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ON CONTINUA OF PERIODIC SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

HEINZ-OTTO PEITGEN*

0. **Introduction.** Recently, certain arguments which originated in asymptotic fixed point theory have been used with remarkable success in the study of functional differential equations by G. S. Jones [**10**], S. N. Chow [**3**] and R. D. Nussbaum [**14, 15, 16**]. Our observations here are in this line and are directly motivated by results of Nussbaum in [**14**]. He considers differential-delay equations which can be transformed to the form $x'(t) = -\lambda f(x(t-1))$.

Our approach here will be to obtain an abstract existence result providing a continuum of non-trivial solutions for a nonlinear eigenvalue problem $F(x, \lambda) = x$ and apply it to obtain a continuum of non-trivial periodic solutions for certain differential-delay equations. Our abstract theorem is based on a kind of asymptotic version of Krasnosel'skiĭ's results in [**11, 12**] on expansions and compressions of a cone in a Banach space and this is due to G. Fournier and the author. The application then substantially relies on earlier work of E. M. Wright [**20**] and R. D. Nussbaum [**14**].

1. **Preliminaries.** We recall a few definitions and results which are essential for the abstract part of our considerations. We shall call a closed, convex subset P of a linear normed space a *cone* (with vertex 0) provided $x \in P$ implies $tx \in P$, $t \geqq 0$, and $x \in P$, $x \neq 0$, implies $-x \notin P$. If $r \in \mathbf{R}_+ = \{t \in \mathbf{R} \mid t > 0\}$ and X is either a cone or an infinite dimensional linear normed space then we fix the notation $B(r) = \{x \in X \mid \|x\| < r\}$, $S(r) = \{x \in X \mid \|x\| = r\}$. If X is a topological space and $A \subset X$, then cl A denotes the closure of A in X. A closed, connected subset of a topological space is called a continuum. Let X, Y be topological spaces and $f: X \to Y$ then $H_*(X)$ denotes the singular homology of X with coefficients in the field of rational numbers and $f_*: H_*(X) \to H_*(Y)$ denotes the induced homomorphism. In what follows an essential use will be made of the notion of the Lefschetz number in the generalized sense as given by J. Leray [**13**] and the fixed point index for metric ANR's developed by A. Granas in [**8**].

Let E be a graded vector space over the field of rational numbers, Φ an endomorphism of degree zero and $N(\Phi) = \bigcup_{n>0}\ker(\Phi^n)$. Then

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Φ is said to be a *Leray endomorphism* provided $\tilde{E} = E/N(\Phi)$ is a graded vector space of finite type. In that case one defines $\mathrm{Tr}(\Phi) = \mathrm{trace}\ (\tilde{\Phi})$, where $\tilde{\Phi} : \tilde{E} \to \tilde{E}$ is the induced endomorphism. The *generalized Lefschetz number* of Φ is denoted by $\Lambda(\Phi)$ and is given by the formula

$$\Lambda(\Phi) = \sum_q (-1)^q \mathrm{Tr}(\Phi).$$

A map $f : X \to X$ is called a *Lefschetz map* provided $f_* : H_*(X) \to H_*(X)$ is a Leray endomorphism and in that case the generalized Lefschetz number of f is given by $\Lambda(f_*) = \Lambda(f : X \to X)$. We note that if $\Phi : E \to E$ is weakly-nilpotent (i.e., for all $e \in E$ there is $n \in \mathbf{N}$ such that $\Phi^n(e) = 0$) then $\Lambda(\Phi) = 0$. A subset A of a topological space X is called an *attractor* for $f: X \to X$ provided for all $x \in X$ $\mathrm{cl}\,(\bigcup_n f^n(x)) \cap A \neq \emptyset$. We say that $f : X \to X$ is a *map of compact attraction* provided f is locally compact and has a compact attractor. For triples (X, f, U) where X is a metric ANR, U is open in X such that $f(x) \neq x$ for all $x \in \partial U$ and $f : X \to X$ is a map of compact attraction one can define a fixed point index $\mathrm{ind}\,(X, f, U)$ which satisfies the standard properties including the strong normalization property $\mathrm{ind}\,(X, f, X) = \Lambda(f : X \to X)$. This is carried through in [5] using earlier work of A. Granas [8] and G. Fournier [6].

Following F. E. Browder [1] we call a subset $E \subset X$, $f : X \to X$, *ejective* for f relative to U an open neighborhood of E provided for all $x \in U \backslash E$ there is $n_x \in \mathbf{N}$ such that $f^{n_x}(x) \notin U$. We recall the following fundamental result from [5].

(1.1) LEMMA (cf. [5], corollary 4.4). *Let X be a metric ANR, $f : X \to X$ a map of compact attraction and E closed in X an ejective set for f relative to U such that $f(x) \neq x$ for all $x \in \partial U$ and $f(X \backslash E) \subset X \backslash E$. Then $f : X \backslash E \to X \backslash E$ is of compact attraction and*

$$\mathrm{ind}\,(X, f, U) = \Lambda(f : X \to X) - \Lambda(f : X \backslash E \to X \backslash E).$$

Finally, we recall that there are deformation retractions $X \to X \backslash B(r)$ and $X \to \mathrm{cl}\, B(r)$. This is observed in [7] in case X is a cone and follows from J. Dugundji [4] in case X is an infinite dimensional linear normed space.

2. A continuum of solutions for a nonlinear eigenvalue problem. We consider a nonlinear eigenvalue problem $F(x, \lambda) = x$ on some interval $J \subset \mathbf{R}$ where the operators $F(\ \cdot, \lambda) : X \to X$ satisfy certain asymptotic conditions studied in [7]. We formulate one possible set of conditions. Our result is directly motivated by a paper of R. E. L.

Turner [18], § 3. However, Turner substantially uses transversality arguments, i.e., differential topology, whereas we can bring everything back to elementary algebraic topological considerations and the Poincaré continuation method.

(2.1) THEOREM. *Let X be either a cone in a linear normed space or an infinite dimensional linear normed space, $J \subset \mathbf{R}$ an interval and $F: X \times J \to X$ a completely continuous operator. Let $a, b \in J$, $a < b$, and $r, R: (a, b) \to \mathbf{R}_+$ continuous such that for all $\lambda \in (a, b)$ either $r(\lambda) < R(\lambda)$ or $r(\lambda) > R(\lambda)$. Assume that for all $\lambda \in (a, b)$*

(2.1.1) *there is $m \in \mathbf{N}$ such that $F^m(\cdot, \lambda)(S(r(\lambda))) \subset B(r(\lambda))$;*

(2.1.2) *for all $x \in B(r(\lambda))$ there is $n_x \in \mathbf{N}$ such that $F^n(x, \lambda) \in B(r(\lambda))$ whenever $n \geqq n_x$;*

(2.1.3) *there is $M \in \mathbf{N}$ such that $F^M(\cdot, \lambda)\ (S(R(\lambda))) \subset X \backslash \mathrm{cl}\, B(R(\lambda))$;*

(2.1.4) *for all $x \in X \backslash \mathrm{cl}\, B(R(\lambda))$ there is $N_x \in \mathbf{N}$ such that $F^N(x, \lambda) \in X \backslash \mathrm{cl}\, B(R(\lambda))$ whenever $N \geqq N_x$.*

Then for any $\epsilon > 0$ there is a continuum $C_\epsilon \subset S = \{(x, \lambda) \in X \times J \mid F(x, \lambda) = x\}$ such that

(2.1.5) *$(x, \lambda) \in C_\epsilon$ implies $r(\lambda) < \|x\| < R(\lambda)$ – 'asymptotic expansion' (resp. $r(\lambda) > \|x\| > R(\lambda)$ – 'asymptotic compression') and $\lambda \in [a + \epsilon, b - \epsilon]$ and*

(2.1.6) *C_ϵ has as its projection on the λ-axis the interval $[a + \epsilon, b - \epsilon]$.*

The proof splits into two parts. The first part is essentially contained in the author's paper with G. Fournier [7]. Therefore we only sketch a proof of part 1. However, we present a slightly different version from that in [7]. It is remarkable that here techniques must be applied which were first developed by J. Leray and A. Granas for the purpose of generalizing the Lefschetz fixed point theorem.

The main ingredients for the proof of theorem (2.1) are:

– the class of mappings which are of *compact attraction* and their generalized index theory (cf. [6], [8] and [5]);
– the notion and properties of *Leray trace, Leray endomorphism* and *Lefschetz map* (cf. [13] and [8]);
– the notion of an *ejective set* and its index characterization (cf. [5]);
– the Whyburn – Lemma (cf. [19], Chap. 1, Theorem 9.3):

Let K be a compact metric space and A and B disjoint, closed subsets of K. Then either there exists a continuum C connecting A and B or there exist disjoint, closed subsets K_A and K_B such that $A \subset K_A$, $B \subset K_B$ and $K = K_A \cup K_B$.

PROOF OF THE THEOREM. Let $U(\lambda) = \{x \in X \mid r(\lambda) < \|x\| < R(\lambda)\}$ *if* $r(\lambda) < R(\lambda)$, *and* $U(\lambda) = \{x \in X \mid r(\lambda) > \|x\| > R(\lambda)\}$ *if* $r(\lambda) > R(\lambda)$.

Part 1. One proves that

$$\operatorname{ind}(X, F(\cdot, \lambda), U(\lambda)) = \begin{cases} -1, & \text{if } r(\lambda) < R(\lambda) \\ +1, & \text{if } r(\lambda) > R(\lambda) \end{cases}$$

for all $\lambda \in (a, b)$.

We sketch a proof and use the abbreviations $F(\cdot, \lambda) \equiv F$, $r(\lambda) \equiv r$, $R(\lambda) \equiv R$ for a fixed $\lambda \in (a, b)$. If s represents one of the symbols r or R we let O_s be $B(r)$, if $s = r$, and O_s be $X \backslash \mathrm{cl}\, B(R)$, if $s = R$. To apply Lefschetz type arguments we define the following open sets in X (hence, ANR's):

$$W_s = \bigcup_{i=1}^{\infty} F^{-i}(O_s), \; s \in \{r, R\}.$$

From (2.1.1)–(2.1.4) it follows that $\mathrm{cl}\, O_s \subset W_s$ and $F(W_s) \subset W_s$. We note that $F : W_s \to W_s$ has no 'good' compactness properties. Therefore, fixed point theoretical arguments cannot be applied to this mapping. Let $\rho_r : X \to X \backslash B(r)$ and $\rho_R \cdot X \to \mathrm{cl}\, B(R)$ denote deformation retractions. Then we have that $F \circ \rho_{s|W_s}$ is homotopic to $F_{|W_s}$. Set $E_R = X \backslash W_R$ and observe that E_R is an ejective set for $F \circ \rho_R$ relative to $X \backslash \mathrm{cl}\, O_R = B(R)$. Moreover, we have that $F \circ \rho_R(X \backslash E_R) \subset X \backslash E_R$. Thus, lemma (1.1) applies to show that $F \circ \rho_R : W_R \to W_R$ is a mapping of compact attraction.

One also shows that $F \circ \rho_r : W_r \to W_r$ is a mapping of compact attraction with compact attractor $M = \mathrm{cl}\,(F(S(r)))$. Thus, according to our remarks in the first section a generalized index theory is available for $F \circ \rho_s$. Hence, the Lefschetz number in the generalized sense is defined and we have that $\Lambda(F \circ \rho_s : W_s \to W_s) = \operatorname{ind}(W_s, F \circ \rho_s, W_s)$. Since $(F \circ \rho_{s|W_s})_* = (F_{|W_s})_*$, we have $\Lambda(F \circ \rho_s : W_s \to W_s) = \Lambda(F : W_s \to W_s)$. Next, we calculate $\Lambda(F : W_s \to W_s)$, which now is well defined. We stress the point that this number only has an algebraic meaning and in general it will not be true that $\Lambda(F : W_s \to W_s) \neq 0$ implies that F has a fixed point. The key for our computation is the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & i_* H_*(O_s) & \to & H_*(W_s) & \to & H_*(W_s)/i_* H_*(O_s) & \to & 0 \\ & & \uparrow \hat{F}_* & & \uparrow F_* & & \uparrow \tilde{F}_* & & \\ 0 & \to & i_* H_*(O_s) & \to & H_*(W_s) & \to & H_*(W_s)/i_* H_*(O_s) & \to & 0 \end{array}$$

