ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 3, Summer 1977

APPROXIMATION AND DECAY OF SOLUTIONS OF SYSTEMS OF NONLINEAR DIFFUSION EQUATIONS

DAVID HOFF

1. **Introduction.** We consider the system of equations

$$
\begin{aligned}
&U_t = DU_{xx} + MU_x + N(U) \\
&U(x, 0) = U_0(x)
\end{aligned}
\tag{1}
$$

where $U \in \mathbf{R}^m$ and $x \in \mathbf{R}$. U_0 and N are assumed to be smooth and $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$ and $M = \operatorname{diag}(\mu_1, \cdots, \mu_m)$ are constant. Furthermore we assume that there is a constant $K_1 > 0$ such that

$$\lambda_i - \frac{K_1}{2}|\mu_i| \geqq 0 \text{ for all } i \tag{2}$$

and that there is a set $S = \Pi_{i=1}^m [a_i, b_i]$ contained in the domain of N such that

$$N(U)^t \nu_S(U) \leqq 0 \text{ for } U \in \partial S \tag{3}$$

where ν_S is the outer normal on S. Under these assumptions S is invariant for (1); this means that if $U_0(x) \in S$ for all x, then $U(x, t) \in S$ for all x and t. Hence solutions of (1) are *a priori* bounded so that (1) has a smooth solution $U(x, t)$ defined for all t. See [1] for details.

We generate approximants to U in the following way: choose increments Δt and Δx, let $t_n = n\Delta t$ and $x_k = k\Delta x$, and approximate $U_k{}^n \equiv U(x_k, t_n)$ by $V_k{}^n$, where

$$
\begin{aligned}
\frac{V_k{}^n - V_k{}^{n-1}}{\Delta t} = D &\left(\frac{V_{k+1}^{n-1} - 2V_k{}^{n-1} + V_{k-1}^{n-1}}{\Delta x^2}\right) \\
&+ M\left(\frac{V_{k+1}^{n-1} - V_{k-1}^{n-1}}{2\Delta x}\right) + N(V_k{}^{n-1}).
\end{aligned}
$$

Letting $\beta = \Delta t/\Delta x^2$ and $\alpha = \Delta t/2\Delta x$, we may write this scheme in the form

$$
\begin{aligned}
V_k{}^n = L_k(V^{n-1}) \equiv (I - 2\beta D)V_k{}^{n-1} &+ (\beta D + \alpha M)V_{k+1}^{n-1} \\
&+ (\beta D - \alpha M)V_{k-1}^{n-1} + \Delta t N(V_k{}^{n-1}).
\end{aligned}
\tag{4}
$$

The local truncation error $\tau_k{}^n$ is defined by $\tau_k{}^n = U_k{}^n - L_k(U^{n-1})$. Since U is smooth it is clear that

This work constitutes a part of the author's Ph.D. thesis research, performed under the direction of Prof. J. A. Smoller.

$$\|\tau_k^n\| \leqq ch\Delta t \sup\|D_x^j U\|$$

where the sup is taken over all x, $t \in [0, t_n]$, and $j = 1, \cdots, 4$. Here as elsewhere $h = \Delta t + \Delta x^2$; $D_x^j = \partial^j/\partial x^j$; c denotes a positive constant independent of Δt, Δx, t and x; and norms are ∞-vector norms. It will be shown in Lemma 6 that the above sups exist, provided that they exist for $n = 0$; moreover, estimates will be obtained to yield

$$\|\tau_k^n\| \leqq ce^{ct_n}h\Delta t. \tag{5}$$

In § 2 we show that if β, Δt and Δx are suitably restricted, then S is invariant for the difference approximations (4). This ensures that the V_k^n are defined for all n and k and enables us to establish that $\|U_k^n - V_k^n\| = O(h)$ for fixed t_n. In § 3 we impose further restrictions on N and U_0 which yield the estimate $\|U_k^n - V_k^n\| = O(h)$ independent of x_k and t_n and which imply that U decays to a root of N as $t \to \infty$. In § 4 we consider two examples and in § 5 we indicate how the theory can be generalized.

2. **Existence and Convergence of Approximants.** In the following, superscripts will be used to designate the components of vectors, and the jth standard basis vector for $\mathbf{R}^m$ will be denoted by $\hat{e}_j$.

From (4) V_k^n is defined provided that V_k^{n-1} is in the domain of N. Theorem 1 yields a sufficient condition for the existence of the approximants.

THEOREM 1. *Choose* $\gamma \in (0, 1)$ *and assume*

$$\beta \leqq \frac{1-\gamma}{2\lambda_i} \text{ for all } i, \tag{6}$$

$$\Delta x \leqq K_1, \tag{7}$$

$$\Delta t \leqq \frac{\gamma}{K_2}, \tag{8}$$

where $K_2 = \sup|\partial N^i/\partial U^i(U)|$, *the* sup *being taken over all* i *and all* $U \in S$. *Then if* $V_k^0 \in S$ *for all* k, $V_k^n \in S$ *for all* n *and* k.

PROOF. Assume that $V_k^{n-1} \in S$ for all k, fix i, and let $c_q^p = (V_q^p)^i$. We need to show that $c_k^n \in [a_i, b_i]$. We have, for some $W \in S$,

$$\begin{aligned} N^i(V_k^{n-1}) &= N^i(V_k^{n-1} + (a_i - c_k^{n-1})\hat{e}_i) \\ &\quad + \frac{\partial N^i}{\partial U^i}(W)(c_k^{n-1} - a_i) \\ &\geqq 0 - K_2(c_k^{n-1} - a_i) \end{aligned}$$

from (3). Therefore from (4),

$$\begin{aligned} c_k^n - a_i &= (1 - 2\beta\lambda_i)(c_k^{n-1} - a_i) + (\beta\lambda_i + \alpha\mu_i)(c_{k+1}^{n-1} - a_i) \\ &\quad + (\beta\lambda_i - \alpha\mu_i)(c_{k-1}^{n-1} - a_i) + \Delta t N^i(V_k^{n-1}) \\ &\geqq (1 - 2\beta\lambda_i - K_2\Delta t)(c_k^{n-1} - a_i) \\ &\quad + (\beta\lambda_i + \alpha\mu_i)(c_{k+1}^{n-1} - a_i) + (\beta\lambda_i - \alpha\mu_i)(c_{k-1}^{n-1} - a_i). \end{aligned}$$

Conditions (6)–(8) guarantee that the right-hand side above is a convex combination of quantities which, by induction, are nonnegative. Therefore $c_k^n \geqq a_i$. The proof that $c_k^n \leqq b_i$ is similar.

$$\text{Let } E_k^n = U_k^n - V_k^n \text{ and } E^n = \sup_k \|E_k^n\|.$$

THEOREM 2. *Assume that* $\sup_x \|D_x^j U_0(x)\| < \infty$ for $j \leqq 4$ (*so that* (5) *holds*), *and assume that* (6)–(8) *are satisfied. Then* $E^n \leqq e^{ct_n}(E^0 + \text{ch})$.

PROOF. $$\begin{aligned} E_k^n &= L_k(U^{n-1}) - L_k(V^{n-1}) + \tau_k^n \\ &= (1 - 2\beta D)E_k^{n-1} + (\beta D + \alpha M)E_{k+1}^{n-1} + (\beta D - \alpha M)E_{k-1}^{n-1} \\ &\quad + \Delta t J E_k^{n-1} + \tau_k^n \end{aligned}$$

where the entries of the matrix J are derivatives of N evaluated at various points in S. Computing the ith component of both sides above, and using (6)–(8) we obtain

$$|(E_k^n)^i| \leqq (1 + c\Delta\, t)E^{n-1} + \|\tau_k^n\|$$

so that, using (5),

$$E^n \leqq (1 + c\Delta t)E^{n-1} + ce^{ct_n}h\Delta t.$$

An easy induction then finishes the proof.

