ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 3, Summer 1977

PERIODIC SOLUTIONS OF AUTONOMOUS DIFFERENTIAL EQUATIONS IN HIGHER DIMENSIONAL SPACES

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1. **Introduction.** The Poincaré-Bendixson theorem supplies a powerful technique for finding periodic orbits of dynamical systems in the plane. As is well known, the theorem does not apply in higher dimensional spaces since it depends on the Jordan curve theorem.

Even though the Poincaré-Bendixson theorem does not apply to higher dimensional systems, some higher-dimensional analogies to results for the plane are hard to disbelieve. For example, suppose a dynamical system in R^n, $n \geqq 3$, leaves the n-disk D^n invariant in the positive direction and has a single rest point which has an unstable manifold of dimension at least two. It seems very likely, at first, that such a system must have a periodic orbit.

Such results are not valid. The method which Schweitzer [11] used to disprove the Seifert conjecture also applies to this and similar situations. Schweitzer's method will be outlined in § 3.

There must be additional hypotheses on the vector field in order to insure the existence of periodic orbits. Rauch [9] and others have used a technique of finding a positively invariant solid torus and showing that a particular flow "twists" around the torus. This "twisting" hypothesis allows one to construct a "first return" map and apply the Brouwer fixed-point theorem to find a periodic orbit.

The process of finding a positively invariant solid torus can be very difficult. For equations that model ecological systems, in particular, it is much more feasible to find a positively invariant disk. In § 2 we state and prove a theorem about systems which have positively invariant disks and satisfy a "twisting" condition and in § 4, we apply the theorem of § 2 to an ecological model of three competing species.

Throughout this paper, R is the set of real numbers, R^n is Euclidean n-space, $C^k(M, N)$ is the set of k-times continuously differentiable maps from M to N, and $C^k(M) = C^k(M, R)$.

I would like to thank Z. Nitecki and J. A. Yorke for a conversation at the 1976 NSF-CBMS conference on topological methods in differential equations, which clarified Schweitzer's method for me. Professor Nitecki must also be credited with the first remark following the statement of the theorem in § 2. I would also like to thank my advisor, Paul Waltman, for suggesting this problem and for his encouragement and patience.

2. **Sufficient conditions for the existence of a periodic orbit.** We will denote a flow on the domain D by a map

$$A : (R \times D) \to D \qquad (2.1)$$
$$A : (t, x) \mapsto A^t(x)$$

where $A^t(A^s(x)) = A^{t+s}(x)$ and $A^0(x) = x$. We can regard the solution of an autonomous differential equation $x' = f(x)$ where $f \in C^1(D, R^n)$ and D is a compact region in R^n as the restriction of a flow to a subdomain of $R \times D$ [**6**, p. 19]. A region $M \subseteq D$ is *positively* invariant under the flow A, or the differential equation $x' = f(x)$, if $x_0 \in M$ implies that $x(t) = A^t(x_0)$ is a solution of the differential equation, which is contained in M for $t \geqq 0$.

Suppose that a differential equation $x' = f(x)$ has 0 as its only critical point in D^n and that this critical point is hyperbolic; that is, the Jacobian $f_x(0)$ has no eigenvalues with vanishing real part. We apply the topological index (also known as the Poincaré-Hopf index) [**2**, **4**] to this situation. If D^n is positively invariant under $x' = f(x)$, then f never points radially outward along the boundary of D^n, and the index of f relative to the boundary of D^n is $(-1)^n$. Now if m is the number of eigenvalues of $f_x(0)$ which have negative real part, the index of the critical point is $(-1)^m$, so $(-1)^m = (-1)^n$. In particular, if $n = 3$, m must be 1 or 3. If $m = 3$, 0 is an attractor. The theorem of this section concerns the general case $m = n - 2$.

Consider the system of differential equations

$$x'_i = f_i(x_1, \cdots, x_n) \qquad (2.2)$$
$$f_i \in C^1(M), \quad 1 \leqq i \leqq n.$$

This system can be transformed into cylindrical coordinates by making the substitution

$$x_1 = y_2 \cos(y_1)$$
$$x_2 = y_2 \sin(y_1) \qquad (2.3)$$
$$x_i = y_i \quad \text{for } 3 \leqq i \leqq n.$$

In making this transformation it is helpful to note that

$$y'_1 = (\cos(y_1)x'_2 - \sin(y_1)x'_1)/y_2 \qquad (2.4)$$
$$y'_2 = \cos(y_1)x'_1 + \sin(y_1)x'_2.$$

It should be noted here that the hypotheses of the theorem will require that $f_i(0, 0, x_3, \cdots, x_n) = 0$ for $i = 1, 2$. If $y_2 = 0$, $\cos(y_1)x'_2 - \sin(y_1)x'_1 = \cos(y_1)f_2(0, 0, y_1, \cdots, y_n) - \sin(y_1)f_1(0, 0, y_1, \cdots, y_n) = 0$.

