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ORTHOGONALITY AND THE HEWITT-YOSIDA THEOREM IN SPACES OF MEASURES

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1. **Introduction.** This paper is concerned with a study of the relationship between the Hewitt-Yosida decomposition theorem and orthogonal decompositions and unique nearest point maps for Banach spaces of measures. During the course of this study, questions relating to the differentiability of the norm in spaces of measures arise. Results of Appling are used to establish connections between the differentiability of this norm and absolute continuity in the concluding section of this paper.

The general setting follows. Let X be a real ($\mathbf{R}$) Banach space, let Σ be an algebra of subsets of a universal space S, and let $ba(\Sigma, X)$ be the Banach space of bounded finitely additive X-valued set functions $\mu : \Sigma \to X$ equipped with the semivariation norm $\|\mu\| = \sup\{|z^*\mu|(S) : z^* \in X^*, \|z^*\| \leqq 1\}$, where $|z^*\mu|(A)$ denotes the total variation of $z^*\mu$ on A. The countably additive members of $\mathrm{ba}(\Sigma, X)$ are denoted by $\mathrm{ca}(\Sigma, X)$; if $X = \mathbf{R}$, then we shorten the notation to $\mathrm{ba}(\Sigma)$ and $\mathrm{ca}(\Sigma)$. If $\mu \in \mathrm{ba}(\Sigma, X)$, then we say that μ is s-additive (= strongly additive) if $\mu(A_i) \to 0$ for each disjoint infinite sequence $(A_i) \subset \Sigma$, and we denote the set of s-additive measures by $\mathrm{sa}(\Sigma, X)$. The space $\mathrm{sa}(\Sigma)$ has been studied by several authors. In particular, we note the papers by Brooks [3] and Uhl [12] and extract from them the following result.

1.1. THEOREM. *Let $\mu \in \mathrm{ba}(\Sigma, X)$. Then the following are equivalent:*

(a) *$\mu \ll \lambda$ for some $0 \leqq \lambda \in \mathrm{ba}(\Sigma)$;*

(b) *μ has conditionally weakly compact range;*

(c) *$\mu \in \mathrm{sa}(\Sigma, X)$.*

Furthermore, if μ is countably additive, then the above conditions are equivalent to the statement that μ has a necessarily unique countably additive extension to $\sigma(\Sigma)$.

If $\mu \in \mathrm{ba}(\Sigma)$, then μ is said to be purely finitely additive (pfa) if $0 \leqq \xi \leqq |\mu|$ and $\xi \in \mathrm{ca}(\Sigma)$ imply that $\xi = 0$. The space $\mathrm{pfa}(\Sigma)$ was studied by Hewitt and Yosida in [13]. In this paper it is shown that each $\mu \in \mathrm{ba}(\Sigma)$ can be written uniquely as $\mu_c + \mu_f$, where $\mu_c \in \mathrm{ca}(\Sigma)$ and $\mu_f \in \mathrm{pfa}(\Sigma)$. The Hewitt-Yosida theorem has been generalized by Brooks [3], Uhl [12], and Huff [8]. (A short version of

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the existence portion of Huff's proof will be presented at the end of the next section.) Specifically, we state the following version of Uhl's theorem [12, p. 675].

1.2. THEOREM. *Let* $\mu \in \mathrm{sa}(\Sigma, X)$. *Then there exist unique vector measure* μ_c *and* μ_f *such that* μ_c *is countably additive,* $x^*\mu_f$ *is purely finitely additive for each* $x^* \in X^*$, *and* $\mu = \mu_c + \mu_f$. *Further,* $\mu_c, \mu_f \in \mathrm{sa}(\Sigma, X)$.

If Y is a Banach space and $x, y \in Y$, then we say that x is orthogonal to $y (x \perp y)$ if $\|x\| = \inf\{\|x + ty\| : t \in \mathbf{R}\}$. If $A \subset Y$, then by $x \perp A$ we mean that $x \perp y$ for each $y \in Y$. Orthogonality has been studied by James [9], [10]. Also, let Y^*_1 denote the closed unit ball of Y^*.

