ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 4, Fall 1977

FACTORALITY IN QUOTIENTS OF LINEAR G_m-ACTIONS

ANDY R. MAGID

Let k be an algebraically closed field of characteristic 0 and let $G_m = GL(1, k)$ be the multiplicative group of k regarded as an algebraic group. Suppose G_m acts rationally on the n-dimensional k vector space V. The purpose of this paper is to examine factorality in the ring $R = k[V]^{G_m}$ of G_m-invariant polynomial functions on V; that is, to compute the divisor class group $C\ell(R)$.

We can always choose a basis $x_1, \cdots, x_n$ of V such that G_m acts diagonally with respect to this basis, i.e. t in G_m sends x_i to $t^{\lambda_i}x_i$. For example, if $n = 2$, $\lambda_1 = 1$, and $\lambda_2 = -1$, then $R = k[x_1x_2]$ and $C\ell(R) = 1$. If $n = 3$, $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -4$, then $R = k[x_1{}^4x_3, x_2{}^2x_3, x_1{}^2x_2x_3]$, so in terms of generators and relations, R is generated by $a = x_1{}^4x_3$, $b = x_2{}^2x_3$, $c = x_1{}^2x_2x_3$ subject to the single relation $c^2 = ab$. Then $C\ell(R) = \mathbf{Z}/2\mathbf{Z}$. If $n = 4$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = -1$, then $R = k[x_1x_3, x_2x_3, x_1x_4, x_2x_4]$ so R is generated by $a = x_1x_3$, $b = x_2x_3$, $c = c_1x_4$, $d = x_2x_4$, subject to the single relation $ad = bc$. Then $C\ell(R) = \mathbf{Z}$. Thus $C\ell(R)$, in these examples, is cyclic, but it can be infinite cyclic, finite cyclic, or trivial.

The general result established here is the following: suppose all λ_i are non-zero with exactly p positive. If p and $n - p$ are larger than 1, $C\ell(R) = \mathbf{Z}$. If $p = 1$ or $n - p = 1$, $C\ell(R)$ is finite cyclic of computable order — the computation depends on some number-theoretic calculations modulo λ_n (if $n - p = 1$) or λ_1 (if $p = 1$). The approach to this determination is via a study of the geometric quotient prevariety $W = (V - 0)/G_m$, whose existence we establish, and we show that $\mathrm{Pic}(W) = \mathbf{Z}$, always.

For convenience, we establish some notational conventions in the beginning of the paper and conserve these throughout for proofs, although the theorems are stated without the conventions.

We assume throughout that k is an algebraically closed field of characteristic zero, and all our pre-varieties are over k. We identify pre-varieties with their k-closed points. We use $k[W]$ to stand for $\Gamma(W, \mathcal{O}_W)$ if W is a pre-variety and $(\ )^*$ to denote the units functor. We also use the relative units functor U_k, whose value on the pre-variety W is $k[W]^*/k^*$.

Received by the editors on December 26, 1975, and in revised form on July 5, 1976.

DEFINITION 1. Let n, m, $\lambda_1, \cdots, \lambda_{n+m}$ be positive integers. Then $[(n, m), \lambda_1, \cdots, \lambda_{n+m}]$ denotes $k^{(n+m-1)} - 0$ with coordinates $x_1, \cdots, x_{n+m}$ and G_m action given by $x_i \to t^{\lambda_i}x_i$, $1 \leqq i \leqq n$, and $x_i \to t^{-\lambda_i}x_i$, $n + 1 \leqq i \leqq n + m$. The action is called reduced if the greatest common divisor of $\lambda_1, \cdots, \lambda_{n+m}$ is 1.

REMARK. Let $d = \text{g.c.d.}(\lambda_1, \cdots, \lambda_{n+m})$ and let $\lambda_i' = \lambda_i/d$. Suppose Z is a geometric quotient of $[(n, m), \lambda_1', \cdots, \lambda_{n+m}']$ by G_m. Then the quotient $[(n, m), \lambda_1, \cdots, \lambda_{n+m}]/G_m$ exists and equals Z: for let $f: G_m \to G_m$ be the dth power map. Then G_m acting on $[(n, m), \lambda_1', \cdots, \lambda_{n+m}']$ via f is equivalent to G_m acting on $[(n, m), \lambda_1, \cdots, \lambda_{n+m}]$. So whenever it is convenient we may take the action to be reduced.

