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SEPARATRICES IN SOLITUDE

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Introduction. Our purpose is to give a characterization of the set of separatrices for a differentiable flow in a neighborhood of solitude for a solitary invariant set in Euclidean space or in a differentiable manifold. We may postpone technical definition of terms and motivate the characterization by considering the following differential equations in the plane:

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= x^2 + y^2 \end{aligned} \tag{0.1}$$

$$\begin{aligned} \dot{x} &= (1 + c^2)xy, \qquad c \neq 0 \\ \dot{y} &= x^2 + y^2. \end{aligned} \tag{0.2}$$

The phase portraits are readily obtained, as shown in Figures 1 and 2, respectively (see [7]). An obvious difference in the portraits occurs in the number of separatrices, there being two in Figure 1, and four in Figure 2, not counting the singularity. However, if one tries to obtain the portraits on a graphics terminal of a digital computer, the distinction is not so easy. Since a typical numerical integration scheme identifies some neighborhood V of $(0, 0)$ with that point, it therefore makes all of V a region of rest points, and hence gives us portraits more like Figure 2 than Figure 1 for both systems, (0.1) and (0.2). More complicated differential equations produce similar, but nontrivial problems. We are led to the question: Is there a relation between the separatrices given by graphics and the actual separatrices determined by a rigorous phase space analysis?

Loosely stated, our answer is: If graphics separatrices are obtained as the flow propagation of extenal tangencies to V, then the actual separatrices are determined as a limit set of such propagation, taken as V shrinks to the origin. For example, if one draws V in Figure 2 as a small disk around the origin, the flow propagation of external tangencies is merely the two trajectories, tangent to V at its sides. As V shrinks to the origin, the limit consists of the four separatrix trajectories, together with the origin itself.

1. **Main Theorem.** Let ϕ be a C^r flow on a C^r manifold $M (r \geqq \dim M)$. A compact ϕ-invariant set I is solitary if there is a compact neighborhood (of solitude) U of I which satisfies: whenever a half-

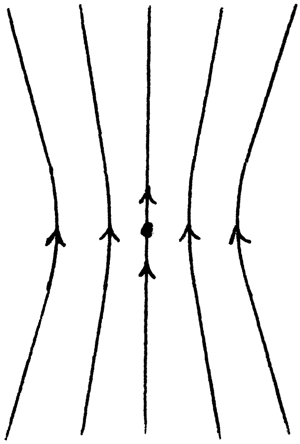

Figure 1

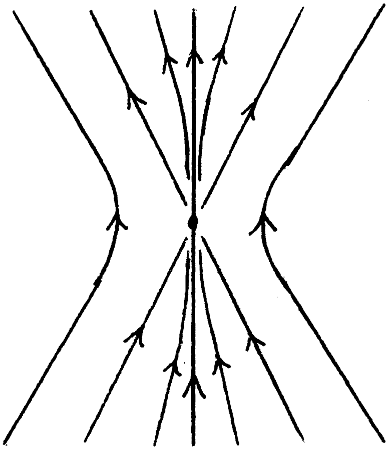

Figure 2

trajectory of ϕ is contained in U, the appropriate α- or ω-limit is contained in I. Solitary invariant sets have been studied in [3] and [5]. We fix ϕ, I, and U for the rest of this section of the paper.

Separatrices in more general settings may be defined to be trajectories where the set valued functions given by the α- and ω-limits are discontinuous or where the time required to arrive near the limits is not uniform near the trajectory in question (see [4], [8]). However, since we are interested only in separatrices which are generated by I in U, we adapt these ideas as follows: A maximum trajectory segment σ in $U - I$ is a primary separatrix generated by I if for some $x \in \sigma$

there are sequences $x_n \to x$ and $t_n \to \infty$ such that $\phi(x_n, [0, t_n]) \subset U$ and $\phi(x_n, t_n)$ converges to a point in $U - I$, or the analogous statement holds for $t_n \to -\infty$. Let $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$ denote the sets of such x in the respective cases $t_n \to +\infty$ and $t_n \to -\infty$, and set $\mathcal{S}_1 = \mathcal{S}_1^+ \cup \mathcal{S}_1^-$. Note that every primary separatrix meets $\mathcal{S}_1$ but is not necessarily contained in it. The separatrix set $\mathcal{S}$ is the closure in $U - I$ of the union of all primary separatrices.