2. Orthogonality.

2.1. LEMMA. *Let* $\mu \in \mathrm{ba}(\Sigma)$. *Then* $\mu \in \mathrm{pfa}(\Sigma)$ *iff* $\mu \perp \mathrm{ca}(\Sigma)$. *Furthermore, the projection* $P : \mathrm{ba}(\Sigma) \to \mathrm{ca}(\Sigma)$ *defined by* $P(\mu) = \mu_c$ *is a unique nearest-point map.*

PROOF. Suppose that $0 \leqq \mu \in \mathrm{pfa}(\Sigma)$ and $\nu \in \mathrm{ca}(\Sigma)$. Then $\|\mu - \nu\| \geqq \|\mu - |\nu|\|$. But since $\mu \wedge |\nu| = 0$, we know that $\|\mu - |\nu|\| = \|\mu + |\nu|\| = \|\mu\| + \|\nu\| \geqq \|\mu\|$. Therefore $\mu \perp \mathrm{ca}(\Sigma)$. For the general case we note that if $|\mu| \in \mathrm{pfa}(\Sigma)$, then $\|\mu\| = \||\mu|\| \leqq \||\mu| - |\nu|\| \leqq \|\mu - \nu\|$ for each $\nu \in \mathrm{ca}(\Sigma)$. Thus $\mu \perp \mathrm{ca}(\Sigma)$.

Conversely, suppose that $\mu \perp \mathrm{ca}(\Sigma)$, and let μ^+ and μ^- be the positive and negative variations, respectively, of μ. Suppose that $0 \leqq \xi \leqq \mu^+$ and that $\xi \in \mathrm{ca}(\Sigma)$. If $\xi(S) > 0$, then $\|\mu - \xi\| \leqq (\mu^+ - \xi)(S) + \mu^-(S) < \mu^+(S) + \mu^-(S) = \|\mu\|$, a contradiction. Therefore $\mu^+ \in \mathrm{pfa}(\Sigma)$. In a similar way one can show that $\mu^- \in \mathrm{pfa}(\Sigma)$ and consequently $\mu \in \mathrm{pfa}(\Sigma)$.

Now suppose that $\mu = (\mu_c + \mu_f) \in \mathrm{ba}(\Sigma)$. If $\nu \in \mathrm{ca}(\Sigma)$, then $\|\mu - \nu\| = \|\mu_f - (\nu - \mu_c)\| \geqq \|\mu_f\|$. Thus $\|\mu - \mu_c\| = \|\mu_f\| =$ distance $(\mu, \mathrm{ca}(\Sigma))$. In fact, it follows that μ_c is the unique nearest point.

The following result shows that this lemma does not extend in general to $\mathrm{ba}(\Sigma, X)$.

2.2. LEMMA. *If* $\Phi \in \mathrm{ba}(\Sigma, X)$ *and* $x^*\Phi \in \mathrm{pfa}(\Sigma)$ *for each* $x^* \in X^* (\Phi \in \mathrm{pfa}(\Sigma, X))$, *then* $\Phi \perp \mathrm{ca}(\Sigma, X)$. *However the converse is false even for finite dimensional* X.

PROOF. Let $\Phi \in \mathrm{pfa}(\Sigma, X)$. By definition, $\|\Phi\| = \sup\{\|x^*\Phi\| : x^* \in X^*_1\}$. But if $x^* \in X^*_1$, then $\|x^*\Phi\| \leqq \|x^*\Phi + x^*\nu\|$ for each $\nu \in$

$\mathrm{ca}(\Sigma, X)$ by Lemma 2.1. Therefore $\|\Phi\| \leqq \|\Phi + \nu\|$ for each $\nu \in \mathrm{ca}(\Sigma, X)$, and $\Phi \perp \mathrm{ca}(\Sigma, X)$.

To establish the last assertion of the lemma, let X be 2-dimensional Euclidean space equipped with the sup norm, and let Σ be any algebra for which each of $\mathrm{ca}(\Sigma)$ and $\mathrm{pfa}(\Sigma)$ is non-trivial. Then there is a non-negative purely finitely measure μ in $\mathrm{ba}(\Sigma)$ so that $\|\mu\| = 1$ and there is $0 \leqq \xi \in \mathrm{ca}(\Sigma)$ so that $\|\xi\| = 1$. Then define $\hat{\mu} : \Sigma \to X$ by $\hat{\mu}(A) = (2\mu(A), \ \mu(A))$. It follows that $\hat{\mu} \in \mathrm{pfa}(\Sigma, X)$, $\nu = \hat{\mu} + (0, \xi)$ is orthogonal to $\mathrm{ca}(\Sigma, X)$, and ν is not purely finitely additive.