Theorem 2 establishes the first-order convergence of the approximants. However, the given bound tends to infinity as $t \to \infty$ with h fixed, whereas Theorem 1 implies that $E^n \leqq \text{diam}(S)$. We can combine Theorems 1 and 2 in the following way:

COROLLARY 3. $E^n \leqq \delta(t_n, h)\ \text{diam}(S) + [1 - \delta(t_n, h)]\, e^{ct_n}(E^0 + ch)$ *where* $\delta \in (0,1)$, $\delta(t,h) = O(h)$ *for fixed* t, $\delta(t,h) \to 1$ *as* $t \to \infty$, and $[1 - \delta(t,h)]\, e^{ct} \to 0$ *as* $t \to \infty$.

PROOF. For any $\delta \in (0,1)$

$$E^n \leqq \delta \text{diam}(S) + (1 - \delta)e^{ct_n}(E^0 + ch).$$

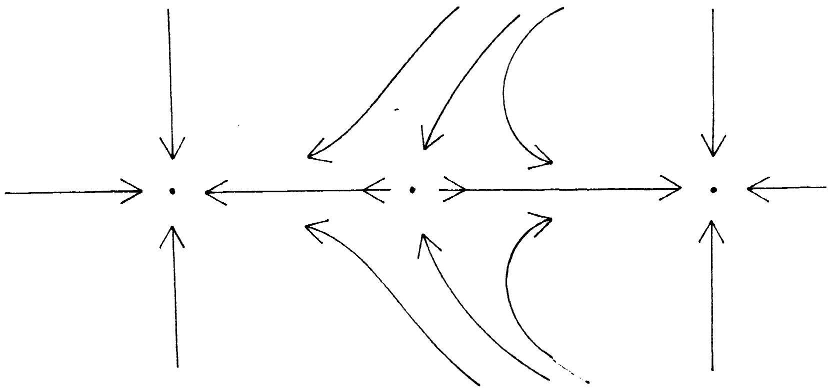
Choose $\delta = h/(h + e^{-c' t_n})$ where $c' > c$ to obtain the result.

Corollary 3 again asserts the first-order convergence of the approximants but now indicates that, for fixed h, E^n may tend to diam(S) as $t_n \to \infty$. We will show that this behavior may actually occur. First note that (1) includes as a subcase the ordinary differential equation

$$U_t = N(U) \tag{9}$$
$$U(0) = U_0.$$

If we take $V_k{}^0 = U_0$ for all k, then (4) becomes Euler's method.

Now suppose $m = 2$ and that S contains three zeroes of N, two of which are attractors and one of which, $\hat{U}$, is a saddle. Then for appropriate U_0, the solution $U(t)$ of (9) tends to $\hat{U}$ exponentially as $t \to \infty$, whereas V^n will tend exponentially to one of the attractors, provided that the stable manifold for $\hat{U}$ is not a line. Thus E^n tends exponentially to a quantity which could be as large as diam(S).



3. **Decay to Equilibrium and Uniform Convergence.** In this section we investigate the convergence of the approximants under hypotheses which preclude instabilities such as that occurring in the above example.

Definition. A square matrix $A = (A_{ij})$ is *diagonally dominant* (d.d.) if

$$A_{ii} \geqq \sum_{j \neq i} |A_{ij}| \text{ for all } i \tag{10}$$

and is strictly diagonally dominant (s.d.d.) if strict inequality holds in (10) for all i.

If $P(U)$ is a vector field we denote by $J_P(U)$ the Jacobian matrix of P

evaluated at U. We continue to assume that (2) and (3) hold.

THEOREM 4. *Suppose* $-J_N$ is s.d.d. *in* S. *Then* N *has one and only one zero in* S.

PROOF. By Gerschgorin's Theorem ([5], pg. 76)

$$\operatorname{sgn} \det J_N(U) = (-1)^m \text{ for all } U \in S.$$

Thus any zeroes of N in a neighborhood of S must be isolated, so that, by enlarging S if necessary, we may assume that $N \neq 0$ on ∂S. We then have

$$\deg(0, N, S) \equiv \sum_{N(U)=0} \operatorname{sgn} \det J_N(U) = (-1)^m p$$

where p is the number of zeroes of N in S. (See [6] for the definition and properties of deg).