Thus if (2.2) transforms to

$$(2.5) \qquad y'_i = g_i(y_1, \cdots, y_n), \quad 1 \leqq i \leqq n,$$

g_1 can be defined when $y_2 = 0$ by taking limits.

Let $c : y \mapsto x$ be the transformation given by (2.3) and let $M^c = \{y \in R^n \mid c(y) \in M\}$. Since $\{y \in M^c \mid 0 \leqq y_1 \leqq 2\pi\}$ is compact if M is and the functions g_i are 2π-periodic in y_1, each g_i attains its maximum and minimum in M^c.

The main result may now be stated.

THEOREM. *Suppose*

(i) *There is a compact neighborhood $M \subset R^n$ which is star-shaped from* 0 *and is positively invariant under* (2.2),

(ii) 0 *is the only critical point of* (2.2) *in M, this critical point is hyperbolic, and exactly two eigenvalues of $f_x(0)$ have positive real parts,*

(iii) *The stable manifold of* 0 *contains $\{x \in R^n \mid x_1 = x_2 = 0\} \cap M$,*

(iv) *System* (2.2) *can be transformed by* (2.3) *to* (2.5) *where $g_i \in C^1(M^c)$ and $g_1 \neq 0$ on M^c.*

Then (2.2) *has a nonconstant periodic orbit in M.*

REMARKS. Hypothesis (iii) can be replaced by the milder requirement that the stable manifold of 0 is contained in $\{x \in R^n \mid x_1 = x_2 = 0\}$ and the set $\{x \in R^n \mid x_1 = x_2 = 0\} \cap M$ is positively invariant under (2.2).

The requirement that M be star-shaped from 0 will be taken to mean that 0 is in the interior of M and that every ray from 0 intersects the boundary of M in exactly one point.

PROOF. Without loss of generality, assume $g_1 > 0$ on M^c. Then there exists a $k > 0$ such that $g_1 \geqq k > 0$ on M^c. Let B be the flow generated by (2.5). For every $y \in M^c$, there exists a unique $t(y) > 0$ such that $(B^{t(y)}(y))_1 = y_1 + 2\pi$. Since B is a C^1 flow, it can be shown that $t(y) \in C^1(M^c)$ by using the implicit function theorem.

Let $\Gamma_\varphi = \{c(y) \mid y \in M^c, y_1 = \varphi\}$. Γ_φ is the intersection of an $n - 1$ dimensional subspace of R^n with M, and since M is star-shaped from 0, Γ_φ is also star-shaped from 0. Let A be the flow which corresponds to (2.2). Define $F : \Gamma_\varphi \to \Gamma_\varphi$ by $F : x \mapsto A^{t(y)}(x)$ where $y \in \{M^c, y_1 = \varphi\}$ is such that $c(y) = x$. F is well defined and $F \in c^1(\Gamma_\varphi, \Gamma_\varphi)$. Since F maps Γ_φ to itself, the map $V : x \mapsto F(x) - x$ is a tangent vector field on Γ_φ. Note that each critical point of V corresponds to a fixed point for F and a periodic orbit for (2.2).

0 is a critical point of V. We will use an index argument to show that there are other critical points. If V has a critical point along the

boundary of Γ_φ, there is nothing to prove. Otherwise the ($n-1$ dimensional) index of V with respect to the boundary of Γ_φ is $(-1)^{n-1}$, since V never points radially outward along the boundary of Γ_φ. We will show that the index of 0 is $(-1)^{n-2}$. This implies the existence of other critical points of V.

Let $t_0 = t(y)$, where $y \in \{M^c \mid y_1 = \varphi\}$ is such that $c(y) = 0$. If $V_x(0)$ and $F_x(0)$ are the Jacobians of F and V at 0, $V_x(0) = F_x(0) - I$, $F_x(0) = \partial A/\partial x(t_0, 0) + \partial A/\partial t(t_0, 0) \cdot \partial t/\partial x(0) = \partial A/\partial t(t_0, 0)$ since $\partial A/\partial t(s, 0) \equiv 0$.