Fix (n, m), $\lambda_1, \cdots, \lambda_{n+m}$, let $V = [(n, m), \lambda_1, \cdots, \lambda_{n+m}]$ and let $V' = [(n, m), 1, \cdots, 1]$. Let $\phi : V' \to V$ send x_i to $x_i^{\lambda_i}$. It is trivial to verify that ϕ is a surjective G_m-equivariant morphism. For any integer ℓ, let $\Gamma(\ell)$ denote the group of ℓth roots of unity in k. Let $G = \Gamma(\lambda_1) \times \cdots \times \Gamma(\lambda_{n+m})$. Then $g = (a_1, \cdots, a_{n+m})$ in G acts on V' by $g(x_1, \cdots, x_{n+m}) = (\alpha_1 x_1, \cdots, \alpha_{n+m}x_{n+m})$. The actions of G and G_m on V' commute and ϕ induces an isomorphism of V'/G with V. For each $i = 1, \cdots, n + m$, let

$$V_i' = V' - (x_i = 0) \text{ and } V_i = V - (x_i = 0).$$

The V_i' are open and G_m-stable in V', and their union covers V'. Similar remarks apply to the V_i. Also, $\phi(V_i') = V_i$.

For each i, let $W_i' = k^{(n+m-1)}$ with coordinates $x_1, \cdots, \hat{x}_i, \cdots, x_{n+m}$, where $\hat{x}_i$ means x_i is deleted. For each i there are G_m-equivariant isomorphisms $\sigma_i : G_m \times W_i' \to V'$ defined as follows:

For $i \leqq n$, $\sigma_i(t, x_1, \cdots, \hat{x}_i, \cdots, x_{n+m})$

$$= (tx_1, \cdots, tx_{i-1}, t, tx_{i+1}, \cdots, tx_n, t^{-1}x_{n+1}, \cdots, t^{-1}x_{n+m}),$$

$$\sigma_i^{-1}(x_1, \cdots, x_{n+m})$$

$$= (x_i, x_i^{-1}x_1, \cdots, \widehat{x_i^{-1}x_i}, \cdots, x_i^{-1}x_n, x_ix_{n+1}, \cdots, x_ix_{n+m}).$$

For $i > n$, $\sigma_i(t, x_1, \cdots, \hat{x}_i, \cdots, x_{n+m})$

$$= (tx_1, \cdots, tx_n, t^{-1}x_{n+1}, \cdots, t^{-1}x_{i-1}, t^{-1}x_{i+1}, \cdots, t^{-1}x_{n+m}),$$

$$\sigma_i^{-1}(x_1, \cdots, x_{n+m})$$

$$= (x_i^{-1}, x_ix_1, \cdots, x_ix_n, x_i^{-1}x_{n+1}, \cdots, \widehat{x_i^{-1}x_i}, \cdots, x_i^{-1}x_{n+m}).$$

For each i, σ_i induces an isomorphism of W_i' with V_i'/G_m. By uniqueness of geometric quotients, this means that the quotient V'/G_m exists and is covered by open sets isomorphic to W_i'. We denote these open sets in the quotient by W_i' also.

Because the actions of G and G_m commute on V', G acts on V'/G_m, and each W_i' is G-stable. Thus all G-conjugates of an element of V'/G_m lie in an open affine, and the quotient $V'/G_m/G$ exists and is covered by the open affines $W_i = W_i'/G$. But $W_i'/G = V_i'/G_m/G = V_i'/G/G_m = V_i/G_m$, and hence $V'/G_m/G$ is a quotient of V by G_m. (The above construction is a special case of the technique of [4, 6.1, p. 543].) We have now shown:

Theorem 2. *The geometric quotient* $[(n, m), \lambda_1, \cdots, \lambda_{n+m}]/G_m$ *exists.*