Now let $N_s(U) = sN(U) + (1 - s)(\tilde{U} - U)$ where $\tilde{U}$ is the midpoint of S. Then $0 \notin N_s(\partial S)$ for any $s \in [0, 1]$ (since $N^t \nu_S \leqq 0$); hence

$$(-1)^m p = \deg(0, N_1, S) = \deg(0, N_0, S) = (-1)^m$$

and $p = 1$.

We denote by $S_h(U)$ the cube $\{V \in \mathbf{R}^m \text{ s.t. } \|U - V\| \leqq h\}$.

THEOREM 5. *Suppose* $N(\hat{U}) = 0$. *If* $-J_N$ *is* d.d. *in a neighborhood* T *of* $\hat{U}$ *and if* $S_h(U) \subseteq T$, *then* $S_h(U)$ *is invariant for* (1). *Conversely, if* $S_h(\hat{U})$ *is invariant for* (1) *for small* $h > 0$, *then* $-J_N(\hat{U})$ *is* d.d.

PROOF. Recall that $S_h(\hat{U})$ is invariant for (1) iff $N^t \nu_{S_h(\hat{U})} \leqq 0$ on $\partial S_h(\hat{U})$. Suppose that $-J_N$ is d.d. in $S_h(\hat{U})$ and let $U \in \partial S_h(\hat{U})$, say $U^i = \hat{U}^i + h$. Then for some $W \in S$

$$\begin{aligned} N^i(U) &= N^i(\hat{U}) + \nabla N^i(W)^t(U - \hat{U}) \\ &= \frac{\partial N^i}{\partial U^i} h + \sum_{j \neq i} \frac{\partial N^i}{\partial U^j}(U^j - \hat{U}^j) \\ &\qquad \partial N^i \qquad\qquad \partial N^i \\ &\leqq \left(\frac{\partial N^i}{\partial U^i} + \sum_{j \neq i} \left| \frac{\partial N^i}{\partial U^j} \right| \right) h \\ &\leqq 0. \end{aligned}$$

Similarly $N^i(U) \geqq 0$ if $U^i = \hat{U}^i - h$. Thus $N^t \nu_{S_h(\hat{U})} \leqq 0$.

Conversely, suppose $S_h(\hat{U})$ is invariant for small h. Fix i and let $\sigma_j = \operatorname{sgn} \partial N^i / \partial U^j(\hat{U})$. For small $h > 0$ define U_h by $U_h{}^j = \hat{U}^j + \sigma_j h$ for $j \neq i$ and $U_h{}^i = \hat{U}^i + h$. Then $U_h \in \partial S_h(\hat{U})$ and

$$0 \geqq N^i(U_h) = N^i(\hat{U}) + \frac{\partial N^i}{\partial U^i}(W_h)h + \sum_{j\neq i} \frac{\partial N^i}{\partial U^j}(W_h)\sigma_j h$$

where $W_h \in S_h(\hat{U})$. Since $N^i(\hat{U}) = 0$ we have

$$0 \geqq \frac{\partial N^i}{\partial U^i}(W_h) + \sum_{j\neq i} \frac{\partial N^i}{\partial U^j}(W_h)\sigma_j.$$

Now let $h \to 0$ to obtain the result.

LEMMA 6. *Assume that* $\sup_x\|D_x^j U_0(x)\| \leqq K$ *for* $1 \leqq j \leqq p$ *and let* $\sigma = \sup((\partial N^i/\partial U^i)(U) + \sum_{j\neq i}|(\partial N^i/\partial U^j)(U)|)$, *where the* sup *is taken over all i and all* $U \in S$. *Then* $\sup_x\|D_x^j U(x, t)\| \leqq Ke^{\sigma t}$ *for* $1 \leqq j \leqq p$.