Now $\partial A/\partial x(t, x)$ satisfies the variational equations [1, p. 25], that is, the matrix initial value problem $d/dt\ \partial A/\partial x(t, x) = f_x(A(t, x)) \cdot \partial A/\partial x(t, x)$, $\partial A/\partial x(0, x) = I$. Since $x = 0$ is a critical point of (2.2), $f_x(A(t, 0)) = f_x(0) \equiv J$. Therefore $\partial A/\partial x(t, 0) = e^{Jt}$ and $F_x(0) = \exp(Jt_0)$.

The next step of the proof is to show that $t_0 = 2\pi/d$ where d is the imaginary part of either of the eigenvalues of J which have positive real parts. Since $M \cap \{x_1 = x_2 = 0\}$ is invariant under (2.2) by hypothesis (iii), $f_i(0, 0, x_3, \cdots, x_n) = 0$ for $i = 1, 2$, and $\partial f_1/\partial x_j(0) = \partial f_2/\partial x_j(0) = 0$ for $3 \leqq j \leqq n$. Let us adopt the notation $f_{ij} = \partial f_i/\partial x_j(0)$. Then J has the form

$$J = \begin{pmatrix} J_0 & 0 \\ J_1 & J_2 \end{pmatrix} \quad \text{where} \quad J_0 = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

0 is a $2 \times (n-2)$ zero matrix, and J_1 and J_2 are $(n-2) \times 2$ and $(n-2) \times (n-2)$ respectively. The eigenvalues of J are the eigenvalues of J_0 together with the eigenvalues of J_2.

By taking limits in (2.4) it follows that $g_1(y_1, 0, 0, \cdots, 0) = f_{21} \cos^2(y_1) + (f_{22} - f_{11}) \cos(y_1) \sin(y_1) - f_{12} \sin^2(y_1) = \cos^2(y_1)(f_{21} + (f_{22} - f_{11}) \tan(y_1) - f_{12} \tan^2(y_1))$ and $g_i(y_1, 0, \cdots, 0) = 0$ for $i = 2, \cdots, n$.

Since $g_1(y_1, 0, \cdots, 0) \neq 0$, the discriminant $(f_{22} - f_{11})^2 + 4f_{21}f_{12} < 0$. This condition forces the eigenvalues of J_0 to be the complex conjugate values $(f_{11} + f_{22})/2 \pm id$ where $d = (-4f_{12}f_{21} - (f_{11} - f_{22})^2)^{1/2}/2$.

If $y(t)$ is a solution of (2.5) which satisfies $y(0) = 0$ then $y_i(t) = 0$ for $i = 2, \cdots, n$ while $y_1'(t) = f_{21} \cos^2(y_1(t)) + (f_{22} - f_{11}) \cos(y_1(t)) \sin(y_1(t)) - f_{12} \sin^2(y_1(t))$. This last equation can be explicitly solved by integration, and its solution satisfies $\tan(y_1(t)) = K_1 \tan(td + K_2) + K_3$ for appropriate constants K_i, $1 \leqq i \leqq 3$. Since $y_1(0) = 0$, $y_1(2\pi/d) = 2\pi$ and $t_0 = 2\pi/d$.

If λ is one of the eigenvalues of J with positive real part, λ is an eigenvalue of J_0 since an eigenvector V of J corresponding to an eigenvalue of J_2 must satisfy $v_1 = v_2 = 0$ and lie in the (complex) stable manifold of (2.2). Since $\mathrm{Im}(\lambda t(0)) = 2\pi$, $\exp(\lambda t_0) - 1$ is a positive real eigenvalue of $V_x(0)$. We can choose φ so that a corresponding eigenvector of $V_x(0)$ is tangent to Γ_φ.

If μ is an eigenvector of J_2, $\exp(\mu t_0) - 1$ is an eigenvector of $V_x(0)$ with negative real part and the corresponding eigenvector is tangent to Γ_φ, since Γ_φ contains the stable manifold of (2.2) and hence the stable manifold of the linearization of (2.2).

Therefore the index of 0 is $(-1)^{n-2}$ and the proof is complete.