For a compact subneighnorhood V, $I \subset V \subset U$, a point $x \in b(V)$ (b = boundary) is a point of external tangency of ϕ to V if for some a, b with $a < 0 < b$, $\phi(x, [a, b]) \subset U - \text{Int}(V)$ and $\phi(x, a), \phi(x, b) \in U - V$. Then the propagation of external tangencies, $e(V)$, is the union of all such trajectory segments, $\phi(x, [a, b])$. If $\mathcal{V} = \{V_n\}$ is a fundamental sequence of such neighborhoods of I, let $e(\mathcal{V}) = (\limsup e(V_n)) - I$, taken as $n \to \infty$. ($\limsup X_n = \{x$: every neighborhood of x intersects infinitely many $X_n\}$.) Let $T_e = \bigcap e(\mathcal{V})$ taken over all fundamental systems $\mathcal{V}$ of compact neighborhoods of I. Finally, if X is any subset of U let $\phi(X)$ denote the union of all trajectory segments $\phi(x, [a, b])$ contained in U which intersect X.

Theorem 1.1. $\mathcal{S} = \phi(T_e)$.

Proof. We first establish $\mathcal{S} \subset \phi(T_e)$. To do this we need notation for exit time:

$$t^+(x) = \sup\{t \geqq 0 \mid \phi(x, [0, t]) \subset U\}$$

$$t^-(x) = \sup\{t \geqq 0 \mid \phi(x, [-t, 0]) \subset U\}.$$

Lemma 1.2. *Let $\mathcal{V} = \{V_m\}$ be a fundamental system of compact neighborhoods of I with $V_m \subset U$ for all m. Then $\mathcal{S}_1^+ \subset e(\mathcal{V})$.*

Proof. Let $x \in \mathcal{S}_1^+$ be given, and let W be an open, connected neighborhood of x. By definition, there exist $y \in U - I$, $x_n \to x$, $t_n \to \infty$ such that $\phi(x_n, t_n) \to y$. We may assume $x_n \in W$ for all n. The trajectory segment $\phi(x_n, [0, t_n])$ meets only finitely many of the V_m; hence for some integer N and all $m > N$, $\phi(x_1, [0, t_1]) \cap V_m = \emptyset$. Letting $m \geqq N$ be fixed, it suffices to show $e(V_m) \cap W \neq \emptyset$.

Let $f: W \to [t_1, \infty]$ be continuous with respect to the usual topology on the extended reals, and such that $f(x_1) = t_1$ and such that $f^{-1}(\infty)$ contains a neighborhood of x.

Let W^i be the set of all $z \in W$ such that $\phi(z, t) \in \text{Int}(V_m)$ for some positive, real $t \leqq \min\{t^+(z), f(z)\}$. Then W^i is open in W. Let W^b be the set of all $z \in W$ such that $\phi(z, t) \in b(V_m)$ and $\phi(z, [0, t]) \subset U - V_m$ for some real $t \leqq \min\{t^+(z), f(z)\}$. Then W^b is closed. Obviously $W^i \subset W^b$. If, for some $z \in W^b$, there are positive times $t_1 < t_2$ such that $\phi(z, [0, t_1]) \subset U - V_m$, $\phi(z, t_1) \in b(V_m)$ and in addition $\phi(z, [t_1, t_2]) \subset U - \text{Int}(V_m)$ with $\phi(z, t_2) \in U - V_m$, then we are done. If this is not the case, then $W^b \subset W^i$, because in solitude every trajectory must either approach I or leave U. In this case $W^b = W^i$ *is* open and closed in W, and hence is either empty or equal to W. But $x_1 \in W - W^b$, and $x \in \mathcal{S}_1{}^t$ implies $x \in W^i$. This contradiction completes the proof of the lemma.

Lemma 1.3. *If* $c\ell(X)$ ($c\ell$ = *closure*) *does not meet* I, *then* $c\ell(\phi(X)) \subset \phi(c\ell(X)) \cup \mathcal{S}_1$.

Proof. Suppose $z \in c\ell(\phi(X))$. Then there are sequences, x_n in X and real t_n such that $\phi(x_n, t_n) \to z$ as $n \to \infty$ with each segment $\phi(x_n, [0, t_n])$ (assuming, without loss of generality, that each $t_n \geqq 0$) is contained in U. By choosing subsequences and relabeling we may assume that $\{x_n\}$ and $\{t_n\}$ have limits x and t_∞ respectively, where perhaps $t_\infty = \infty$. In the latter case $z \in \mathcal{S}_1$ since $x \in U - I$. But if t_∞ is finite, then $z = \phi(x, t_\infty)$ because of continuity in initial conditions, and it is easily checked that $\phi(x[0, t_\infty] \subset U$ for the same reason.

Notice that the reverse inclusion, $\phi c\ell(X)) \subset c\ell(\phi(X))$, is false whenever ∂U has internal tangencies with ϕ. Furthermore the conclusion of the lemma does not hold if $c\ell(X)$ meets I, as can be seen when I is a rest point in the plane and X consists of a sequence of points, contained in a fan uniformly attracted to I, and converging to I.

An easy application of (1.2) shows that $\mathcal{S}$, equal by definition to $c\ell(\phi(\mathcal{S}_1{}^+ \cup \mathcal{S}_1{}^-)) - I$, is contained in $c\ell(\phi(e(\mathcal{V}))) - I$ for any fundamental system $\mathcal{V}$. In trying to apply (1.3) to $X = e(\mathcal{V})$, we encounter some trouble because $c\ell(e(\mathcal{V}))$ may meet I. The trouble can be overcome, and below we will use (1.3) to establish

$$c\ell(\phi(e(\mathcal{V}))) - I \subset \phi(c\ell(e(\mathcal{V})) - I) \cup \mathcal{S}_1. \tag{1.3a}$$

Since superior limits are closed, $e(\mathcal{V})$ is closed in $U - I$ and (1.3a) simplifies to

$$c\ell(\phi(e(\mathcal{V}))) - I \subset \phi(e(\mathcal{V})) \cup \mathcal{S}_1. \tag{1.3b}$$

Finally, we apply (1.2) again to find that $\mathcal{S} \subset \phi(e(\mathcal{V}))$.

Toward establishing (1.3a), suppose $\{z_n\}$ is a sequence in $\phi(e(\mathcal{V}))$ convergent to $z \notin I$. If infinitely many of the z_n are in $e(\mathcal{V})$, then it is easy to see we are done. If only finitely many z_n are in $e(\mathcal{V})$ then we may as well assume that every $z_n \in \phi(e(\mathcal{V})) - e(\mathcal{V})$.

We say that a point x of ∂U is a point of internal tangency if $\phi(x, (-\epsilon, \epsilon)) \subset U$ for some positive ϵ.

LEMMA 1.4. *If* $y \in \phi(e(\mathcal{V})) - e(\mathcal{V})$ *then* $\phi(y)$ *contains a point of internal tangency.*

PROOF. If $y \in \phi(e(\mathcal{V})) - e(\mathcal{V})$ then there is a point $\bar{y} \in \phi(y) \cap e(\mathcal{V})$ with $y = \phi(\bar{y}, \bar{t})$. It will suffice to show that $y \in e(\mathcal{V})$ whenever the open trajectory segment between y and $\bar{y}$ does not meet ∂U. We shall restrict ourselves to the case $\bar{t} > 0$. If S is a sufficiently small local surface of section to ϕ at $\bar{y}$, then the connected component of $\bar{y}$ in $\phi(S, (0, \bar{t})) \cap U$ is an open tubular neighborhood N of the open segment $\phi(\bar{y}, (0, \bar{t}))$, with $N \subset U$. If $\{y_i\}$ is a sequence in N convergent to $\bar{y}$, then there is a real number sequence $\{\epsilon_i\}$ convergent to zero such that $\phi(y_i, \bar{t} + \epsilon_i)$ converges to y and $\phi(y_i, (0, \bar{t} + \epsilon_i)) \subset N$. Since $\bar{y} \in e(\mathcal{V})$, it is the limit of a sequence $\{y_{m_i}\}$ with $y_{m_i} \in V_{m_i}$ for $i = 1, 2, \cdots$. There is a real number sequence $\{\delta_i\}$ convergent to zero such that $\{\phi(y_{m_i}, \delta_i)\}$ is in N and convergent to $\bar{y}$. Hence we may choose $\{\epsilon_i\}$ as above to obtain $\phi(y_{m_i}, \bar{t} + \delta_i + \epsilon_i) \to y$ as $i \to \infty$. It is now easily seen that $y \in e(\mathcal{V})$ as claimed.

COROLLARY 1.5. *Under the conditions of* (1.4) $y = \phi(y', t)$ *and* $\phi(y', [0, t]) \subset U$ *for some point of internal tangency* y' *which is in* $e(\mathcal{V})$.

Finally, we apply (1.5) to obtain $z_n = \phi(z_n', t_n)$ (z_n as above) with $z_n' \in \partial U \cap e(\mathcal{V})$ and $\phi(z_n', [0, t_n]) \subset U$. Let X be the point set of the sequence $\{z_n'\}$ and apply (1.3) to find that $y \in c\ell(\phi(X)) \subset \phi(c\ell(X)) \cup \mathcal{S}_1 \subset \phi(c\ell(e(\mathcal{V}))) \cup \mathcal{S}_1$. Hence (1.3a) is established, and with it, the containment of $\mathcal{S}$ in $\phi(e(\mathcal{V}))$.

LEMMA 1.6. *Given any point* x *of* $U - (I \cup \mathcal{S})$, *there is a fundamental system* $\mathcal{V} = \{V_n\}$ *of compact neighborhoods of* I *such that* x *is not contained in* $e(\mathcal{V})$.

PROOF. If $t^+(x)$ is finite, then for any fundamental system $\{V_m\}$, it cannot occur that $x = \lim x_i$ as $i \to \infty$ such that $\phi(x_i, [0, t^+(x_i)])$ contains an external tangency of some V_{m_i} with $m_i \to \infty$, because of the compactness of $\phi(x, [0, t^+(x)])$. Hence if both $t^+(x)$ and $t^-(x)$ are finite, we have $x \notin e(\mathcal{V})$ for arbitrary $\mathcal{V}$. We will confront the situation where both $t^+(x)$, $t^-(x)$ are infinite, and leave the adaptions for the remaining cases to the reader.

In this case, the condition that $x \in U - (I \cup \mathcal{S})$ is equivalent to the statement: there exists a local surface of section σ to ϕ through x such that for every $\epsilon > 0$ there corresponds a $T_\epsilon > 0$ satisfying $d(\phi(y, t), I) < \epsilon$ (d = distance) if $y \in \sigma$ and $|t| > T_\epsilon$. Let p denote the dimension of the ambient manifold M, and B_r denote the closed ball in Euclidean $(p - 1)$-space with radius r and centered at the origin. We may choose $\sigma = h(B_2)$ for some diffeomorphism h, and with $x = h(0)$. Let $\sigma^* = h(B_1)$. We will construct $\{V_m\}$ so that for every $y \in \sigma^*$ and for each index m, the trajectory through y is transverse to the boundary of V_m, thus completing the proof.