The pathology in the preceding lemma occurs because elements of $\mathrm{pfa}(\Sigma, X)$ do not necessarily have unique nearest points in $\mathrm{ca}(\Sigma, X)$. The following theorems and corollaries show that this aberrance is not possible in a fairly general setting.

If x belongs to the Banach space Y, then we say that x is a smooth point if the norm is Gateaux differentiable at x, and we call Y smooth if each point on the surface of the unit ball is a smooth point. We use the notation $D(x, y)(D^+(x, y), D^-(x, y))$ to denote the Gateaux derivative (one sided derivatives) of the norm function at the point x in the direction y.

2.3. THEOREM. *Suppose that* $\mu \in \mathrm{ba}(\Sigma, X)$, $z^* \in X^*_1$, $\|z^*\mu\| = \|\mu\|$, *and* z^* *is a smooth point. Then* $D(z^*\mu, \ y^*\mu) = 0$ *whenever* $D(z^*, y^*) = 0$.

PROOF. Suppose that $D(z^*, y^*) = 0$, let $p(t) = (z^* + ty^*)/\|z^* + ty^*\|$, $t \in \mathbf{R}$, and set $F(t) = \|p(t)\mu\|$. Then $F(t) = \|p(t)\mu\| \leqq \|\mu\| = F(0)$ for each t.

Further, the function p is differentiable at $t = 0$; in fact, $p'(0) = y^*$. Then by two applications of the chain rule,

$$F'(0+) = D^+(p(0)\mu, p'(0)\mu) = D^+(z^*\mu, y^*\mu),$$

and

$$F'(0-) = D^-(p(0)\mu, p'(0)\mu) = D^-(z^*\mu, y^*\mu).$$

Therefore since $F(0)$ is the maximum value of F, $D^+(z^*\mu, \ y^*\mu) \leqq 0 \leqq D^-(z^*\mu, \ y^*\mu)$. But because of the convexity of the norm, $D^-(z^*\mu, y^*\mu) \leqq D^+(z^*\mu, y^*\mu)$. Thus $D^+(z^*\mu, y^*) = D^-(z^*\mu, y^*\mu) = 0$.

2.4. COROLLARY. *Suppose that* $\mu \in \mathrm{ba}(\Sigma, X)$, z^* *is a smooth point of* X^*_1, *and* $\|z^*\mu\| = \|\mu\|$. *Then* $x^*\mu \ll z^*\mu$ *for each* $x^* \in X^*$.

PROOF. The existence of $D(z^*, x^*) \equiv f(x^*)$ for each $x^* \in X^*$ implies that $f \in X^{**}$, and hence $\ker(f)$ is a closed maximal subspace of X^*. Since $z^*\mu$ is obviously absolutely continuous with respect to $z^*\mu$, the conclusion then follows from Theorem 3.1, *infra*.

2.5. THEOREM. *Suppose that* μ, ξ, $\eta \in \mathrm{ba}(\Sigma, X)$, $\mu = \xi + \eta$, $|x^*\xi| \wedge |y^*\eta| = 0$ *for all* x^*, $y^* \in X^*$, *and that* $\|\mu\| = \|\eta\|$. *Suppose also that* z^* *is a smooth point of* X^*_1 *so that* $\|z^*\eta\| = \|\eta\|$. *Then* $\xi = 0$.

PROOF. Suppose that μ, ξ, η, and z^* satisfy the hypothesis of the theorem. Then since $|z^*\xi| \wedge |z^*| = 0$, we see that $\|\mu\| \geqq \|z^*\mu\| = \|z^*\xi\| + \|z^*\eta\| = \|\eta\| = \|\mu\|$. Hence $z^*\xi = 0$. And by the preceding corollary, $x^*\xi + x^*\eta = x^*\mu \ll z^*\mu (= z^*\eta)$ and $x^*\eta \ll z^*\eta$ for each $x^* \in X^*$. Therefore $|x^*\xi| \wedge |z^*\eta| = 0$ and $x^*\xi \ll z^*\eta$, which implies that $x^*\xi = 0$ for each x^*.

Before establishing our characterization of orthogonality, we need a lemma. Let $U(\Sigma)$ be the uniform closure of the real valued simple functions over Σ, let $T : U(\Sigma) \to X$ be an operator (= bounded linear map), and let $\mu : \Sigma \to X$ be the unique finitely additive X-valued measure with finite semivariation so that $T(f) = \int f d\mu$, $f \in U(\Sigma)$. For $x^* \in X^*$, define $\zeta(x^*)$ to be $\|z^*\mu\|$; then ζ is a seminorm on X^*.