Example. A system in R^3 which satisfies the hypothesis of the theorem is

$$\begin{aligned} x_1' &= x_1 - 5x_2 - x_1{}^3 - x_1x_2{}^2 \\ x_2' &= x_1 + x_2 - x_1{}^2x_2 - x_2{}^3 \\ x_3' &= -(1 + x_1{}^2x_2{}^2)x_3. \end{aligned} \tag{2.6}$$

In verifying the hypotheses of the theorem it is useful to observe that the condition $g_1 \neq 0$ implies that there are no critical points x in M which do not satisfy $x_1 = x_2 = 0$. If hypothesis (iii) holds, as it clearly does in this example, there are no critical points $x \in M$ other than 0 with $x_1 = x_2 = 0$.

The system (2.6) transforms to

$$\begin{aligned} y_1' &= \cos^2(y_1) + 5\sin^2(y_1) \\ y_2' &= y_2(1 - 4\sin(y_1)\cos(y_1)) - y_2{}^3 \\ y_3' &= -(1 + y_2{}^4\sin^2 y_1 \cos^2 y_1)y_3. \end{aligned}$$

The region $M = \{x \mid \max|x_i| \leqq (6)^{1/2}\}$ is positively invariant under (2.6) and the eigenvalues of the linearization of (2.6) are -1 and $1 \pm (5)^{1/2}i$.

The theorem implies the existence of a nonconstant periodic orbit of (2.6). In fact, since the expression for y_1' does not depend on y_2 or y_3, the proof of the theorem implies that the period is $2\pi/(5)^{1/2}$.

Since the plane $x_3 = 0$ is invariant, a periodic solution of (2.6) could have been deduced from the Poincaré-Bendixson theorem. A more serious example is presented in § 4.

Remark. It is not hard to generalize the theorem to the situation in which g_1 is nonnegative in M^c but $g_1 = 0$ at a set of isolated points (not including 0) in M^c. The theorem is not true in general if $g_1 = 0$ is allowed on some set which is invariant under (2.2). In particular, those sets could be exceptional minimal sets as discussed in the next section.

3. **A counterexample to a more general conjecture.** If $n = 2$, the theorem of section 2 follows from the Poincaré-Bendixson theorem, and hypothesis (iv) is not required. While the Poincaré-Bendixson theorem is only valid in R^2, we might nevertheless believe that hypothesis (iv) could be deleted even for $n > 2$. This is not the case.

To find a counterexample, we first construct a system such as

$$x_1' = x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$x_2' = -x_1 + x_2(1 - x_1^2 - x_2^2)$$
$$x_3' = -x_3$$

which satisfies all of the hypotheses of the theorem and has a *unique* nonconstant periodic orbit [**6**, p. 356]. Uniqueness is only for convenience. The generalization is false if we can perturb such a system so as to "break" the periodic orbit, add no new periodic orbits or critical points, and maintain the hypotheses of the generalized conjecture.

The construction of such perturbations is given in an important paper by Schweitzer [**11**]. Schweitzer's paper provides counterexamples to the Seifert conjecture that every non-vanishing vector field on the three-sphere S^3 has a periodic orbit. This technique also disproves other possible higher-dimensional analogies of the Poincaré-Bendixson theorem. A brief outline of Schweitzer's method of "breaking" an isolated periodic orbit follows.

Let N be the two-torus with a small disk removed, $N = T^2 - D^2$, and let $\bar{N}$ be its closure. $\bar{N}$ can be deformed to the highway overpass pictured in figure 1. Clearly $\bar{N}$ can be embedded as a transversal to a vector field on any manifold of dimension at least three.

We now follow $\bar{N}$ along the integral curves of the vector field for some short time interval. As in the short-tube theorem [**6**, p. 333], if the time interval is chosen short enough, we have embedded a copy of $\bar{N} \times [-1, 1]$ in the manifold.

Recall Denjoy's [3, 7, pp. 47–49, **11** pp. 396–399] well known C^1 vector field on T^2 which has an exceptional minimal set C. Let the small disk we removed from T^2 to define N be contained in $T^2 - C$.

Schweitzer's construction changes the vector field on a compact subset of the embedded copy of $N \times (-1, 1)$ so as to accomplish the following three things. (1) The new flow will have minimal sets which are copies of C at $N \times \{1/2\}$ and $N \times \{-1/2\}$. (2) Every orbit in $\bar{N} \times [-1, 1]$ will either be asymptotic to one of these copies of C or it will intersect the boundary of $\bar{N} \times [-1, 1]$ at $(n, -1)$ and $(n, 1)$. (3) If the periodic orbit had contained the points $(n, -1)$ and $(n, 1)$ on the boundary of $\bar{N} \times [-1, 1]$, the new orbit through those points would be among those which are asymptotic to C. Because we have embedded $N \times [-1, 1]$ by following integral curves of the original system, the points $(n, -1)$ and $(n, +1)$ were on the same orbit in the original flow. Clearly then, we have broken the old periodic orbit and added no new periodic orbits.

