is a sequentially complete K-space in which every sequence has unconditionally bounded sections, then* $E = c(E)$.

THEOREM 7. *Let E be an FK space. The following are equivalent:*

(i) $E = c(E)$

(ii) $\sum |x_k|\,|f(\delta^k)| < \infty$ *for every* $x \in E, f \in E'$

(iii) $E \subseteq E_{UAB}$

(iv) $H(x)$ *is bounded for every* $x \in E$.

PROOF. The equivalence of (ii), (iii), and (iv) follows from Theorem 1, and Corollary 6.1 shows that (iii) $\Rightarrow$ (i). If $E = c(E)$, then the mapping $T_x : c \to E$ defined by $T_x(\tau) = \tau(x)$ is a continuous linear mapping between FK spaces (for example, it can be viewed as an infinite diagonal matrix transformation), consequently it maps bounded sets into bounded sets. Since C_λ is bounded in c, $T_x(C_\lambda) = C_\lambda(x)$ is bounded in E for every $x \in E$. In particular, $H(x)$ is bounded for every $x \in E$. Thus (i) $\Rightarrow$ (iv).

It is clear from Theorem 7 that every solid FK space has unconditionally bounded sections. We thus have

COROLLARY 7.1. *Every sequence in a solid FK space has UFAK.* Goes [6] proved that every solid FK space has FAK.

COROLLARY 7.2. *If E is a solid FK space, then* $E_{UAK} = C_0(E)$.

PROOF. This follows from Theorems 5 and 7.

COROLLARY 7.3. *In a solid FK space, the series* $\sum x_k\delta^k$ *converges if and only if it converges unconditionally.*

PROOF. This follows from Corollary 7.2 and Garling's result, mentioned in the proof of Theorem 5, that $E_{AK} \subseteq C_0(E)$.

For other results along the lines of those given in Theorems 4, 5, 6, and 7 the reader is referred to the fundamental work of Garling [5] (e.g., [5], Thm. 4, p. 1006) and to the papers of Buntinas [2, 3] on Cesàro- and T-section boundedness and convergence (e.g., [3], Theorem 11, p. 458).

Finally we observe that in a weakly sequentially complete K-space the property of having unconditionally bounded sections is especially strong.

THEOREM 8. *Let E be a weakly sequentially complete K-space and let x be any sequence. Then x has unconditionally bounded sections in E if and only if* $\sum x_k\delta^k$ *converges to x unconditionally in E* (*i.e.,* $E_{UAB} = E_{UAK}$).

PROOF. If $H(x)$ is bounded then $H(x)$ is weakly Cauchy, from Theorem 1. It follows that $P(x)$ is weakly Cauchy, hence weakly convergent to x. By Theorem 2, $\sum x_k\delta^k = x$ unconditionally in $\sigma(E, E')$ and, since $(E, \sigma(E, E'))$ is sequentially complete, $\sum x_k\delta^k$ is weakly subseries convergent to x ([4], p. 59). But then by the Orlicz, Pettis, Grothendieck theorem ([7], [8], p. 153) $\sum x_k\delta^k$ is subseries convergent, hence unconditionally convergent, to x in the initial topology.

5. **Some Examples.** The space c of convergent sequences is an FK space with the property that every element has unconditionally bounded sections and, according to Theorem 7, is a particularly significant one. Professor Goes has pointed out, in private correspondence, that the space of strongly Cesàro summable sequences is another nonsolid FK space with unconditional section boundedness. The space of almost periodic sequences (see [9]) provides yet another example. As observed earlier, it follows from Theorem 7 that every solid FK space enjoys the property.

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SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA