ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 3, Summer 1977

NON-SINGULAR FLOWS ON S^3 WITH HYPERBOLIC CHAIN-RECURRENT SET

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0. Introduction. An important area of investigation in the study of smooth dynamical systems on compact manifolds is the structure of flows which are structurally stable, i.e., flows whose orbit structure is preserved under small C^1 perturbations. All known such flows have a hyperbolic chain-recurrent set (see § 1 for all definitions) or equivalently satisfy Axiom A and the no-cycle property ([9], [6]). The local structure of such flows has been much studied in [1], [2] and [9] and in particular a great deal is known about the basic sets or indecomposable pieces of the chain-recurrent set. In this article we are concerned with the global structure of the flow, especially the interplay of the local structure and topology of the manifold. This is a very complex question in general and we will limit our investigation to non-singular flows on the three-sphere and will also assume that the flow and its inverse have no strange attractors, which is equivalent to the assumption that all basic sets are one-dimensional.

With these assumptions the flow on each basic set is known to be a suspension of a subshift of finite type (see § 1) and is determined by a matrix B whose entries are 0 and ± 1 (see (1.1)). Although this matrix is far from unique we know ([8] and (1.3)) that the *Parry-Sullivan number* $\psi = |\det(I - B)|$ is an invariant and independent of the choice of the matrix B.

THEOREM 1. *Suppose f_t is a non-singular flow on S^3 with hyperbolic chain-recurrent set and basic sets $\Lambda_0, \Lambda_1, \cdots, \Lambda_{\ell+1}$, all one-dimensional. Then if Λ_0 and $\Lambda_{\ell+1}$ are respectively the only attracting and only repelling basic sets, the linking number of Λ_0 and $\Lambda_{\ell+1}$ equals $\Pi_{i=1}^{\ell} \psi_i$ where ψ_i is the Parry-Sullivan number of Λ_i.*

The linking number of the closed orbits Λ_0 and $\Lambda_{\ell+1}$ is always taken to be positive, i.e., we don't take the orientation of Λ_0 and $\Lambda_{\ell+1}$ into account. If $\ell = 0$ in the above theorem then the vacuous product $\Pi\, \psi_i$ is taken to be 1. Using the fact that for a closed orbit the Parry-Sullivan number ψ is 0 or 2 depending on whether or not the unstable manifold is orientable we immediately obtain the following.

*Research supported in part by NSF Grant MPS74-06731 A01.

Corollary 1. *If f_t is a non-singular Morse-Smale flow on S^3 with exactly one attracting closed orbit and one repelling closed orbit, then the linking number of these orbits is zero or a power of* 2. *If it is 2^k then there are exactly k other hyperbolic closed orbits.*

Corollary 2. *If f_t is a flow on S^3 with an attracting closed orbit and a repelling closed orbit whose linking number is not* 1, *then there are other chain-recurrent orbits in addition to these two.*

For suspensions of subshifts of finite type another invariant introduced by Zeeman [**11**] is defined to be the dimension of $\ker(I - A)$ where A is a $k \times k$ matrix representing the subshift and $I - A$ is thought of as acting on a k-dimensional vector space. The number obtained depends on the field of the vector space, and for our purposes it is useful to use the field $Z/2$, the integers mod two. Thus we define the *Zeeman number* ν to be the dimension of the kernel of $B : (Z/2)^k \to (Z/2)^k$ where B is the mod two reduction of $I - A$.

Theorem 2. *Suppose f_t is a non-singular flow on S^3 with hyperbolic chain-recurrent set and one-dimensional basic sets. Then if there are m attracting closed orbits and n repelling closed orbits, $m + n - 1 \geqq \nu_i$ for all i, where ν_i is the Zeeman number of the i-th basic set.*

This result should be seen as the existence of a topological obstruction to existence of certain basic sets on S^3 when a bound on the number of attracting and repelling closed orbits is given.

1. **Background and Definitions.** If f_t is a smooth flow on a compact manifold M, it is said to be structurally stable provided that for any sufficiently close C^1 approximation g_t there is a homeomorphism $h : M \to M$ carrying orbits of f to orbits of g and preserving the sense of orbits. All known examples of structurally stable flows have a hyperbolic chain-recurrent set so we now define these concepts.