The geometry of the quotient may be quite complicated. We make a few remarks regarding it. Let $S = \{(i, j) \mid 1 \leqq i \leqq n, 1 \leqq j \leqq m\}$, let $V_0' = U(V_i' \cap V'_{j+n})$ (union over (i, j) in S) and let $V_0 = U(V_i \cap V_{j+n})$. Then V_0' and V_0 are open G_m-stable subsets of V' and V, respectively. Let $C_0' = V_0'/G_m$ and $C_0 = V_0/G_m$. Clearly $C_0 = C_0'/G$. To determine C_0', we consider the subset C' of $k^{(nm)}$ with coordinates t_{ij} which is the locus of $t_{ij}t_{k\ell} - t_{i\ell}t_{kj}$ for all pairs (i, j) and (k, ℓ) in S. It is easy to see that the morphism $V_0' \to C'$ by $t_{ij} = x_ix_{j+n}$ defines an isomorphism of C_0' with $C' - 0$, and thus C_0' is an open subset of the affine variety C'. C' is usually called the Segre cone of signature (n, m), because C' is the cone in $k^{(nm)}$ lying over the image $P^{(n-1)} \times P^{(m-1)}$ in $P^{(nm-1)}$ under the Segre embedding. The action of G extends to C', and if $C = C'/G$, C_0 is isomorphic to an open subset of C whose complement is a single point (the image in C of 0 in C').

We now describe the complements of C_0' and C_0. First $V' - V_0' = Z_1' \cup Z_2'$, where $Z_1' = (k^{(n)} - 0) \times 0$ and $Z_2' = 0 \times (k^{(m)} - 0)$, and $V - V_0 = Z_1 \cup Z_2$ where $Z_i = \phi(Z_i')$. Now Z_i' is G_m-stable, and $Z_1'/G_m = P^{(n-1)}$ and $Z_2'/G_m = P^{(m-1)}$, so $V'/G_m = C_0' \cup P^{(n-1)} \cup P^{(m-1)}$, and $V/G_m = C_0 \cup P^{(n-1)}/G \cup P^{(m-1)}/G$, so the complements of C_0' and C_0 are complete and, if $n > 1$ or $m > 1$, not affine. (If $n = m = 1$, it is easy to see that V'/G_m is the affine line with the origin doubled so V'/G_m is not separated).

Finally, we note that V'/G_m is non-singular since it is covered by the non-singular open sets W_i' and that V/G_m is normal since it is covered by the open sets $W_i = W_i'/G$.

We now begin the calculation of the Picard groups of the quotients.

Proposition 3. $\mathrm{Pic}([(n, m), 1, \cdots, 1]/G_m) = \mathbf{Z}$.

Proof. In the established notation, $V' \to V'/G_m$ is a locally trivial fibration with fibre G_m: the inverse image of W_i' being $G_m \times W_i'$. We employ the exact sequence of [2, Theorem 5], noting that the demonstration in the reference does not require separation of the base. Since $U_k(V') = 1$ and $\mathrm{Pic}(V') = 1$, the exact sequence gives $1 \to U_k(G_m) \to \mathrm{Pic}\,(V'/G_m) \to 1$ exact, and hence the result.

To calculate $\mathrm{Pic}\,(V/G_m)$, we use the fact [2, Lemma 2] that $\mathrm{Pic}\,(Y) = H^1(Y, G_m) = \mathrm{H}^1(Y, U_k)$ (cohomology in the Zariski topology).

THEOREM 4. $\mathrm{Pic}([(n, m), \lambda_1, \cdots, \lambda_{n+m}]/G_m) = \mathbf{Z}$.

PROOF. Let $\mathcal{U}$ be the open cover of V/G_m by the W_i and let $\mathcal{U}'$ be the open cover of V'/G_m by the W_i'. The Cech-to-derived functor cohomology spectral sequences give rise to the following exact sequences of low degree:

$$(*) \qquad 1 \to H^1(\mathcal{U}, U_k) \to H^1(V/G_m, U_k) \to H^0(\mathcal{U}, \mathrm{Pic})$$

$$(**) \qquad 1 \to H^1(\mathcal{U}', U_k) \to H^1(V'/G_m, U_k) \to H^0(\mathcal{U}', \mathrm{Pic})$$

We begin by analyzing $(**)$. $\mathrm{Pic}(W_i') = 1$ for each i, hence $H^0(\mathcal{U}, \mathrm{Pic}) = 1$ and $H^1(\mathcal{U}', U_k) = H^1(V'/G_m, U_k) = \mathrm{Pic}(V'/G_m) = \mathbf{Z}$, using proposition 3.

Next we calculate $H^1(\mathcal{U}, U_k)$. We have the following commutative diagrams:

$$(***) \qquad \begin{array}{ccccccccc} 1 \to & B^1(\mathcal{U}, U_k) & \to & Z^1(\mathcal{U}, U_k) & \to & H^1(\mathcal{U}, U_k) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & B^1(\mathcal{U}', U_k) & \to & Z^1(\mathcal{U}', U_k) & \to & H^1(\mathcal{U}', U_k) & \to 1 \end{array}$$

$$(****) \qquad \begin{array}{ccccccc} 1 \to & Z^1(\mathcal{U}, U_k) & \to & \prod U_k(W_i \cap W_j) & \rightrightarrows & \prod U_k(W_i \cap W_j \cap W_k) \\ & & & \downarrow f & & \downarrow g & \downarrow \\ 1 \to & Z^1(\mathcal{U}', U_k) & \to & \prod U_k(W_i' \cap W_j') & \rightrightarrows & \prod U_k(W_i' \cap W_j' \cap W_k') \end{array}$$

Since $W_i' = k^{(n+m-1)}$ and $W_i = W_i'/G$, $U_k(W_i') = U_k(W_i) = 1$. Thus in $(***)$, $B^1(\mathcal{U}, U_k) = B^1(\mathcal{U}', U_k) = 1$, and hence $Z^1(\mathcal{U}, U_k) = H^1(\mathcal{U}, U_k)$ and $Z^1(\mathcal{U}', U_k) = H^1(\mathcal{U}', U_k) = \mathbf{Z}$. Now $U_k(W_i' \cap W_j') = \mathbf{Z}$, and $U_k(W_i \cap W_j)$ is a non-trivial subgroup of $U_k(W_i' \cap W_j')$, since $W_i \cap W_j = (W_i' \cap W_j')/G$. Similarly, $U_k(W_i \cap W_j \cap W_k)$ is contained in $U_k(W_i' \cap W_j' \cap W_k')$. Now in the diagram $(****)$, the vertical maps are injections, and the cokernel of f injects into the cokernel of g. But the cokernel of g is torsion by the above remarks, and hence the cokernel of f is torsion, hence $Z^1(\mathcal{U}, U_k)$ is infinite cyclic. Thus $H^1(\mathcal{U}, U_k) = \mathbf{Z}$.

Now we return to the sequence $(*)$. For each i, W_i is affine and $\mathrm{Pic}(W_i)$ is contained in $\mathcal{C}\ell(W_i)$. Since $W_i = W_i'/G$ and $\mathcal{C}\ell(W_i') = 1$, by [1, 16.1, p. 82] $C\ell(W_i)$ is contained in the finite group $H^1(G, k[W_i']^*) = \mathrm{Hom}(G, k^*)$. Thus $\mathrm{Pic}(W_i)$ is finite, and hence $H^0(\mathcal{U}, \mathrm{Pic})$, which is a subgroup of $\prod \mathrm{Pic}(W_i)$, is finite. It follows from (*) that $H^1(V/G_m, U_k)$ is finitely generated of rank one. To complete the proof, we must show that $H^1(V/G_m, U_k) = \mathrm{Pic}(V/G_m)$ is torsion free. But this follows from [3, Lemma 4] (again, we note that the reference does not use separation in the proof).

We can now compute the relevant rings of invariants. We first observe that, in the notation of the discussion following Theorem 2, $k[V']^{G_m} = k[C']$. It follows that $k[V]^{G_m} = (k[V']^{G_m})^G = k[C']^G = k[C]$. Also, $C - C_0$ and $C' - C_0'$ are single points. Since C and C' have dimension $n + m - 1$, if $n > 1$ or $m > 1$, $C\ell(C_0) = C\ell(C)$.

THEOREM 5. *Suppose $n > 1$ and $m > 1$. Then $k[[(n, m), \lambda_1, \cdots, \lambda_{n+m}]]^{G_m}$ has class group* $\mathbf{Z}$.