PROOF. We give the proof for $p = 1$. Let $V(x, t) = \eta e^{-\sigma' t}U_x(x, t)$ where $\sigma' > \sigma$ and $\eta > 0$ is to be chosen. By differentiating (1) we obtain

$$V_t = DV_{xx} + MV_x + \{J_N(U) - \sigma'\}V.$$

Let $\tilde{D} = D \oplus D$, $\tilde{M} = M \oplus M$, $\tilde{U} = \begin{bmatrix} U \\ V \end{bmatrix}$, and $\tilde{N}(\tilde{U}) = \begin{bmatrix} N(U) \\ \{J_N(U) - \sigma'\}V \end{bmatrix}$. Then

$$\tilde{U}_t = \tilde{D}\tilde{U}_{xx} + \tilde{M}\tilde{U}_x + \tilde{N}(\tilde{U}). \tag{11}$$

A simple computation shows that

$$J_{\tilde{N}}\left(\begin{bmatrix} U \\ 0 \end{bmatrix}\right) = J_N(U) \oplus [J_N(U) - \sigma'].$$

Thus for $U \in S$ and $\|V\| \leqq h_0$, $-J_{\tilde{N}}(\begin{bmatrix} U \\ V \end{bmatrix})$ satisfies the diagonal dominance condition in the last m rows, where h_0 depends only on N, S, and σ'. An argument similar to that given in the proof of Theorem 5 then shows that $S \times S_{h_0}(0)$ is invariant for (11). Now choose $\eta = h_0/K$ so that $\|V(x, 0)\| \leqq h_0$. Then $\|V(x, t)\| \leqq h_0$ for all x and t, and $\|U_x(x, t)\| \leqq (h_0/\eta)e^{\sigma' t} = Ke^{\sigma' t}$ for any $\sigma' > \sigma$.

Lemma 6 establishes (5). Note that if $-J_N$ is s.d.d. in S then the σ occurring above is negative.

LEMMA 7. *Let* $R = \{R_k\}$ *and* $T = \{T_k\}$ *be sequences in* S. *Then if* $-J_N$ *is* s.d.d. *in* S *and* (6)–(8) *hold,*

$$\|L_k(R) - L_k(T)\| \leqq (1 - c\Delta t) \sup_j \|R_j - T_j\|.$$

PROOF. Let $E_j = R_j - T_j$. Then using (6)–(8),

$$|[L_k(R) - L_k(T)]^i| \leqq (1 - 2\beta\lambda_i + \Delta t \frac{\partial N^i}{\partial U^i}(W))|(E_k)^i|$$
$$+ (\beta\lambda_i + \alpha\mu_i)|(E_{k+1})^i| + (\beta\lambda_i - \alpha\mu_i)|(E_{k-1})^i|$$
$$+ \Delta t \sum_{j\neq i} \left| \frac{\partial N^i}{\partial U^j}(W) \right| |(E_k)^j|$$
$$\leqq (1 - c\Delta t) \sup_j \|E_j\|$$

where c is defined by

$$-\frac{\partial N^i}{\partial U^i} \geqq c + \sum_{j\neq i} \left| \frac{\partial N^i}{\partial U^j} \right| \text{in S.}$$

Recall the definitions $E_k^n = U_k^n - V_k^n$ and $E^n = \sup_k \|E_k^n\|$.

THEOREM 8. *Let* $-J_N$ *be* s.d.d. *in* S, *assume that* (6)–(8) *hold, and assume that* $\sup_x \|D_x^j U_0(x)\| < \infty$ *for* $j \leqq 4$. *Then if* $\hat{U}$ *is the unique zero of* N *in* S,

(a) $$E^n \leqq \left(1 - \frac{ct_n}{n}\right)^n (E^0 + ch),$$

(b) $$\|V_k^n - \hat{U}\| \leqq \left(1 - \frac{ct_n}{n}\right)^n \sup_k \|V_k^0 - \hat{U}\|,$$

(c) $$\|U(x, t) - \hat{U}\| \leqq e^{-ct} \sup_x \|U_0(x) - \hat{U}\|.$$

PROOF. We have $\|\tau_k^n\| \leqq ce^{-ct_n}h\Delta t$ from Lemma 6, so that from Lemma 7

$$\|E_k^n\| \leqq \|L_k(U^{n-1}) - L_k(V^{n-1})\| + \|\tau_k^n\|$$
$$\leqq (1 - c\Delta t)E^{n-1} + ce^{-ct_n}h\Delta t.$$

(a) then follows by induction.