4. **An application in ecology.** Consider the system

$$N_1' = N_1(1 - N_1 - \alpha N_2 - \beta N_3) + \epsilon$$
$$N'_2 = N_2(1 - \beta N_1 - N_2 - \alpha N_3) + \epsilon \tag{4.1}$$
$$N_3' = N_3(1 - \alpha N_1 - \beta N_2 - N_3) + \epsilon.$$

This is a model of the competition of three populations based on a model of May and Leonard [5]. The ϵ represents immigration of each of the populations into the arena of competition. The model is somewhat artificial because of its symmetry, but one can expect similar behavior for perturbations of the system. We are interested here in the case $0 < \beta < 1 < \alpha, \alpha + \beta > 2$, and $0 < \epsilon \ll 1$.

The following elementary lemma establishes the existence of an invariant disk for (4.1).

LEMMA. *There exist $\delta, B > 0$ such that $M_0 = \{(N_1, N_2, N_3) \mid \delta \leqq N_1 \leqq B\}$ is positively invariant under* (4.1).

The proof (which is omitted) consists of showing, by dropping terms and using inequalities, that $N'_i \leqq 0$ if $N_i = B = (1 + (1 + 4\epsilon))^{1/2}/2$ and $N_j \geqq 0$ for $j \neq i$; and then, by a continuity argument, choosing $\delta > 0$ so that $N'_i \geqq 0$ if $N_i = \delta$ and $0 \leqq N_j \leqq B$ for $j \neq i$.

Now observe that system (4.1) is unchanged if we cyclically permute the subscripts $1 \to 2 \to 3 \to 1$. This suggests that the line $N_1 = N_2 = N_3$ is invariant. In fact, it is easy to show that the portion of this line contained in M is contained in the stable manifold of the critical point (γ, γ, γ) where $\gamma = (1 + (1 + 4\epsilon\rho))^{1/2}/2\rho$ and $\rho = 1 + \alpha + \beta$. This, by the same argument as in the example of section 2, is the only critical point of (4.1) in M.

The eigenvalues of the critical point are

$$\lambda_1 = -\sqrt{1 + 4\epsilon\rho} < 0$$
$$\lambda_{2,3} = 1 - \gamma(1 + \rho) + (\alpha\gamma + \beta\gamma)/2 \pm i\sqrt{3}(\beta\gamma - \alpha\gamma)/2.$$

It can be shown that $\mathrm{Re}(\lambda_{2,3}) > 0$ if

$$\rho_1 = (1 - 3\epsilon - \sqrt{1 - 12\epsilon})/\epsilon < \rho < (1 - 3\epsilon + \sqrt{1 - 12\epsilon})/\epsilon = \rho_2.$$

Now $\rho_1 \to 3$ and $\rho_2 \to \infty$ as $\epsilon \to 0$, so for $\alpha + \beta > 2$ and ϵ sufficiently small, the critical point will have one negative real eigenvalue and two eigenvalues with positive real part. Assume that α, β, and ϵ have been so chosen.

Before the system is transformed to cylindrical coordinates, the affine transformation

$$x_1 = (-N_1 + N_3)/\sqrt{2}$$
$$x_2 = (-N_1 + 2N_2 - N_3)/\sqrt{6}$$
$$x_3 = (N_1 + N_2 + N_3)/\sqrt{3} - \sqrt{3}\gamma$$

is applied.

This transformation translates the critical point (γ, γ, γ) to the origin and moves the stable manifold $N_1 = N_2 = N_3$ to the line $x_1 = x_2 = 0$.

Since this transformation is affine it maps the convex region M_0 to a convex region M in the coordinate system. It is also easy to verify that eigenvalues of the critical point $x = 0$ are the same as those of the critical point of the original system. Therefore the new system satisfies hypotheses i, ii, and iii of the theorem.