PROOF. Let C_{00} be the non-singular locus of C_0. Since C_0 is normal, codim $(C_0 - C_{00}) \geqq 2$, so $C\ell(C_0) = C\ell(C_{00})$, and since C_{00} is non-singular, $C\ell(C_{00}) = \mathrm{Pic}(C_{00})$. Thus we need to compute $\mathrm{Pic}(C_{00})$. Since $n > 1$ and $m > 1$, codim $(V - V_0) \geqq 2$, hence $U_k(V_0) = 1$. Let V_{00} be the inverse image of C_{00} in V_0; since $C_{00} = V_{00}/G_m$ and codim $(C_0 - C_{00}) \geqq 2$, codim $(V_0 - V_{00}) \geqq 2$, so $U_k(V_{00}) = 1$. By [3, Lemma 4], $\mathrm{Pic}(C_{00})$ is torsion free. Let C_{00}' be the inverse image of C_{00} in C_0', so $C_{00} = C_{00}'/G$. Let V_{00}' be the inverse image of C_{00}' in V_0', so $V_{00}'/G_m = C_{00}'$. As before, codim $(C_0' - C_{00}') \geqq 2$, so $U_k(V_{00}') = 1$. Now $V_{00}' \to C_{00}'$ is a fibration with fibre G_m, so, as in Proposition 3 above, [2, Theorem 5] shows that $\mathrm{Pic}(C_{00}') = \mathbf{Z}$. Now we consider the diagram

$$\begin{array}{ccc} \mathrm{Pic}(V/G_m) & \to & \mathrm{Pic}(V'/G_m) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(C_{00}) & \to & \mathrm{Pic}(C_{00}'). \end{array}$$

Since codim $(V'/G_m - C_{00}') = \mathrm{codim}(V/G_m - C_{00}) \geqq 2$, the vertical arrows are injections, and in the proof of Theorem 4 we saw that the top horizontal arrow is an injection. Since $C_{00}' \to C_{00} = C_{00}'/G$ is finite, the bottom horizontal arrow has torsion kernel. But we know $\mathrm{Pic}(C_{00})$ to be torsion free. Since $\mathrm{Pic}(V/G_m) = \mathbf{Z}$ and $\mathrm{Pic}(C_{00}') = \mathbf{Z}$, it follows that $\mathrm{Pic}(C_{00})$ is also infinite cyclic, which completes the proof.

In order to treat the case $n = 1$ or $m = 1$, we will need the following number-theoretic observation

Lemma 6. *Let a, b, c be relatively prime positive integers. The g.c.d. (b, c) is the smallest positive integer e such that there is a pos tive integer f with $ea + fb \equiv 0 \pmod{c}$.*

Since the cases $n = 1$ and $m = 1$ are symmetric, we focus on th case $m = 1$.

Theorem 7. *Let $\lambda_1, \cdots, \lambda_{n+1}$ be relatively prime. Then $k[[(n, 1), \lambda$ $\cdots, \lambda_{n+1}]]^{G_m}$ has finite cyclic class group of order N, where N defined as follows: let $e_i =$ g.c.d. (g.c.d. $(\lambda_1, \cdots, \hat{\lambda}_i, \cdots, \lambda_n), \lambda_{n+1})$ for i = $1, 2, \cdots, n$. Then $N = \lambda_{n+1}/e_1 \cdots e_n$.*

Proof. We first consider the case $\lambda_{n+1} = 1$. For $i = 1, 2, \cdots,$ let $t_i = x_i x_{n+1}^{\lambda_i}$, and let $Y' = [(n, 1), \lambda_1, \cdots, \lambda_n, 1]$. Then it is easy t check that $k[Y']^{G_m} = k[t_1, \cdots, t_n]$ is a polynomial ring and hence h trivial class group. Now let $Y = [(n, 1), \lambda_1, \cdots, \lambda_{n+1}]$ and let $\psi : Y' \rightarrow$ be $\psi(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_n, x_{n+1}^{\lambda_{n+1}})$. $H = \Gamma(\lambda_{n+1})$ acts on Y' b $\beta(x_1, \cdots, x_{n+1}) = (x_1, \cdots, \beta x_{n+1})$, and ψ induces an isomorphism Y'/H with Y. Also, ψ is a morphism of spaces with G_m-action, and th actions of H and G_m on Y' commute. Let $S = k[Y']^{G_m}$ and let R = $k[Y]^{G_m}$. It follows that $S^H = R$, so that by [1, 16.1, p. 82] there is a exact sequence $1 \rightarrow C\ell(R) \rightarrow H^1(H, k^*) \rightarrow \mathrm{Div}(S)^H/\mathrm{Div}(R) \rightarrow 1$; th right hand 1 coming from the fact that $C\ell(S) = 1$. H is cyclic of ord λ_{n+1} and hence so is $H^1(H, k^*)$: if α is a generator of H, α sends t_i $\alpha^{\lambda_i} t_i$ and since g.c.d. $(\lambda_1, \cdots, \lambda_{n+1}) = 1$, H acts faithfully on S.