2.6. LEMMA. *The operator T is compact iff* (X^*_1, ζ) *is a compact space.*

PROOF. Suppose that (X^*_1, ζ) is compact, let $(z^*_n) \subset X^*_1$, and let z^* be a ζ-cluster point of (z^*_n). Then

$$\begin{aligned} &\|T^*(z^*_n) - T^*(z^*)\| \\ &\quad = \sup\{|(f, T^*(z^*_n) - T^*(z^*))| : f \in U(\Sigma), \|f\| \leqq 1\} \\ &\quad = \sup\{|(T(f), z^*_n - z^*)|\} \\ &\quad = \|(z^*_n - z^*)\mu\|. \end{aligned}$$

Therefore T^* is compact and hence T is compact.

Conversely, suppose that T is compact (therefore T^* is compact) and let (z^*_α) be a net in X^*_1. Without loss of generality, suppose that $z^*_\alpha \xrightarrow{w^*} z^* \in X^*_1$. Then by Theorem 6, p. 486, of [6], $T^*(z^*_\alpha) \to T^*(z^*)$ in the norm topology. As a consequence of this

$$\begin{aligned} &\sup\{|(T(f), z^*_\alpha - z^*)| : f \in U(\Sigma), \|f\| \leqq 1\} \\ &\quad = \|(z^*_\alpha - z^*)\mu\| \to 0 \text{ with } \alpha, \end{aligned}$$

and (X^*_1, ζ) is compact.

Let $\mathcal{C}(\Sigma, X)$ denote those members of $\mathrm{sa}(\Sigma, X)$ with conditionally compact range; ca $\mathcal{C}(\Sigma, X)$ and pfa $\mathcal{C}(\Sigma, X)$ are defined to be the countably additive and purely finitely additive members, respectively, of $\mathcal{C}(\Sigma, X)$.

2.7. THEOREM. *Suppose that X is a Banach space so that X^* is smooth. If* $\mu \in \mathcal{C}(\Sigma, X)$, *then* $\mu \in \text{pfa } \mathcal{C}(\Sigma, X)$ *iff* $\mu \perp \text{ca } \mathcal{C}(\Sigma, X)$.

Furthermore, the Hewitt-Yosida projection $P : \mathcal{C}(\Sigma, X) \to \mathrm{ca}\ \mathcal{C}(\Sigma, X)$ is a unique-nearest-point map.

PROOF. We begin by noting that by Uhl's proof of the Hewitt-Yosida theorem in $\mathrm{sa}(\Sigma, X)$ [12], it follows that $\mu_c(\Sigma)$ and $\mu_f(\Sigma)$ are subsets of $\overline{\mathrm{co}}(\mu(\Sigma))$. Therefore if $\mu \in \mathcal{C}(\Sigma, X)$, then $\mu_c, \mu_f \in \mathcal{C}(\Sigma, X)$.

Now suppose that $\mu \perp \mathrm{ca}\ \mathcal{C}(\Sigma, X)$, and write μ as $\mu_c + \mu_f$. Then $\|\mu\| = \inf\{\|\mu + \nu\| : \nu \in \mathrm{ca}\ \mathcal{C}(\Sigma, X)\} = \|\mu_f\|$. Now since $\mu_f(\Sigma)$ is conditionally compact, the operator $T_f : U(\Sigma) \to X$ defined by $T_f(g) = \int g\, d\mu_f$, $g \in U(\Sigma)$, is compact [6], p. 497. Consequently, if ζ_f is the seminorm associated with T_f by Lemma 2.7, then $(X^*_1, \zeta f)$ is compact. Thus there is some $z^* \in X^*_1$ so that $\|z^*\| = 1$ and $\|z^*\mu_f\| = \|\mu_f\|$. Hence by Theorem 2.5, $\mu_c = 0$, and it follows that $\mu = \mu_f \in \mathrm{pfa}\ \mathcal{C}(\Sigma, X)$. The converse implication was established in Lemma 2.2.