A point x of M is called chain-recurrent ([4] or [5]) for f_t provided that corresponding to any ϵ, $T > 0$ there exist points $x = x_0, x_1, \cdots, x_n = x$ and real numbers $t_0, t_1, \cdots, t_{n-1}$ all greater than T such that $d(f_{t_i}(x_i), x_{i+1}) < \epsilon$ for all $0 \leqq i \leqq n - 1$. The set of all such points, called the chain-recurrent set $\mathcal{R}$, is a compact set invariant under the flow.

A compact invariant set K for a flow f_t is said to have a hyperbolic structure provided that the tangent bundle of M restricted to K is the Whitney sum of three bundles $E^s \oplus E^u \oplus E^c$ each invariant under Df_t for all t and that

(a) The vector field tangent to f_t spans E^c.

(b) There are $C, \lambda > 0$, such that

$$\|Df_t(\nu)\| \leqq Ce^{-\lambda t}\|\nu\| \quad \text{for } t \geqq 0 \text{ and } \nu \in E^s$$

and

$$\|Df_t(\nu)\| \geqq Ce^{\lambda t}\|\nu\| \quad \text{for } t \geqq 0 \text{ and } \nu \in E^u.$$

It is shown in [6] that the condition that a flow have hyperbolic chain-recurrent set is equivalent to Axiom A of Smale [9] and the no-cycle property. Results of Smale [9] then show that the chain-recurrent set $\mathscr{R}$ is the union of a finite number of disjoint, compact, invariant pieces called *basic sets,* each of which contains a dense orbit. A result of Bowen [1] then gives a complete description of one-dimensional basic sets by showing they are topologically conjugate to suspensions of subshifts of finite type, which we now describe.

Given an $n \times n$ matrix $A = (a_{ij})$ of 0's and 1's one considers

$$\Sigma_A = \left\{ \underline{x} = (x_k)_{k=-\infty}^{\infty} \in \prod_{-\infty}^{\infty} \{1, 2, \cdots, n\} \mid a_{x_k x_{k-1}} = 1 \text{ for all } k \right\},$$

and the shift map $\sigma : \sum_A \to \sum_A$ defined by $\sigma(\underline{x}) = \underline{y}$ where $y_k = x_{k+1}$. Then $\sum_A$ is a compact zero-dimensional space and σ is called a *subshift of finite type.* The *suspension flow* of σ is defined as follows: let $X_A = \sum_A \times [0, 1]/\sim$ where $\sim$ identifies $(\underline{x}, 1)$ with $(\sigma(\underline{x}), 0)$ and the flow ϕ_t on X_A is defined by $\phi_t(\underline{x}, s) = (\underline{x}, s + t)$ for $t + s \in [0, 1)$ and for other t by using identification.

By the result of Bowen [1] mentioned above each one dimensional basic set is topologically conjugate to a suspension of a subshift of finite type and hence described by the matrix A. Although the matrix A is far from unique it is shown in [8] that $\det(I - A)$ is an invariant of the topological conjugacy class and it is a variant of this invariant which we use in Theorem 1.

We first alter the matrix A so that it includes more information, namely the structure of the bundles E^u and E^s. If $h : x_A \to \Omega_i$ is the topological conjugacy we consider the cross section to the flow f_t given by $h(\sum_A)$, where $\sum_A = \sum_A \times \{0\} \subset X_A$, and the first return map $\rho : h(\sum_A) \to h(\sum_A)$ under the flow f_t ($h \mid \sum_A$ will be a topological conjugacy between σ and ρ). The bundle E^u restricted to $h(\sum_A)$ is trivial since $h(\sum_A)$ is zero dimensional so we can choose an orientation for it. If we let $C_i = \{a \in \sum_A \mid a_0 = i\}$ and A is chosen so the C_i are sufficiently small, then the function

$$\Delta(x) = \begin{cases} 1 \text{ if } D\rho_x : E_x{}^u \to E^u_{\rho(x)} \text{ preserves orientation} \\ -1 \text{ if } D\rho_x : E_x{}^u \to E^u_{\rho(x)} \text{ reverses orientation} \end{cases}$$

is constant on C_i and we define its value on C_i to be Δ_i.

(1.1) DEFINITION. The matrix $B = (b_{ij})$ defined by $b_{ij} = \Delta_j a_{ij}$ will be called a *structure matrix* for the basic set.

Clearly the matrix B contains all the information of A and it is not difficult to see that the bundle E^u is isomorphic to the bundle $\sum_A \times [0,1] \times R^k/\sim$, where $k =$ fiber dim E^u and $\sim$ identifies $(\underline{a}, 1, v)$ with $(\sigma(\underline{a}), 0, \tau(\underline{a}, v))$ where $\tau(\underline{a}, v) = v$ if $b_{a_1 a_0} = 1$ and $\tau(v)$ (τ an orientation reversing involution of R^k) if $b_{a_1 a_0} = -1$. If the ambient manifold is orientable the matrix B also determines the isomorphism class of E^s.

If a flow has hyperbolic chain-recurrent set (and hence satisfies Axiom A and the no cycle property) it possesses a *filtration*. That is, there are $M_0 \subset M_1 \subset \cdots \subset M_\ell = M$, submanifolds (with boundary) of the same dimension as M, invariant under f_t for $t \geqq 0$ such that $\bigcap_{t \in R} f_t(M_i - M_{i-1}) = \Omega_i$ where Ω_i is a basic set for f_t (see [5] for example).

If Ω_i is one-dimensional then it can be described by a structure matrix B as above and a theorem (which is (5.3) from [3]) shows B is closely related to $H_*(M_i, M_{i-1})$.

(1.2) THEOREM [3]. *Let B be a structure matrix for the one-dimensional basic set Λ_i, then if G is any abelian group and $u =$* fiber dim E_i^u,

$$H_{u+1}(M_i, M_{i-1}; G) = \ker(I - B \text{ on } G^n)$$

$$H_u(M_i, M_{i-1}; G) = G^n/(I - B)G^n$$

and

$$H_k(M_i, M_{i-1}; G) = 0 \qquad k \neq u, u+1.$$

(1.3) REMARK. If we take $G = Z$, we note that if $\det(I - B) \neq 0$, then $|\det(I - B)|$ equals the order of the group $H_u(M_i, M_{i-1}; Z)$ and $\det(I - B) = 0$ if and only if $H_u(M_i, M_{i-1}; Z)$ is infinite. Hence $|\det(I - B)|$ is independent of the choice of structure matrix B. We thus obtain a slight variation of an invariant of Parry and Sullivan [8].

(1.4) DEFINITION. The *Parry-Sullivan number* of a one-dimensional basic set Λ_i is $\psi_i = |\det(I - B)|$ where B is a structure matrix for Λ_i.

If Λ_i consists of a single closed orbit, then the structure matrix B is either (1) or (-1) depending on whether or not the bundle E^u is orientable. Hence in this case the Parry-Sullivan number is either zero or two.

The invariant used in Theorem 2 has the advantage that it can be computed from either a structure matrix B or the unsigned matrix A representing the subshift of finite type. This is because $b_{ij} = \pm a_{ij}$ and hence A and B agree mod 2.

(1.5) DEFINITION. If Λ_i is topologically conjugate to the suspension of $\sigma : \sum_A \to \sum_A$ and has structure matrix B we define the *Zeeman number* $\nu = \dim\ker(I - A \text{ on } (Z/2)^n) = \dim\ker(I - B \text{ on } (Z/2)^n) = \dim H_{u+1}(M_i, M_{i-1}; Z/2)$.

The last equality follows from (1.2).

2. **Proof of Theorems.**

(2.1) LEMMA. *If f_t is a non-singular flow on S^3 with hyperbolic chain-recurrent set having a single attracting basic set and a single repelling basic set, both closed orbits, and if $M_0 \subset M_1 \subset \cdots \subset M_{\ell+1}$ is a filtration for f_t, then*

$$|H_1(M_\ell, M_0)| = \prod_{i=1} |H_1(M_i, M_{i-1})|$$

where $|\ |$ *denotes the order of a group.*

If not all the groups involved are finite this is to be interpreted to mean $H_1(M_\ell, M_0)$ is infinite if and only if at least one of the $H_1(M_i, M_{i-1})$ is infinite.

PROOF. We first note that the set of points which eventually flow into M_0 is dense in M, so since M_k is invariant under the flow, it follows that each M_k is connected. Consider now the case that $H_1(M_i, M_{i-1})$ is finite for all $1 \leqq i \leqq \ell$. Then by Theorem (1.2), $H_1(M_{k+1}, M_k)$ finite implies $H_2(M_{k+1}, M_k) = 0$ and $H_0(M_k, M_0) = 0$ since M_k is connected, so the long exact sequence of the triple (M_{k+1}, M_k, M_0) becomes

$$0 \to H_1(M_k, M_0) \to H_1(M_{k+1}, M_0) \to H_1(M_{k+1}, M_k) \to 0.$$

Now it follows that

$$|H_1(M_{k+1}, M_0)| = |H_1(M_k, M_0)|\,|H_1(M_k, M_{k-1})|,$$

so by induction on k

$$|H_1(M_\ell, M_{0)}| = \prod_{i=1}^{\ell} |H_1(M_i, M_{i-1})|,$$

which is the desired result. This also shows that if $H_1(M_\ell, M_0)$ is infinite, then at least one of the $H_1(M_i, M_{i-1})$ is infinite.

Suppose now that $H_1(M_{k+1}, M_k)$ is infinite then consideration of the exact sequences

$$H_2(M_{k+1}, M_k) \to H_1(M_k, M_0) \to H_1(M_{k+1}, M_0) \to H_1(M_{k+1}, M_k) \to 0$$

and

$$H_2(M_{k+2}, M_{k+1}) \to H_1(M_{k+1}, M_0) \to H_1(M_{k+2}, M_0)$$

shows that $H_1(M_{k+1}, M_0)$ is infinite and hence (from the second sequence) either $H_1(M_{k+2}, M_0)$ is infinite or $H_2(M_{k+2}, M_{k+1})$ is infinite. But by Theorem (1.2), $H_2(M_{k+2}, M_{k+1})$ infinite implies $H_1(M_{k+2}, M_{k+1})$ infinite and (replacing k by $k+1$ in the first sequence) we get $H_1(M_{k+2}, M_0)$ infinite in this case too. Continuing inductively we have $H_1(M_\ell, M_0)$ infinite.

(2.2) PROOF OF THEOREM 1. We note that since the single closed orbit attractor is a deformation retract of M_0, $H_1(M_0) = Z$ and similarly $H_1(M - M_\ell) = Z$ and so by Alexander duality (see [10], p. 296) $H_1(M_\ell) = Z$. The linking number of two curves C_1 and C_2 in S^3 is defined to be the degree of the map $i_* : H_1(C_1) \to H_1(X)$ where X is the complement of C_2 and $i : C_1 \to X$ is the inclusion. Clearly in our case this is the same as the degree of $i_* : H_1(M_0) \to H_1(M_\ell)$.

Now considering the exact sequence of the pair (M_ℓ, M_0),

$$\begin{array}{ccccccc} H_1(M_\ell) & \xrightarrow{xm} & H_1(M_0) & \to & H_1(M_\ell, M_0) & \to & \tilde{H}_0(M_0) \\ \| & & \| & & & & \| \\ Z & & Z & & & & 0 \end{array}$$

we see that the linking number is m if and only if

$$H_1(M_\ell, M_0) = Z/mZ \quad (Z \text{ if } m = 0).$$

If $m \neq 0$, then by Lemma (2.1) and (1.3)

$$m = |H_1(M_\ell, M_0)| = \prod_{i=1}^{\ell} |H_1(M_i, M_{i-1})| = \prod_{i=1}^{\ell} \psi_i.$$

And if $m = 0$, then $H_1(M_\ell, M_0)$ is infinite so at least one of the $H_1(M_i, M_{i-1})$ is infinite and hence one of the $\psi_i = 0$.

(2.3) EXAMPLE: It is not difficult to construct an embedding $g : D^2 \to D^2$ whose chain-recurrent set consists of an attracting periodic point of period 2 and a hyperbolic fixed point whose unstable manifold has its orientation reversed by g. Suspending g (and rounding off corners) gives a flow on the solid torus $S^1 \times D^2$ which flows inward on the boundary and whose chain-recurrent set is a single attracting closed orbit which runs twice around the torus and a hyperbolic closed orbit with unoriented unstable manifold.

If we do surgery to replace a neighborhood of the attracting closed orbit with another copy of this solid torus, the new attracting orbit will wrap 4 times around, or by iteration, any power of 2. Finally attaching another solid torus with an outward vector field which has a single closed orbit repeller inside we obtain a flow on S^3 illustrating Theorem 1.

(2.4) LEMMA. *If ϕ is a non-singular flow on S^3 with hyperbolic chain-recurrent set and with each attracting or repelling basic set a closed orbit, and if $M_0 \subset M_1 \subset \cdots \subset M_{\ell+1} = S^3$ is a filtration for ϕ with M_0 containing the attracting basic sets and no others and $M_{\ell+1} - M_\ell$ containing the repelling basic sets and no others, then*

$$\dim H_1(M_\ell, M_0; Z/2) \geqq \dim H_1(M_i, M_{i-1}; Z/2)$$

for all $1 \leqq i \leqq \ell$.