Choose a fundamental system $\{U_n\}$ of neighborhoods of I such that each U_n is a smooth submanifold with boundary ∂U_n in U. This is possible by using regular values of a C^r real valued function which is 0 on I and 1 off of U. We may assume $x \notin U_n$ and $U_{n+1} \subset \text{Int}(U_n)$ for all n. Let n be fixed. Choose $\tau_n > 0$ such that $\phi(\sigma, t) \subset \text{Int}(U_{n+1})$ if $|t| > \tau_n$. Using the density of transversal intersections (see, for example, [1]), we may perturb the embedding of ∂U_n to obtain ∂W_n bounding W_n and satisfying both: (a) $U_{n+1} \subset \text{Int}(W_n)$ and $W_n \subset \text{Int}(U_{n-1})$, and (b) ∂W_n is transverse to $\phi(x, [-\tau_n, \tau_n])$. By openness of transversality, there is a subdisk $\sigma_n \subset \sigma$ such that ∂W_n is transverse to $\phi(y, [-\tau_n, \tau_n])$ for all y in σ_n. We want to alter W_n to obtain the last transversality condition on a subdisc of σ which is independent of n. Choose a smooth isotopy $f_s : \sigma \to \sigma$ so that (a) $f_s(x) = x$ and f_s is the identity near $\partial\sigma$ for all s, (b) f_0 is the identity, and (c) $\sigma^* \subset f_1(\sigma_n)$. Define the isotopy F_s on $\phi(\sigma, \mathbf{R})$ by $F_s(\phi(y, t)) = \phi(f_s(y), t)$. Finally, we extend F_1 to U (smoothly, off of I) as the identity off of $\phi(\sigma, \mathbf{R})$, and set $V_m = F_1(W_m)$. Since σ^* was fixed and there are no tangencies of $\phi(y, \mathbf{R})$ with ∂V_m for any $y \in \sigma^*$ or m, we conclude $x \notin e(\mathcal{V})$ for $\mathcal{V} = \{V_m\}$. The proofs of Lemma 1.6 and Theorem 1.1 are complete.

2. **Examples.** There is a subtlety in our formulation of separatrices that should be exposed. Intuitively, it arises because we are interested only in those separatrices which are "generated" by the invariant set I. For example, consider the homogeneous planar system

$$\begin{aligned} \dot{x} &= x^2 - y^2 \\ \dot{y} &= xy \end{aligned} \tag{2.1}$$

which has the easily obtained phase portrait in Figure 3. This system can be extended to the 2-sphere in two distinct ways which are of interest here. In both, we obtain the sphere as a smooth one point compactification of the plane. Extend system (2.1) to the sphere in the following ways:

(2.2) The vector field has a zero at ∞.

(2.3) The vector field is nonzero at ∞ in such a way as to smoothly join the trajectories of (2.1) given by the open positive and negative rays of the x-axis. (In appropriate local u, v coordinates near the point at infinity, $\dot{u} \equiv 1$ and $\dot{v} \equiv 0$).