We need to compute the order of $\mathrm{Div}(S)^H/\mathrm{Div}(R)$. Choose positi integers $b_1, \cdots, b_n$ such that $\sum b_i\lambda_i \equiv 1\ (\lambda_{n+1})$, and let $u = t_1^{b_1} \cdots t_n^{b_n}$ Then $\alpha(u) = \alpha u$, so u is a primitive element for the quotient field S over the quotient field of R, with minimal polynomial $f = T^d - u$ where $d = \lambda_{n+1}$. Then $f'(u) = du^{d-1}$. By [1, 16.3, p. 84], the heig one primes of S which ramify are among $P_1 = (t_1), \cdots, P_n = (t_n)$. L e_i be the ramification index of P_i. The above exact sequence shows th $C\ell(R)$ is cyclic of order $d/e_1 \cdots e_n$. To complete the proof, we mu show that e_i is the desired g.c.d.

Now R is generated over k by invariant monomials, and hence is the least positive integer such that there is an invariant monomial degree e_i in t_i. For convenience, we may assume $i = 1$, and let w = $t_1^{e_1} t_2^{b_2} \cdots t_n^{b_n}$ be an invariant monomial. Then $e_1\lambda_1 + \sum b_i\lambda_i \equiv$ (mod d), so e_1 is at least as large as g.c.d. (g.c.d. $(\lambda_2, \cdots, \lambda_n), d)$. If co versely e is that g.c.d., there are positive integers $c_2, \cdots, c_n$ such th $e\lambda_1 + \sum c_i\lambda_i \equiv 0(d)$, and hence $w' = t_1^e t_2^{c_2} \cdots t_n^{c_n}$ is an invaria monomial, so $e_1 \leqq e$. Thus $e_1 = e$ and the result follows.

The assumption in Theorem 7 that g.c.d. $(\lambda_1, \cdots, \lambda_{n+1}) = 1$ is th the action is reduced, and, by the remark following definition 1, this can always be assumed.

We conclude with an illustration of Theorem 7: $\mathcal{C}\ell(k[[(2, 1), 1, 1, n]]) = \mathbf{Z}/n\mathbf{Z}$ for any positive n: for g.c.d. $(1, n) = 1$. A. Geramita and M. Krusemeyer (unpublished) have computed generators and relations of $k[[(2, 1), 1, 1, n]]^{G_m} = R$: let $S = k[s_0, \cdots, s_n]$ map to R by $s_0 \to x_2^n x_3$, $s_1 \to x_1 x_2^{n-1} x_3$, $\cdots$, $s_{n-1} \to x_1^{n-1} x_2 x_3$, $s_n \to x_1^n x_3$. This is surjective, and the kernel is the ideal generated by the 2×2 minors of

$$\begin{bmatrix} s_0 s_1 \cdots s_{n-1} \\ s_1 s_2 \cdots s_n \end{bmatrix} .$$

References

1. R. Fossum, *The Divisor Class Group of a Krull Domain*, Springer-Verlag, New York, 1973.

2. A. Magid, *The Picard sequence of a fibration*, Proc. Amer. Math. Soc. **53** (1975), 37–40.

3. ———, *Fundamental, Class, and Picard groups of rings of invariants*, preprint.

4. C. Seshadri, *Quotient spaces modulo reductive algebraic groups*, Amer. J. Math. **95** (1972), 511–556.

UNIVERSITY OF OLKAHOMA, NORMAN, OKLAHOMA 73069