To verify hypothesis iv of the theorem it may be useful to note that the cylindrical coordinates (2.3) correspond to

$$y_1 = \arctan(x_2/x_1)$$
$$y_2 = \sqrt{x_1^2 + x_2^2}$$
$$y_3 = x_3.$$

The entire system can be written in this coordinate system, but only the expression for y_1 is needed.

$$y_1' = \sqrt{3}/2 \, \frac{\left[\begin{array}{l}(\beta - 1)(N_1^2N_2 + N_2^2N_3 + N_3^2N_1) \\ + (1 - \alpha)(N_1N_2^2 + N_2N_3^2 + N_3N_1^2) \\ + 3(\alpha - \beta)\, N_1N_2N_3\end{array}\right]}{N_1^2 + N_2^2 + N_3^2 - N_1N_2 - N_2N_3 - N_3N_1} \tag{4.2}$$

$$\begin{aligned} &= 2(\beta - \alpha)\gamma + 2/\sqrt{3}(\beta - \alpha)y_3 \\ &\quad + \sqrt{6}(\beta - \alpha)y_2 \sin y_1 \cos^2 y_1 \\ &\quad + \sqrt{2}(\alpha + \beta - 2)y_2 \sin^2 y_1 \cos y_1 \\ &\quad + \sqrt{2}/\sqrt{3}(\alpha - \beta)y_2 \sin^3 y_1 \\ &\quad - \sqrt{2}/3\,(\alpha + \beta - 2)y_2 \cos^3 y_1. \end{aligned} \tag{4.3}$$

To show that $y_1' \neq 0$ in M^c, it suffices to show that (4.2) is nonzero for $(N_1, N_2, N_3) \in M_0$.

Since $N_1{}^2 + N_2{}^2 + N_3{}^2 - N_1N_2 - N_2N_3 - N_3N_1 = 3/2(x_2{}^2 + x_3{}^2)$, the denominator of (4.2) is positive except on the line $x_1 = x_2 = 0$, which corresponds to $N_1 = N_2 = N_3$. The numerator U of (4.2) is nonpositive on each of the faces of the positive octant, and the directional derivative $(\partial/\partial N_1 + \partial/\partial N_2 + \partial/\partial N_3)U = (\beta - \alpha)(N_1{}^2 + N_2{}^2 + N_3{}^2 - N_1N_2 - N_2N_3 - N_3N_1) < 0$ unless $N_1 = N_2 = N_3$. If $N_1 = N_2 = N_3$ then $y_2 = 0$. Since $y_3 > -\sqrt{3}y$ if $y \in M^c$, (4.3) guarantees that $y_3' < 0$ for $y \in M^c$.

The hypotheses of the theorem are satisfied and thus there is a nonconstant periodic orbit for (4.1) which is contained in M_0.

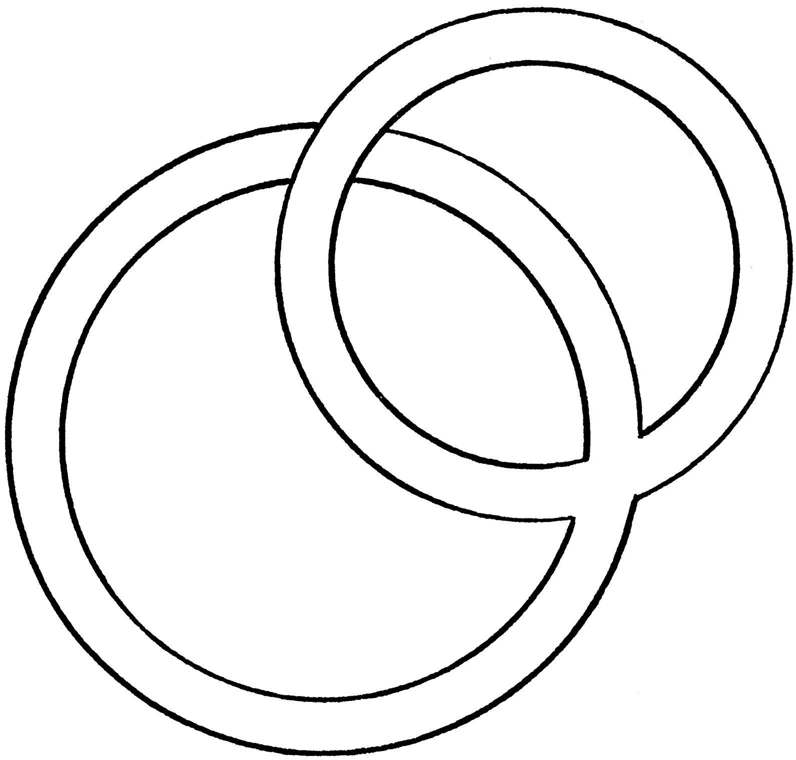

Figure 1. The highway overpass.

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