Once the commutativity is proved a generalization of the 'dimension formula' of the form $\Lambda(F_*) = \Lambda(\hat{F}_*) + \Lambda(\tilde{F}_*)$ holds. Since O_s is connected and acyclic in H_* we have that $i : O_s \to W_s$ induces an injection $i_* : H_*(O_s) \to H_*(W_s)$. Using the facts that O_s is acyclic, ρ_s is a deformation retraction and that O_s absorbs compact sets of W_s under $(F^k \circ \rho_s)$ for suitable $k \in \mathbf{N}$ one shows that $F_*(i_* H_*(O_s)) \subset i_* H_*(O_s)$ and $F_*|_{i_* H_*(O_s)} = Id$. Thus, letting $\hat{F}_*$ be the restriction of F_* and $\tilde{F}_*$ their quotient homomorphism, one has the commutativity of the diagram and $\Lambda(\hat{F}_*) = 1$. Next, one observes that $\tilde{F}_*$ is weakly-nilpotent. Hence, $\Lambda(F : W_s \to W_s) = \Lambda(F_*) = \Lambda(\hat{F}_*) + \Lambda(\tilde{F}_*) = 1 + 0 = 1$. This purely algebraic computation provides us a fixed point result:

$$\operatorname{ind}(W_s, F \circ \rho_s, W_s) = \Lambda(F \circ \rho_s : W_s \to W_s) = \Lambda(F : W_s \to W_s) = 1.$$

Using that E_R is ejective and applying lemma (1.1) we obtain $\operatorname{ind}(X, F, B(R)) = \operatorname{ind}(X, F \circ \rho_R, B(R)) = \Lambda(F \circ \rho_R : X \to X) - \Lambda(F \circ \rho_R : W_R \to W_R = 1 - 1 = 0$.

Now, if $s = r$, we obtain as a consequence of the properties of the fixed point index: $\operatorname{ind}(X, F, B(r)) = \operatorname{ind}(X, F \circ \rho_r, B(r)) = \operatorname{ind}(W_r, F \circ \rho_r, B(r)) = \operatorname{ind}(W_r, F \circ \rho_r, W_r) = \Lambda(F \circ \rho_r : W_r \to W_r) = 1$.

Finally, if $r < R$ (i.e., asymptotic expansion), $\operatorname{ind}(X, F, U) = \operatorname{ind}(X, F, B(R)) - \operatorname{ind}(X, F, B(r)) = 0 - 1 = -1$ and, if $r > R$ (i.e., asymptotic compression), $\operatorname{ind}(X, F, U) = \operatorname{ind}(X, F, B(r)) - \operatorname{ind}(X, F, B(R)) = 1 - 0 = +1$.

Part 2. We shall obtain a continuum of solutions by combining the Poincaré continuation method and the Whyburn — Lemma. This idea is very much analogous to the proof of Rabinowitz's global bifurcation theorem in [17]. Assume that $r(\lambda) < R(\lambda)$ for all $\lambda \in (a, b)$ and choose $\epsilon > 0$. If Ω is any subset in $X \times J$ we let $\Omega_\lambda = \{x \in X \mid (x, \lambda) \in \Omega\}$ be the λ-section. We have the open set $U = \{(x, \lambda) \in X \times J \mid r(\lambda) < \|x\| < R(\lambda), \lambda \in J\}$ in $X \times J$. Let $K = \{(x, \lambda) \in U \mid a + \epsilon \leqq \lambda \leqq b - \epsilon\} \cap S$ be the set of solutions in U with $\lambda \in [a + \epsilon, b - \epsilon]$. Since F is completely continuous it follows that K is compact in $U \subset X \times J$. Now suppose that there is no continuum connecting $A = K_{a+\epsilon} \times \{a + \epsilon\}$ with $B = K_{b-\epsilon} \times \{b - \epsilon\}$. Then the Lemma of Whyburn implies that there exist disjoint compacta $K_A \supset A$ and $K_B \supset B$ such that $K = K_A \cup K_B$. Choose Ω bounded and open in $X \times J$ such that $K_A \subset \Omega$, $\Omega \cap K_B = \emptyset$, $\Omega \subset X \times (-\infty, b - \epsilon)$ and $\Omega \subset U$. Then $F(x, \lambda) \neq x$ for all $(x, \lambda) \in \partial\Omega$ such that $\lambda \geqq a + \epsilon$. For $r \geqq 0$ define $h_r : X \times J \to X \times \mathbf{R}$ by $h_r(x, \lambda) = (F(x, \lambda), a + \epsilon + r)$ and observe that $h_r(x, \lambda) \neq (x, \lambda)$ for all $(x, \lambda) \in \partial\Omega$. Thus, $\operatorname{ind}(X \times \mathbf{R}, h_r, \Omega) =$ constant for all $r \geqq 0$. For r large enough we clearly have $h_r(x, \lambda) \neq (x, \lambda)$ for all $(x, \lambda) \in \Omega$ since Ω is bounded. Thus, $\operatorname{ind}(X \times \mathbf{R}, h_r, \Omega) = 0$. However, as a consequence of the properties of the fixed point index we obtain the identities $\operatorname{ind}(X \times \mathbf{R}, h_0, \Omega) = \operatorname{ind}(X \times \{a+\epsilon\}, h_0, \Omega \cap X \times \{a+\epsilon\}) = \operatorname{ind}(X, F(\cdot, a+\epsilon), \Omega_{a+\epsilon}) = \operatorname{ind}(X, F(\cdot, a+\epsilon), U_{a+\epsilon})$, and it follows from part 1 that the last number equals -1. This is a contradiction. The case of 'asymptotic compression' is proved analogously.

3. **A continuum of periodic solutions for $x'(t) = -\lambda f(x(t-1))$.** The following considerations are substantially based on earlier work of E. M. Wright [20] and R. D. Nussbaum [14]. As an example of an application of theorem (2.1) we wish to study the structure of the set of periodic solutions of the equations

$$(3.1) \qquad \begin{cases} x'(t) = -\lambda f(x(t-1)), & \text{for } t \geqq 0 \\ x(t) = \phi(t), & \text{for } -1 \leqq t \leqq 0, \end{cases}$$

where λ varies in some interval $J \subset \mathbf{R}_+$. Our goal in this section is to place further conditions on f which guarantee as an application of theorem (2.1) that (3.1) has a continuum of nontrivial periodic solutions.

Let C be the Banach space of continuous, real-valued functions ϕ defined on the interval $[-1, 0]$; the norm on C is the sup-norm. If $\lambda \in \mathbf{R}$, $\phi \in C$ and f is continuous, it is clear that there exists a unique continuous, real-valued function $x = x(t; \phi, \lambda)$ defined for $t \geqq -1$ and such that $x(t) = \phi(t)$ for $-1 \leqq t \leqq 0$ and $x'(t) = -\lambda f(x(t-1))$ for $t \geqq 0$. We can therefore identify $\phi \in C$ with $x(\cdot; \phi, \lambda)$ and speak of periodic solutions in C.

We have the following result.

(3.2) THEOREM. *Assume that $f: \mathbf{R} \to \mathbf{R}$ is continuous, $x \cdot f(x) > 0$ for all $x \neq 0$, $\lim_{x \to 0}(f(x)/x) = \alpha > 0$, $\lim_{|x| \to \infty}(f(x)/x) = \beta < \infty$ and $0 < a = (\pi/2)(\beta^{-1}) < b = (\pi/2)(\alpha^{-1}) < \infty$. Then for any $\epsilon > 0$ there is a continuum C_ϵ of nontrivial periodic solutions for (3.1) in $C \times (a, b)$. Furthermore, if $(x, \lambda) \in C_\epsilon$ then $x(-1) = 0$ and x is monotonic increasing on $[-1, 0]$ and C_ϵ has as its projection on the λ-axis the interval $[a+\epsilon, b-\epsilon]$.*