The final assertion in the theorem is that if $\mu = \mu_c + \mu_f$, then $\|\mu - \mu_c\| = \text{distance } (\mu, \mathrm{ca}\ \mathcal{C}(\Sigma, X))$ and μ_c is the unique member of $\mathcal{C}(\Sigma, X)$ which produces this equality. Since $\mu - \mu_c = \mu_f$ and $\inf\{\|\mu + \nu\| : \nu \in \mathrm{ca}\ \mathcal{C}(\Sigma, X)\} = \inf\{\|\mu_f + \nu\| : \nu \in \mathrm{ca}\ \mathcal{C}(\Sigma, X)\} = \|\mu_f\|$, it follows that μ_c is a nearest point. And if $\nu \in \mathrm{ca}\ \mathcal{C}(\Sigma, X)$ and $\|\mu + \nu\| = \|\mu_f + (\mu_c + \nu)\| = \|\mu_f\|$, then $\mu_c + \nu = 0$ by the argument in the preceding paragraph.

In the following propositions we give alternative constructions of the Hewitt-Yosida theorem in $\mathrm{sa}(\Sigma, X)$ and in $\mathcal{C}(\Sigma, X)$. Our approach in 2.8 uses the operators studied by Huff in [8] and the Tychonoff theorem directly instead of appealing to semigroup techniques. Our approach in 2.9 is motivated by, and relies upon, a theorem of D. Lewis [11], p. 206.

We remark that if $\mu \in \mathrm{ba}(\Sigma)$, then μ is purely finitely additive iff for each $\epsilon > 0$ there is a countable partition (A_i) of S so that $|\Sigma \mu(A \cap A_i)| < \epsilon$ for each $A \in \Sigma$.

2.8. THEOREM. *Suppose that $\mu \in \mathrm{sa}(\Sigma, X)$. Then μ can be written as $\mu_c + \mu_f$, where μ_c is countably additive and $x^*\mu_f$ is purely finitely additive for each $x^* \in X^*$.*

PROOF. Let P denote the directed set (by refinement) of all countable Σ-partitions. For $\alpha = (A_i) \in P$ and $\mu \in \mathrm{sa}(\Sigma, X)$, let $T_\alpha(\mu) = \Sigma \mu \cdot A_i$, where $\mu \cdot A(B) = \mu(A \cap B)$. Then $T_\alpha(\mu) \in \mathrm{sa}(\Sigma, X)$ since μ is s-additive, and the range of $\sigma_\mu(\sigma_\mu(A) \equiv \mu \cdot A)$ is conditionally weakly compact in $\mathrm{sa}(\Sigma, X)$ by Theorem 1.1. Further, μ is countable additive iff $T_\alpha(\mu) = \mu$ for each $\alpha \in P$. Let $w(\sigma_\mu)$ denote the weak closure of $\sigma_\mu(\Sigma)$, let w denote the weak topology, and let $W = \Pi(w(\sigma_\mu), w)$, $\mu \in \mathrm{sa}(\Sigma, X)$. Then (T_α) is a net in the space W, and by the Tychonoff theorem some subnet (suppose the full net), must converge to a point $T \in W$. It follows easily that $T : \mathrm{sa}(\Sigma, X) \to \mathrm{sa}(\Sigma, X)$ is a continuous linear projection. Further, since $T_\alpha(T(\mu)) = T(\mu)$ for each α, then $T(\mu)$ is countably additive. Also if $T(\mu) = 0$, $x^* \in X^*$, and $\epsilon > 0$, then there is an $\alpha_0 \in P$ so that if $\alpha \geqq \alpha_0$, then $|\Sigma_\alpha x^*\mu(A_i)| < \epsilon$. Thus it follows that if $\alpha \geqq \alpha_0$ and $A \in \Sigma$, then $|\Sigma_\alpha x^*\mu(A \cap A_i)| \leqq 2\epsilon$; hence $x^*\mu$ is purely finitely additive.

If U and V are Banach spaces, then λ will denote the least cross-norm on $U \otimes V$, and $U \otimes_\lambda V$ will denote the λ-completion of $U \otimes V$. If $T : U \to X$ and $L : V \to Y$ are operators, then $T \otimes_\lambda L$ will denote the tensor product operator from $U \otimes_\lambda V$ to $X \otimes_\lambda Y$, e.g., see Holub [7]. The canonical mapping of $\mathrm{ba}(\Sigma) \otimes X$ into $\mathcal{C}(\Sigma, X)$ will be denoted by Φ, Φ_1 will denote the restriction of Φ to $\mathrm{ca}(\Sigma) \otimes X$, and Φ_2 will denote the restriction of Φ to $\mathrm{pfa}(\Sigma) \otimes X$.