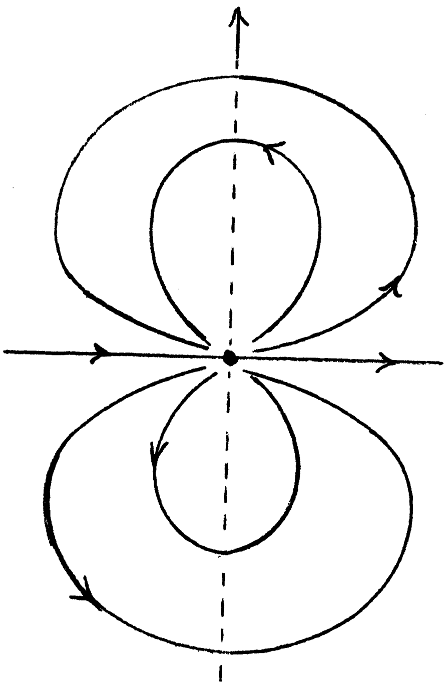

Figure 3

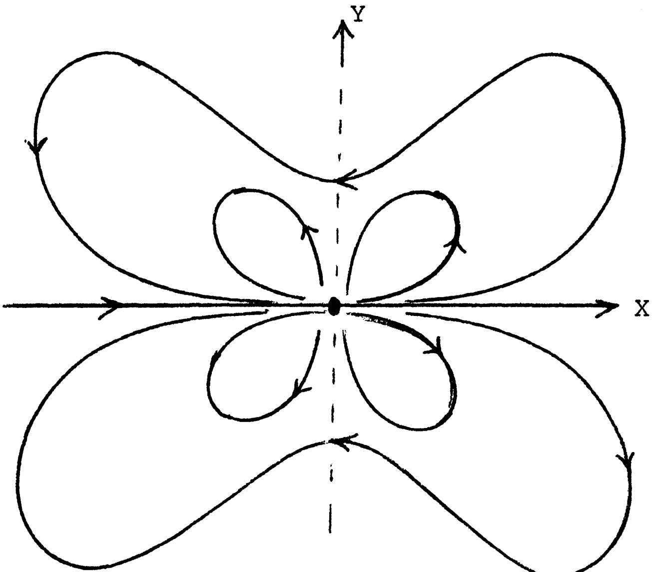

Figure 4

The system (2.3) has no separatrices (unless one declares the original fixed point to be one). Hence it is our point of view that it should be said that neither (2.2) nor (2.3) have separatrices generated by the original rest point of (2.1). Clearly (2.2) has two separatrices, namely on the finite x-axis, but these are generated by the rest point at ∞.

A second example is portrayed in Figure 4. Here the separatrices are the four "leaves" of the inner "clover." Although the author does not know of a simple expression for the vector field, the separatrices are easily found by applying the theorem. In fact, if $\mathcal{V}$ is the fundamental system of neighborhoods which are concentric disks centered at the origin, then $\mathcal{S} = \phi(e(\mathcal{V}))$. We emphasize that in this example, it is sufficient to consider only one fundamental system of neighborhoods. We shall see that this is always true if I is an isolated rest point of a planar analytic system. Furthermore, Poincaré-Bendixson considerations show that it is trivially true when I is an isolated periodic orbit of a planar system. The author does not know whether solitary periodic solutions in higher dimensions are also nice in this respect.

A final example easily shows that the fundamental system of concentric disks about an isolated critical point at the origin of the plane is not always useful. In contrast with Figure 4 is Figure 5, generated by the system

$$\begin{aligned} \dot{x} &= x + y^2 \\ \dot{y} &= -2xy. \end{aligned} \tag{2.4}$$

Let U be large enough to contain all of Figure 5, and let $\mathcal{V}$ be a fundamental sequence of concentric disks about the origin, with V denoting a typical member of $\mathcal{V}$. Then the external tangencies to V are at the intersection of ∂V with the curves $x = 0$ and $x = 2y^2$. The propagation of the tangencies on $x = 2y^2$ by the solution curves of (2.4) limit to the positive x-axis as V shrinks to the origin. So $e(\mathcal{V}) \neq \emptyset$ for this choice of $\mathcal{V}$, yet the origin is asymptotically stable so that $\mathcal{S} = \emptyset$. We conclude that $\mathcal{V}$ is not useful in determining the separatrix structure.

3. **Analytic systems in the plane.** Consider the system

$$\begin{aligned} \dot{x} &= X(x, y) \\ \dot{y} &= Y(x, y) \end{aligned} \tag{3.1}$$

where X and Y are real analytic and $(0, 0)$ is an isolated critical point. Our goal is to prove Theorem 3.2. Assume that $0 = (0, 0)$ is not a center.

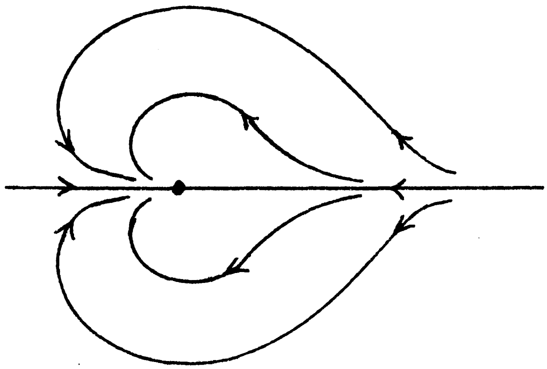

Figure 5

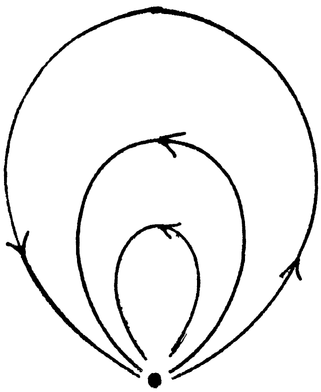

Figure 6

By a *generalized Lyapunov function* for a solitary invariant set I in solitude U, we mean a continuous function V defined on U with non-negative values such that $V(z) = 0$ if and only if $z \in I$ and such that the time derivatives, $\dot{V}(z) = d/dt\, V(\phi(z, t))$ exist at $t = 0$ for all z in U(cf. [9]).

THEOREM 3.2. *For X, Y as above, 0 is solitary and there is a generalized Lyapunov function for 0 such that the fundamental system $\mathcal{V} = \{V_n\}$ of neighborhoods of 0, given by $V_n = V^{-1}([0, 1/n])$, satisfies $\mathcal{S} = e(\mathcal{V})$.*

PROOF. We summarize the first few steps which are given by Lefschetz [6, chapter X]. Let $D(r) = \{(x, y) \mid x^2 + y^2 \leqq r^2\}$. For r_0 sufficiently small we find within $D(r_0)$ that the locus $xX + yY = 0$ ($\dot{r} = 0$) is a finite number of curves, intersecting only at the origin. Furthermore, no trajectory segment of (3.1) within $D(r_0) - \{0\}$ is tan-

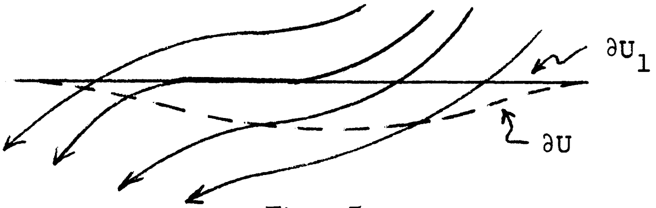

Figure 7

gent to one of these curves. By a further shrinking of r_0, we can guarantee that any trajectory of the elliptic type (i.e., the complete trajectory is in $D(r_0)$) lies in one of a finite number of simple elliptic sectors, each trajectory equivalent with that in Figure 6. We also must make r_0 small enough that each elliptic sector intersects the boundary of $D(r_0)$. Although Lefschetz does not mention this step, it is necessary in order to ascertain that points in the "fan" regions (i.e., where trajectories limit to 0 in one direction and leave $D(r_0)$ in the other) do not accumulate at 0 except in the obvious way (example: [2, Figure 1]). If r_0 is slightly shrunk one final time, then we guarantee that each elliptic sector is bordered on either side by a fan region. More precisely, if σ is an elliptic trajectory which bounds an elliptic sector in $D(r_0)$ (r_0 as finally chosen), then σ is internally tangent to $\partial D(r_0)$ in at least one point. In the case of only one, then because of the last adjustment of r_0, the elliptic sector in $D(r_0)$ is surrounded by trajectories which, because they were elliptic for previous r_0, now comprise two fans, one on either side of the elliptic sector. In the case of more than one internal tangency of σ, we still find the elliptic sector surrounded as before, but between successive internal tangencies there are sectors of hyperbolic trajectory segments (those which leave $D(r_0)$ in both directions) with the entire sector separated from 0. Note that successive internal tangents along a trajectory in a fan can also trap a hyperbolic sector which is separated from 0 by the fan. Denote by U_1 the neighborhood of 0 obtained by deleting from $D(r_0)$ all these hyperbolic sectors which are separated from 0 by an elliptic sector or a fan. Now each elliptic sector is bordered in U_1 only by positively and negatively attracted fans, and there are no internal tangencies of fan trajectories with ∂U_1, except along a trajectory segment next to its point of entry to (or exit from) U_1. It is geometrically clear that we can shave off part of a fan near such a prolonged tangency next to an entry (or exit) point (Figure 7). If U denotes the result of shaving U_1 in this manner, then U retains the desired properties of U_1 and additionally has no internal tangencies with ∂U within fan sectors.

It is now clear that any half-trajectory contained in U has its appropriate α- or ω-limit at 0. Hence U, or any compact subneighborhood, is a neighborhood of solitude for 0.