PROOF. To reduce the problem of finding continua of solutions to a nonlinear eigenvalue problem of the form $F(x, \lambda) = x$ one considers the cone $P = \{\phi \in C \mid \phi(-1) = 0 \text{ and } \phi(t_1) \leqq \phi(t_2) \text{ for all } t_1, t_2 \text{ such that } t_1 \leqq t_2\}$ in C. If J denotes the interval $(0, \infty)$ one defines a map $F: P \times J \to P$ as follows: If $\phi(0) > 0$ and $\lambda \in J$ let $x(t; \phi, \lambda)$ be the corresponding solution of (3.1). Define $z_1 = z_1(\phi, \lambda) = \inf\{t > 0 \mid x(t; \phi, \lambda) = 0\}$ and $z_2 = z_2(\phi, \lambda) = \inf\{t > z_1 \mid x(t; \phi, \lambda) = 0\}$. If z_1 or z_2 is infinite define $F(\phi, \lambda) = 0$; and if z_1 and z_2 are finite, define $F(\phi, \lambda)(t) = x(z_2 + 1 + t; \phi, \lambda)$ for $-1 \leqq t \leqq 0$. Finally, define $F(0, \lambda) = 0$.

Now F is a completely continuous map from $P \times J$ to P. This is proved in [14, Lemma 2.7]. Moreover, if $F(\phi, \lambda) = \phi$ for $\phi \neq 0$, then $x(t; \phi, \lambda)$ is a nontrivial periodic solution of (3.1) of period $z_2(\phi, \lambda) + 1$. To prove that F defined as above satisfies the asymptotic conditions (2.1.1)–(2.1.4) for suitable functions r, $R : (a, b) \to \mathbf{R}_+$ one analyses the 'stability' of the solutions $x(t; \phi, \lambda)$. This is done explicitly for λ fixed in [14, § 2]. Observe that for $\lambda \in (a, b)$ it follows that $\lambda\beta > \pi/2$ and $\lambda\alpha < \pi/2$. Nussbaum shows that under these assumptions for λ fixed there exist a constant $r(\lambda) > 0$ and an integer N_1 such that if $\phi \in P$, $\|\phi\| \leqq r(\lambda)$ and $n \geqq N_1(\lambda)$ then $\| F^n(\phi, \lambda)\| < r(\lambda)$. Moreover, there exist a constant $R(\lambda) > r(\lambda)$ and an integer $N_2(\lambda)$ such that if $\phi \in P$, $\|\phi\| \geqq R(\lambda)$ and $n \geqq N_2(\lambda)$ then $\| F^n(\phi, \lambda)\| > R(\lambda)$. To apply theorem (2.1) it remains to show that r and R can be chosen as continuous functions of $\lambda \in (a, b)$. Nussbaum's work in [14, § 2] is based on the linearizations of (3.1) at '0' and '∞'. To study the linear equations $x'(t) = -\gamma(\lambda)\, x(t-1)$ (either $\gamma(\lambda) = \lambda\beta > \pi/2$ or $\gamma(\lambda) = \lambda\alpha < \pi/2$ for $\lambda \in (a, b)$) one considers the associated transcendental characteristic equation $s + \gamma(\lambda) \exp(-s) = 0$. This equation has been studied in detail by E. M. Wright in [20]. An analysis of Nussbaum's elaborate estimates shows that in order to obtain $R : (a, b) \to \mathbf{R}_+$ as a continuous function it suffices to combine them with assertion (3.3.1) in the lemma below. To obtain $r : (a, b) \to \mathbf{R}_+$ as a continuous function one observes that by standard stability theory (cf. [2], [9]) it suffices to have assertion (3.3.2) about the roots of $s + \gamma(\lambda) \exp(-s) = 0$ in the lemma below.

(3.3) Lemma.

(3.3.1) *Let λ be in $(\pi/2, d)$ and $d > \pi/2$. Then for any $\epsilon > 0$ there exists a pair of continuous curves $\tau_1, \tau_2 : [\pi/2 + \epsilon, d - \epsilon] \to \mathbf{R}_+$ such that if $s \in \mathbf{C}$ is a solution of $s + \lambda \exp(-s) = 0$, $0 < \mathrm{Im}(s) < \pi$ and $\lambda \in [\pi/2 + \epsilon,\ d - \epsilon]$ then $\mathrm{Re}(s) = \tau_1(\lambda)$ and $\mathrm{Im}(s) = \tau_2(\lambda)$.*

(3.3.2) *Let λ be in $(0, \pi/2)$. Then for any $\epsilon > 0$ there exists a constant $d_\epsilon > 0$ such that if $s \in C$ is a solution of $s + \lambda \exp(-s) = 0$ and $\lambda \in [\epsilon, \pi/2 - \epsilon]$ then $\mathrm{Re}(s) < -d_\epsilon$.*