2.8. PROPOSITION. (i) Φ *extends to an isometry of* $\mathrm{ba}(\Sigma) \otimes_\lambda X$ *onto* $\mathcal{C}(\Sigma, X)$;

(ii) Φ_1 *extends to an isometry of* $\mathrm{ca}(\Sigma) \otimes_\lambda X$ *onto* ca $\mathcal{C}(\Sigma, X)$;

(iii) Φ_2 *extends to an isometry of* $\mathrm{pfa}(\Sigma) \otimes_\lambda X$ *onto* pfa $\mathcal{C}(\Sigma, X)$;

(iv) *if* $P : \mathrm{ba}(\Sigma) \to \mathrm{ca}(\Sigma)$ *is the nearest point projection and* $I : X \to X$ *is the identity, then the mapping* θ *of Huff is given by*

$$\Phi_1 \cdot (P \otimes_\lambda I) \cdot \Phi^{-1}.$$

PROOF. We establish (i) by showing that $\Phi : \mathrm{ba}(\Sigma) \otimes X \to \mathcal{C}(\Sigma, X)$ is an isometry with dense range. Let $u = \Sigma\, \mu_i \otimes x_i$, let $z^* \in X^*{}_1$, and let (A_j) be a Σ-partition of S. Then

$$\Sigma_j\, |\Sigma_i\, z^*(x_i)\, \mu_i(A_j)| \leqq |z^*(\Sigma\, \mu_i x_i)|(\textstyle\bigcup A_j),$$

which implies that $\|\Sigma\, \mu_i \otimes x_i\|_\lambda \leqq \|\Sigma \mu_i x_i\| = \|\Phi(u)\|$. Conversely,

$$\Sigma_j |z^*(\Sigma_i\, \mu_i x_i) A_j| \leqq \|\Sigma z^*(x_i)\mu_i\| \leqq \|\Sigma\, \mu_i \otimes x_i\|_\lambda.$$

Therefore $\|\Phi(u)\| = \|u\|$.

To show that Φ is a dense embedding, let $\bar{\Sigma}$ be the Stone algebra, let $\bar{\nu} \in \mathcal{C}(\bar{\Sigma}, X)$, let $\bar{\nu}_1$ be the unique countably additive extension of ν_1 to $\sigma(\bar{\Sigma})$ which is given by Theorem 1.1, and let $\epsilon > 0$. An investigation of the argument in Uhl [12] shows that $\bar{\nu}_1 \in \mathrm{ca}\ \mathcal{C}(\sigma(\Sigma), X)$ so that by Theorem 3.1 of D. Lewis [11], there is an element $\bar{u}_1$ in $\mathrm{ca}(\sigma(\bar{\Sigma})) \otimes X$ so that $\|\Phi(\bar{u}_1) - \bar{\nu}_1\| < \epsilon$. If $\Phi(\bar{u})$ is the restriction of $\Phi(\bar{u}_1)$ to $\bar{\Sigma}$, then $\Phi(\bar{u}) \in \Phi(\mathrm{ba}(\bar{\Sigma}) \otimes X)$ and $\|\Phi(\bar{u}) - \bar{\nu}\| < \epsilon$. Since $\mathcal{C}(\Sigma, X)$ and $\mathcal{C}(\bar{\Sigma}, x)$ are canonically isometric, part (i) follows.

Parts (ii) and (iii) follow from (i) and the observations that

(a) $\mathrm{ca}(\Sigma) \otimes_\lambda X(\mathrm{ca}\ \mathcal{C}(\Sigma, X))$ is a closed subspace of $\mathrm{ba}(\Sigma) \otimes_\lambda X(\mathcal{C}(\Sigma, X))$,

(b) pfa $(\Sigma) \otimes_\lambda X(\text{pfa } \mathcal{C}(\Sigma, X))$ is a closed subspace of $\text{ba}(\Sigma) \otimes_\lambda X(\mathcal{C}(\Sigma, X))$,

(c) Φ_1 maps $\text{ca}(\Sigma) \otimes_\lambda X$ into ca $\mathcal{C}(\Sigma, X)$,

(d) Φ_2 maps $\text{pfa}(\Sigma) \otimes_\lambda X$ into pfa $\mathcal{C}(\Sigma, X)$,

and

(e) $\text{ba}(\Sigma) = \text{ca}(\Sigma) \oplus \text{pfa}(\Sigma)$.