To prove (b) take $R_k = V_k^n$ and $T_k = \hat{U}$ in Lemma 7 and note that $L_k(T) = \hat{U}$ for all k. (c) follows from (b) and the convergence of the approximants.

COROLLARY 9. *The hypothesis* "$-J_N$ s.d.d." *may be replaced in Theorems* 4, 5 *and* 8 *by "there is a constant diagonal matrix* $A > 0$ *such that* $-AJ_NA^{-1}$ *is* s.d.d."

PROOF. Make the change of variable $W = AU$ in (1) and (4).

We note, however, that the constants change and that the invariant cubes of Theorem 5 become invariant parallelepipeds.

If B is a square matrix let $\tilde{B}$ be the matrix obtained by replacing the off-diagonal entries of B by their absolute values. The following gives a simple criterion for applying Corollary 9 when $m = 2$.

THEOREM 10. *Let B be a 2×2 matrix. Then there is a diagonal matrix $A > 0$ such that $-ABA^{-1}$ is* s.d.d. *iff $\tilde{B}$ has negative diagonal elements and positive determinant.*

PROOF. Let $\tilde{B} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and assume $a, d < 0$ and ad $-$ $bc > 0$. Then $-a/b > c/-d > 0$, so that if $-a/b > \eta > -c/d$, then $-a > \eta b$ and $-d > (1/\eta)c$. That is,

$$- \begin{bmatrix} \eta & 0 \\ 0 & 1 \end{bmatrix} B \begin{bmatrix} \eta^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

is s.d.d. The converse is proved in a similar manner.

4. **Examples.** In this section we give two examples illustrating the application of Theorems 8 and 10. It will be seen that our hypothesis on N implying the decay of U is the appropriate generalisation of conditions found earlier for these particular examples in [4] and [2] respectively.

Consider first the FitzHugh-Nagumo equations (see [4]),

$$v_t = v_{xx} + f(v) - u$$

$$u_t = \epsilon u_{xx} + \sigma v - \gamma u$$

where $\epsilon \geqq 0$; $\sigma, \gamma > 0$; and $f(v) = -v(v-a)(v-b)$ with $a > b > 0$. In [4] the existence of a family of invariant rectangles of arbitrarily large diameter containing $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is proved.

We have that $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is a zero of $N(U)$ and that

$$\tilde{J}_N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} f'(0) & 1 \\ \sigma & -\gamma \end{bmatrix}.$$

By Theorem 10, Corollary 9 holds provided that $-f'(0) > \sigma/\gamma$. In this case there is a family of invariant rectangles R_h centered at $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$; moreover, if $U_0(x) \in R_h$ for all x where $-f'(v) > \sigma/\gamma$ holds in R_h, then the solution $U(x, t)$ decays exponentially in t, uniformly in x, to $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, and the errors E^n tend to 0 uniformly in t_n as $h \to 0$.

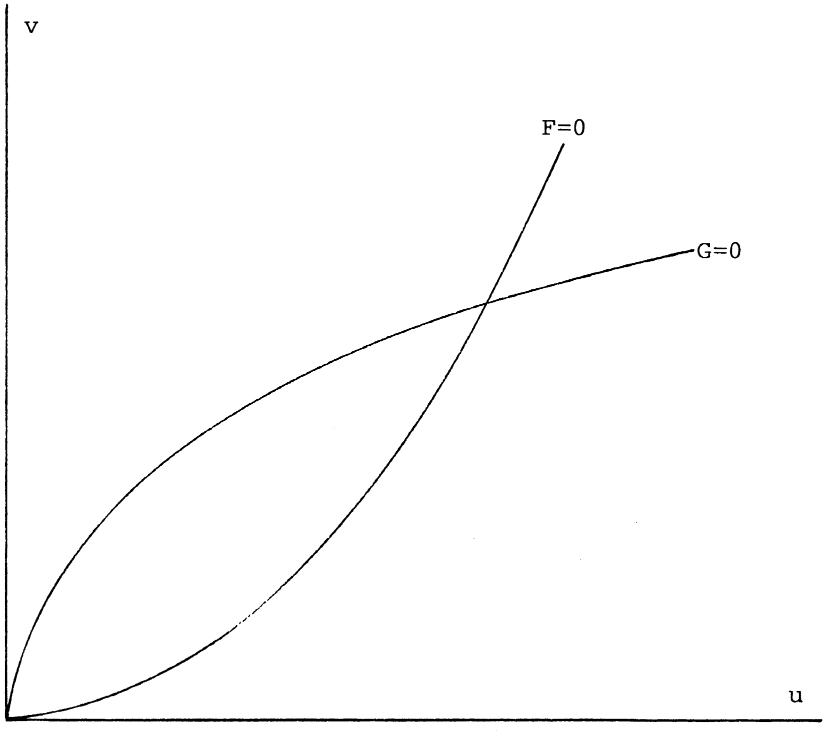
A similar example is provided by the following model for the coexistence of two species whose population densities u and v obey the system