Proof. From E. M. Wright [20, Theorem 5] we have the following results: If $\lambda > \exp(-1)$ then the roots of the equation $s + \lambda \exp(-s) = 0$ occur in conjugate complex pairs $s_p, \bar{s}_p$ *and*

$$s_p = \sigma_p + it_p,\ 2p\pi < t_p < (2p+1)\pi,\ p = 0, 1, 2, 3, \cdots \tag{3.3.3}$$

If $0 < \lambda < \exp(-1)$, the same is true except that s_0, $\bar{s}_0$ are replaced by a double root at $s = -1$ for $\lambda = \exp(-1)$ and by two real roots $s =$

σ_0', σ_0 for $\lambda < \exp(-1)$ and

$$0 > \sigma_0' > -1 > \log \lambda > \sigma_0. \tag{3.3.4}$$

If $\lambda < \pi/2$ then every root has its real part negative. Moreover, $\sigma_0 = 0$ for $\lambda = \pi/2$ and $\sigma_0 > 0$ for $\lambda > \pi/2$. Finally, we extract from Wright's results that $\sigma_{p+1} < \sigma_p$ for all λ and all $p \geqq 0$.

Let $g : \mathbf{R} \times \mathbf{C} \to \mathbf{C}$ be the mapping $g(\lambda, s) = s + \lambda \exp(-s)$. To obtain assertion (3.3.1) one considers g on the open set $(\pi/2, d) \times X_1$, where $X_1 = \{s \in \mathbf{C} \mid \operatorname{Re}(s) > 0 \text{ and } 0 < \operatorname{Im}(s) < \pi\}$ and applies the implicit function theorem for pairs $(\lambda_0, s_0) \in (\pi/2, d) \times X_1$ such that $s_0 + \lambda_0 \exp(-s_0) = 0$. Pasting local parametrizations together one obtains a continuous curve $\tau : [\pi/2 + \epsilon, d - \epsilon] \to \mathbf{C}$ such that $g(\lambda, \tau(\lambda)) = 0$ and $0 < \operatorname{Im}(\tau(\lambda)) < \pi$ and $\operatorname{Re}(\tau(\lambda)) > 0$. The assertion (3.3.1) is then obvious from the above. To obtain assertion (3.3.2) one considers g on the open sets $(0, \exp(-1)) \times X_2$ and $(\exp(-1), \pi/2) \times X_3$, where $X_2 = \{s \in \mathbf{C} \mid \operatorname{Re}(s) > -1\}$ and $X_3 = \{s \in \mathbf{C} \mid \operatorname{Re}(s) < 0$ and $0 < \operatorname{Im}(s) < \pi\}$. In both sets one applies the implicit function theorem for pairs (λ_0, s_0) such that $g(\lambda_0, s_0) = 0$. Pasting local parametrizations together one obtains from (3.3.3) and (3.3.4) a continuous curve $\tau : ([\epsilon, \exp(-1) - \delta] \cup [\exp(-1) + \delta, \pi/2 - \epsilon]) \to \mathbf{C}$ for some $\delta > 0$ such that $g(\lambda, \tau(\lambda)) = 0$ and if s is any root for a fixed λ then $\operatorname{Re}(s) \leqq \operatorname{Re}(\tau(\lambda)) < 0$. Clearly, $\lim_{\lambda \to \exp(-1)} \tau(\lambda) = -1$ and therefore τ can be continuously extended to the entire interval $[\epsilon, \pi/2 - \epsilon]$. Assertion (3.3.2) is then obvious.

ADDED IN PROOF. R. D. Nussbaum has pointed out to the author that his paper [*Global Bifurcation of Periodic Solutions of Some Autonomous Functional Differential Equations,* J. Math. Anal. Appl. **55** (1976) 699–725] contains an argument for (3.2) based on abstract bifurcation theory.

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INSTITUT FÜR ANGEWANDTE MATHEMATIK DER RHEINISCHEN FRIEDRICH-WILHELMS-UNIVERSITÄT, WEGELERSTRASSE 6, 53 BONN, FEDERAL REPUBLIC OF GERMANY