For the final assertion, it is clear that $P \otimes_\lambda I$ fixes all of $\text{ca}(\Sigma) \otimes X$ and therefore all of $\text{ca}(\Sigma) \otimes_\lambda X$. Hence $Q = \Phi_1 \cdot (P \otimes_\lambda I) \cdot \Phi^{-1}$ is a projection from $\mathcal{C}(\Sigma, X)$ onto ca $\mathcal{C}(\Sigma, X)$. And $Q(\mu) = 0$ iff $P \otimes_\lambda I(\Phi^{-1}(\mu)) = 0$. But $N(P \otimes_\lambda I) = \text{pfa}(\Sigma) \otimes_\lambda X$, and hence $\mu \in$ pfa $\mathcal{C}(\Sigma, X)$.

3. **Differentiability of the Norm and Absolute Continuity.** Derivatives of the semivariation norm in $\text{ba}(\Sigma)$ may be described as refinement-type integrals of real functions on Σ. The definition of the integral is

$$\int \mu = \lim \Sigma\, \mu(A_i),$$

the limit being taken over all Σ-partitions of S, directed by refinement. The reader is referred to the papers of Appling, especially [1], [2], for further details. We need the formula for the absolutely continuous part of ν with respect to μ, where $\nu, \mu \in \text{ba}(\Sigma)$:

$$\nu_a(A) = \int_A (\text{sgn}\, \nu)|\nu|_a$$

$$= \lim_{k\to\infty} \int_A (\text{sgn}\, \nu)\min\{|\nu(\,\cdot\,)|, k|\mu(\,\cdot\,)|\}.$$

We decompose ν into $\nu_a + \nu_s$, $\nu_a \ll \mu$ and $|\nu_s| \wedge |\mu| = 0$.

3.1. THEOREM. *If* $\nu, \mu \in \text{ba}(\Sigma)$, *then*

$$D^+(\mu, \nu) = \int (\text{sgn}\, \mu)\nu_a + \|\nu_s\|,$$

and

$$D^-(\mu, \nu) = \int (\text{sgn}\, \mu)\nu_a - \|\nu_s\|.$$

Consequently, $D(\mu, \nu)$ *exists iff* $\nu \ll \mu$, *in which case* $D(\mu, \nu) = \int (\text{sgn}\, \mu)\nu$.

PROOF. We first show that $D^+(\mu, \nu) - D^-(\mu, \nu) = 2\|\nu_S\|$. Now

$$D^+(\mu, \nu) - D^-(\mu, \nu) = \lim_{t\to 0^+} (\|\mu + t\nu\| + \|\mu - t\nu\| - 2\|\mu\|)/t$$

$$= \lim_{k\to\infty} \int (|k\mu(\,\cdot\,) + \nu(\,\cdot\,)| + |k\mu(\,\cdot\,) - \nu(\,\cdot\,)| - 2k|\mu(\,\cdot\,)|).$$

This integrand may be written as

$$|x + y| + |x - y| - 2|x| = 2\max\{|x|, |y|\} - 2|x|\} = 2(|y| - \min\{|y|, |x|\}).$$

Therefore

$$\begin{aligned} &D^+(\mu, \nu) - D^-(\mu, \nu) \\ &\quad = 2\lim_{k\to\infty} \int (|\nu(\,\cdot\,)| - \min\{|\nu(\,\cdot\,)|, k|\mu(\,\cdot\,)|\}) \\ &\quad = 2(|\nu|(S) - \lim_{k\to\infty} \int \min\{|\nu(\ )|,\ k|\mu(\ )|\}) \\ &\quad = 2(|\nu|(S) - |\nu|_a(S)) \\ &\quad = 2\|\nu_s\|. \end{aligned}$$

Next we show that

$$D^+(\mu, \nu) + D^-(\mu, \nu) = 2\int (\operatorname{sgn}\mu)\nu_a,$$

and our stated conclusions follow at once.

Now

$$\begin{aligned} &D^+(\mu, \nu) + D^-(\mu, \nu) \\ &\quad = \lim_{k\to\infty} \int (|k\mu(\,\cdot\,) + \nu(\,\cdot\,)| - |k\mu(\,\cdot\,) - \nu(\,\cdot\,)|). \end{aligned}$$

Since $|x + y| - |x - y| = 2(\operatorname{sgn} x)(\operatorname{sgn} y)\min\{|x|, |y|\}$,

$$\begin{aligned} &D^+(\mu, \nu) + D^-(\mu, \nu) \\ &\quad = 2\lim_{k\to\infty} \int (\operatorname{sgn}\mu)(\operatorname{sgn}\nu)\min\{|\nu(\,\cdot\,)|, k|\mu(\,\cdot\,)|\}. \end{aligned}$$