$$u_t = \epsilon_1 u_{xx} + uF(u, v)$$

$$v_t = \epsilon_2 v_{xx} + vG(u, v)$$

where $\epsilon_i \geqq 0$, $F_u < 0 < F_v$, and $G_v < 0 < G_u$. See [2]. To be specific take $F(u, v) = v - u(a - u)^{-1}$ and $G(u, v) = u - v(b - v)^{-1}$ where $a, b > 0$. Assuming that $ab > 1$, the field $N = \begin{bmatrix} uF \\ vG \end{bmatrix}$ has a zero in the first quadrant at

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = (ab - 1) \begin{bmatrix} (1 + b)^{-1} \\ (1 + a)^{-1} \end{bmatrix}.$$



A simple computation shows that at $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$

$$J_N = (ab - 1) \begin{bmatrix} \dfrac{-a(1 + b)}{(1 + a)^2} & \dfrac{1}{1 + b} \\ \dfrac{1}{1 + a} & \dfrac{-b(1 + a)}{(1 + b)^2} \end{bmatrix}.$$

Thus Theorem 10 applies because $ab > 1$ and we conclude that $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ is an attracting equilibrium in the sense described in the first example.

5. **Concluding Remarks.** The results of § 2 and § 3 remain valid for the boundary value problem for (1), for the case that D, M and N depend on x and t, and for the case of several space variables. Furthermore, the results of § 2 can be proved in case D and M depend on U (the proof of Theorem 1 is in fact unchanged); and the results of § 3 can be proved if D and M are sufficiently weak functions of U. Here we use the same difference equation as (4), but with D and M evaluated at V_k^{n-1}.

In approximating the solution of a boundary value problem for (1) it is feasible to employ the implicit scheme

$$
\begin{aligned}
\frac{V_k^n - V_k^{n-1}}{\Delta t} = D \Bigg[& (1-\theta)\left(\frac{V_{k+1}^{n-1} - 2V_k^{n-1} + V_{k-1}^{n-1}}{\Delta x^2}\right) \\
& + \theta \left(\frac{V_{k+1}^{n} - 2V_k^{n} + V_{k-1}^{n}}{\Delta x^2}\right)\Bigg] \\
& + M \left[(1-\theta)\left(\frac{V_{k+1}^{n-1} - V_{k-1}^{n-1}}{2\Delta x}\right) + \theta \left(\frac{V_{k+1}^{n} - V_{k-1}^{n}}{2\Delta x}\right)\right] \\
& + N(V_k^{n-1}).
\end{aligned}
\tag{12}
$$

Again, the stability and error estimates in § 2 and § 3 can be obtained using similar techniques. We note, however, that to obtain the invariance of S for (12), it remains necessary to restrict β as in (6) unless $\theta = 1$. Compare the case $N \equiv 0$ (heat equation) in which a weaker form of stability can be obtained unconditionally (i.e., with no restriction on β) provided $\theta \geqq 1/2$; see [5] pp. 16–18. In other words, the strong stability of Theorem 1 is obtained unconditionally only for the pure implicit scheme $\theta = 1$. For details and proofs see [3].

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University of Michigan, Ann Arbor, Michigan 48109