PROOF: Since the set of points in M_i which flow into M_0 is dense and open and M_i is invariant under the flow, we have $H_0(M_i, M_0) = 0$ for all i. (In this proof all homology will be with $Z/2$ coefficients). We consider now the exact sequence of the triple (M_k, M_{k-1}, M_0)

$$\begin{aligned}\to H_2(M_k, M_0) \xrightarrow{j_*} H_2(M_k, M_{k-1}) &\to H_1(M_{k-1}, M_0) \to H_1(M_k, M_0) \\ &\to H_1(M_k, M_{k-1}) \to H_0(M_{k-1}, M_0).\end{aligned}$$

From this we obtain the exact sequence

$$\begin{aligned}O \to G \to H_2(M_k, M_{k-1}) &\to H_1(M_{k-1}, M_0) \\ &\to H_1(M_k, M_0) \to H_1(M_k, M_{k-1}) \to 0,\end{aligned}$$

where $G = j_*(H_2(M_k, M_0))$.

Since the alternating sum of the dimensions of vector spaces in an exact sequence is zero (see [7], p. 99) we have $g - x + y - z + x = 0$, where $g = \dim G$, $y = \dim H_1(M_{k-1}, M_0)$, $z = \dim H_1(M_k, M_0)$, and $x = \dim H_2(M_k, M_{k-1}) = \dim H_1(M_k, M_{k-1})$ (this equality follows from Theorem (1.2) since we are using coefficients in $Z/2$, a field). Thus we have $g + y = z$, or $z \geqq y$ so $\dim H_1(M_k, M_0) \geqq \dim H_1(M_{k-1}, M_0)$ for all $1 \leqq k \leqq \ell$. It follows inductively that $\dim H_1(M_\ell, M_0) \geqq \dim H_1(M_k, M_0)$ and since it is clear from the exact sequence above that $\dim H_1(M_k, M_0) \geqq \dim H_1(M_k, M_{k-1})$ we have the desired result.

(2.5) PROOF OF THEOREM 2. All homology in this proof is with $Z/2$ coefficients. We first choose a filtration satisfying the hypothesis of (2.4). So if there are m attracting closed orbits, M_0 can be chosen to be

m disjoint solid tori and we will have $\dim H_1(M_0) = m$ and $\dim \tilde{H}_0(M_0) = m - 1$ ($\tilde{H}_0$ denotes reduced homology). Similarly $\dim H^1(S^3 - M_\ell) = n$ and so by Alexander duality (see [10], p. 296) we have $\dim H_1(M_\ell) = n$.

We now consider the exact sequence of the pair (M_ℓ, M_0),

$$\cdots \to H_1(M_0) \xrightarrow{i_*} H_1(M_\ell) \to H_1(M_\ell, M_0) \to \tilde{H}_0(M_0) \xrightarrow{i_*} \tilde{H}_0(M_\ell) \to \cdots$$

from which we obtain

$$0 \to G_1 \to H_1(M_\ell) \to H_1(M_\ell, M_0) \to \tilde{H}_0(M_0) \to G_2 \to 0$$

where $G_1 = i_*(H_1(M_0)) \subset H_1(M_\ell)$ and $G_2 = i_*(\tilde{H}_0(M_0)) \subset \tilde{H}_0(M_\ell)$. We again use the fact that the alternating sum of the dimensions of vector spaces in an exact sequence is zero, and obtain

$$g_1 - n + \dim H_1(M_\ell, M_0) - (m - 1) + g_2 = 0$$

where $g_i = \dim G_i$. Thus

$$\dim H_1(M_\ell, M_0) + g_1 + g_2 = n + m - 1,$$

and it follows that

$$n + m - 1 \geqq \dim H_i(M_\ell, M_0).$$

By lemma (2.4) $\dim H_1(M_\ell, M_0) \geqq \dim H_1(M_i, M_{i-1})$ for all $1 \leqq i \leqq \ell$. Now by (1.5) we have

$$\nu_i = \dim H_1(M_i, M_{i-1}) \leqq \dim H_1(M_\ell, M_0) \leqq n + m - 1.$$

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