According to the Kolmogoroff principle of differential equivalence (for example, [**2**] Cor. 2.K), the last integral is

$$\int (\operatorname{sgn}\mu)(\operatorname{sgn}\nu)\xi_k$$

where $\xi_k(A) = \int_A \min\{|\nu(\,\cdot\,)|, k|\mu(\,\cdot\,)|\}$, $A \in \Sigma$. Now $\|\xi_k - |\nu|_a\| \to 0$; hence

$$D^+(\mu, \nu) + D^-(\mu, \nu) = 2\int (\operatorname{sgn}\mu)(\operatorname{sgn}\nu)|\nu|_a.$$

A second application of the Kolmogoroff principle completes the proof.

The differentiality criterion for absolute continuity is not valid for the semivariation norm on $\mathrm{ba}(\Sigma, X)$. Rather, the variation norm proves to be appropriate for extending this criterion. We are able to verify the criterion for countably additive set functions of bounded variation defined on a σ-algebra and having values in a smooth space with RNP. Before proceeding to this verification we offer a counterexample in $\mathrm{ba}(\Sigma, X)$.

EXAMPLE. Let X be two-dimensional Hilbert space and let Σ be an algebra such that $\mathrm{ba}(\Sigma)$ contains disjoint members μ_1 and μ_2 of unit norm. Then $(0, \mu_2)$ is not absolutely continuous with respect to $(\mu_1, 0)$, yet $D((\mu_1, 0), (0, \mu_2))$ exists because

$$\|(\mu_1, 0) + t(0, \mu_2)\| = (1 + t^2)^{1/2}.$$

On the other hand,

$$(\mu_1, \mu_1) \ll (\mu_1, \mu_2),$$

$$D^+((\mu_1, \mu_2), (\mu_1, \mu_1)) = (2)^{1/2},$$

and

$$D^-((\mu_1, \mu_2), (\mu_1, \mu_1)) = 0.$$

3.2. THEOREM. *Suppose X is a Banach space with the Radon-Nikodym property, Σ is a σ-algebra, and $\nu, \mu \in \mathrm{cabv}(\Sigma, \gamma)$. If $D(\mu, \nu)$ exists then $\nu \ll |\mu|$. If X is smooth and $\nu \ll |\mu|$, then $D(\mu, \nu)$ exists and*

$$D(\mu, \nu) = \int D\left(\frac{d\mu}{d|\mu|}, \frac{d\nu}{d|\mu|}\right) d|\mu|.$$

PROOF. Suppose that $D(\mu, \nu)$ exists, and decompose ν into $\nu_a + \nu_s$, where $\nu_a \ll |\mu|$ and $|\nu_s| \cap |\mu| = 0$. Then

$$D^+(\mu, \nu) = D^+(\mu, \nu_a) + \|\nu_s\|$$

and

$$D^-(\mu, \nu) = D^-(\mu, \nu_a) - \|\nu_s\|.$$

Therefore, since $D^+(\mu, \nu) = D^-(\mu, \nu)$, we have

$$0 \leqq D^+(\mu, \nu_a) - D^-(\mu, \nu_a) = -2\|\nu_s\|;$$

hence $\nu_s = 0$.

Now suppose X is smooth and $\nu \ll |\mu|$. The Radon-Nikodym property insures the existence of f, $g \in L^1(|\mu|, X)$ such that $\mu(A) = \int_A f d|\mu|$ and $\nu(A) = \int_A g d|\mu|$, $A \in \Sigma$. Now $\|f(x)\| = 1$ almost everywhere ($|\mu|$); hence $D(f(x), g(x))$ exists almost everywhere. Define

$$P(x, t) = (\|f(x) + tg(x)\| - \|f(x)\|)/t, t > 0.$$

Because of the convexity of the norm, $P(x, t)$ converges monotonically to $D(f(x), g(x))$, whenever the latter exists, as $t \to 0^+$. Also,

$$|P(x, t)| \leqq \|g(x)\|.$$

By the Beppo-Levi Theorem [5], p. 133,

$$\lim_{t \to 0^+} \int P(x, t)\, d\mu = \int D(f(x), g(x))\, d|\mu|.$$

This limit is precisely $D^+(\mu, \nu)$. The same argument, modified slightly, shows that $D^-(\mu, \nu)$ is also

$$\int D(f(x), g(x)) d|\mu|.$$

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