We now construct the function V, first on elliptic sectors, then on fans and hyperbolic sectors, finally obtaining V as a convex combination of the pieces. We have seen that each elliptic sector is equivalent by a trajectory preserving homeomorphism to one as presented in Figure 6. By more careful argument, given by the methods in [8], it is seen that the trajectory equivalence homeomorphism is differentiable along trajectories. If we use such an equivalence to pull back the distance from 0 on Figure 6, we obtain a function V on the elliptic sector which has a unique relative maximum on each trajectory and satisfies the conditions of the theorem. We will assume that V is scaled to have maximum value 1 on the elliptic sector.

Let A denote a fan sector, and for definiteness assume that A is positively attracted to 0. Let α denote the arc of $\partial U \cap A$. If $\limsup \phi(\alpha, t)$, taken as $t \to \infty$, is 0, then $V(x) = (t+1)^{-1}$ for $x \in \phi(\alpha, t)$ is a function of the desired characteristics. But if $y \neq 0$ and $y \in \limsup \phi(\alpha, t)$, it is easily seen that y must be a boundary point between A and an elliptic sector which is separated from ∂U by A. We have arranged things so that this cannot happen.

Now each fan has nonempty interior, and elliptic sectors only border fans. Where such bordering occurs, we paste the pieces of V together by a convex combination with the elliptic sectors. This is clearly possible and satisfactory toward achieving our ends.

It remains to consider a hyperbolic sector H. From our previous constructions, we know that H cannot be separated from 0, so the boundary of H consists of 0, an arc along ∂U and two half-trajectories, say σ_1 and σ_2 with one positively attracted to 0, and the other negatively attracted. Hence V is already defined on $\sigma_1 \cup \sigma_2$, with $V \equiv 1$ on $(\sigma_1 \cup \sigma_2) \cap \partial U$, and with V tending to 0 along σ_1 or σ_2 near 0. Set $V \equiv 1$ on $H \cap \partial U$ and extend smoothly to $V : H \cup \sigma_1 \cup \sigma_2 \to (0, 1]$.

It is now clear from the construction that if $V_n = \{x \mid V(x) \leqq 1/n\}$ and $\mathcal{V} = \{V_n\}$, then $e(\mathcal{V}) = b(H) - \{0\} \subset \mathcal{S}$. But from Theorem 1.1, $\mathcal{S} \subset e(\mathcal{V})$. The proof is complete.

4. **Conclusion and query.** Finally we use a generalized Lyapunov point of view to interpret some results above and to state a query. If V is a generalized Lyapunov function for an invariant set I in solitude U, let $\mathcal{V} = \{V_n\}$ where $V_n = \{x \in U \mid V_n(x) \leqq 1/n\}$. Re-examination of the proof of (1.6) shows that if σ is a trajectory segment in $U - \mathcal{S}$ then there is a generalized Lyapunov function V such that $\sigma \subset U$

$-\, e(\mathcal{V})$. Of course the choice of V depends on σ. The author would like to know when V can be chosen to depend only on I and U. If this were the case, then one could proceed to further investigation of separatrix structure using generalized Lyapunov techniques analogous to those used for isolated invariant sets [9].

References

1. R. Abraman and J. Robbin, *Transversal mappings and flows*, Benjamin: 1967.

2. J. Auslander, N. P. Bhatia, and P. Seibert, *Attractors in dynamical systems*, Bol. Soc. Mat. Mex. **9** (1964), 55–66.

3. R. Elderkin, *Separatrices and solitary periodic solutions*, to appear in J. Diff. Eq.

4. ———, *Separatrix structure for elliptic flows*, Amer. J. of Math. **97** (1975), 221-247.

5. R. Elderkin and F. W. Wilson, *Solitary invariant sets*, Dynamical Systems, An International Symposium, Vol. 2 (J. P. LaSalle, ed.), Academic Press, (1976) 13-18.

6. S. Lefschetz, *Differential equations, geometric theory*, Interscience: 1957.

7. G. Sansoni and R. Conti, *Non-linear differential equations*, Macmillan: 1964.

8. F. W. Wilson, *Elliptic flows are trajectory equivalent*, Amer. J. of Math. **92** (1970), 779-792.

9. F. W. Wilson and J. A. Yorke, *Lyapunov functions and isolating blocks*, J. Diff. Eq. 13 (1973), 106-123.

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