ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 3, Summer 1977

STABLE HOMOTOPY AND ORDINARY DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

JEAN MAWHIN

1. **Introduction.** This paper is a continuation of [1] where coincidence degree arguments have been used to give fairly general existence theorems for nonlinear boundary value problems relative to ordinary differential equations. In contrast with [1] where the case of nonlinear perturbations of Fredholm mappings of index zero has been treated, we consider here the case where this index is positive.

Following Nirenberg [5] we use stable homotopy arguments to get a continuation theorem which was announced in [4] and is given here with complete proof for reader's convenience (Section 2). This continuation result leads in Section 3 to a fairly general existence theorem for boundary value problems. An interesting specialization and an example are given in Section 4.

2. **A continuation theorem for some nonlinear perturbations of Fredholm mappings with non-negative index.** Let X and Z be real normed spaces and $L : \operatorname{dom} L \subset X \to Z$ a linear mapping such that $\operatorname{Im} L$ is closed and

$$q = \operatorname{codim} \operatorname{Im} L \leqq \dim \ker L = p.$$

We shall call L a *Fredholm mapping of index* $p - q$. Let $R > 0$ and $N : \bar{B}(R) \subset X \to Z$ *be L-compact* on the closed ball $\bar{B}(R)$ of center 0 and radius R. That means [4] that if $P : X \to X$ and $Q : Z \to Z$ denote continuous projectors such that the sequence

$$X \xrightarrow{P} \operatorname{dom} L \xrightarrow{L} Z \xrightarrow{Q} Z$$

is exact, and if

$$K_{P,Q} = (L \mid \operatorname{dom} L \cap \ker P)^{-1}(I - Q),$$

then QN is continuous on $\bar{B}(R)$, $QN(\bar{B}(R))$ is bounded and $K_{P,Q}N : \bar{B}(R) \to X$ is compact. It is known [4] that those conditions are independent of the choice of P and Q. If now

$$\Gamma : R^p \to \ker L, \Gamma' : \operatorname{Im} Q \to R^q$$

are isomorphisms, we shall define the mapping ν by

$$\nu(u) = \frac{\Gamma' QN\Gamma(Ru)}{|\Gamma' QN\Gamma(Ru)|} \tag{2.1}$$

for all points u where $QN\Gamma(Ru) \neq 0$. We shall denote by S^r the unit sphere in R^{r+1}.

THEOREM 2.1. *Assume that the following conditions hold.*

1. *For each* $\lambda \in]0, 1[$ *and each* $x \in \operatorname{dom} L \cap \partial B(R)$, *one has*

$$Lx \neq \lambda Nx. \tag{2.2}$$

2. *For each* $x \in \ker L \cap \partial B(R)$, $Nx \notin \operatorname{Im} L$, *i.e.*, $QNx \neq 0$.
3. *The mapping* $\nu : S^{p-1} \rightarrow S^{q-1}$ *defined by* (2.1) *has nontrivial stable homotopy.*

Then the equation

$$Lx = Nx \tag{2.3}$$

has at least one solution $x \in \operatorname{dom} L \cap \bar{B}(R)$.

PROOF. If there exists $x \in \operatorname{dom} L \cap \partial B(R)$ such that (2.3) holds, the theorem is proved. Hence one can assume that (2.2) holds for all $\lambda \in]0, 1]$. As shown in [3], for all linear one-to-one $J : \operatorname{Im} Q \rightarrow \ker L$ and all $\lambda \in]0, 1]$, (2.2) is equivalent to

$$x = M(x, \lambda)$$

with

$$M(x, \lambda) = Px + (JQ + \lambda K_{P,Q})Nx,$$

and

$$x = M(x, 0)$$

is equivalent to

$$x \in \ker L, \; QNx = 0.$$

Therefore, by conditions 1 and 2,

$$x \neq M(x, \lambda)$$

for any $\lambda \in [0, 1]$ and $x \in \partial B(R)$. Also, $M : \bar{B}(R) \times [0, 1] \rightarrow X$ is clearly compact and hence its restriction to $\partial B(R) \times [0, 1]$ is a permissible deformation in the sense of Nirenberg ([**6**], p. 128). Consequently $I - M(\cdot, \lambda)$ will have a zero in $B(R)$ for any $\lambda \in [0, 1]$ if the restriction of $I - M(\cdot, 0)$ to $\partial B(R)$ is essential, i.e., if any extension of this map to $\bar{B}(R)$ in the class of compact perturbations of the identity has a zero [**2**, **6**]. But the mapping $F : \partial B(R) \rightarrow X$ defined by

$$I - M(\cdot, 0) = I - P - JQN$$

restricted to $\partial B(R)$ is clearly homotopic to the mapping $F_0 : \partial B(R) \to X$ defined by

$$F_0 = I - P - JQNP$$

which is of the type considered in Proposition 4.1.1 of [6] with

$$X_0 = \ker P, W = \ker L, \quad \Phi = -JQN.$$

Therefore, using this Proposition, F_0, and hence F, will be essential if and only if the map ν defined in (2.1) has nontrivial stable homotopy (see [6], p. 29, for a definition). The result then follows using assumption 3.

REMARK. When $p = q$, assumption 3 is equivalent to requiring that the degree of ν is nonzero, i.e., that the Brouwer degree

$$d_B[JQN \mid \ker L, B(R), 0]$$

is nonzero.

3. A continuation theorem for ordinary differential equations with nonlinear boundary conditions. Let $I = [0, 1]$ and

$$f : I \times R^n \times \cdots \times R^n \to R^n$$

$$(t, x^1, x^2, \cdots, x^m) \mapsto f(t, x^1, x^2, \cdots, x^m)$$

be continuous. Let X be the (Banach) space $C^{m-1}(I, R^n)$ of mappings $x : I \to R^n$ which are continuously differentiable up to the order $m - 1$, with the norm (we use the Euclidian norm in R^n)

$$|x| = \max\{\max_{t \in I}|x(t)|, \cdots, \max_{t \in I} |x^{(m-1)}(t)|\},$$

and let $g : X \to R^q$ be continuous and such that it takes bounded sets into bounded sets. We shall be interested in the nonlinear boundary value problem

$$(3.1) \qquad \begin{aligned} x^{(m)} &= f(t, x, x', \cdots, x^{(m-1)}) \\ g(x) &= 0. \end{aligned}$$

If we denote by Z the (Banach) space

$$Z = C(I, R^n) \times R^q$$

with $C(I, R^n)$ the (Banach) space of continuous mappings $x : I \to R^n$ with the usual supremum norm $| \cdot |_0$, and if we denote by dom L the subspace of X of m-times continuously differentiable mappings $x : I \to R^n$, it is clear that (3.1) is equivalent to the operator equation in dom L

$$(3.2) \qquad Lx = Nx$$

when we define L and N respectively by

$$(3.3) \qquad \begin{aligned} &L : \operatorname{dom} L \subset X \to Z, x \mapsto (x^{(m)}, 0) \\ &N : X \to Z, x \mapsto (f(t, x, x', \cdots, x^{(m-1)}), g(x)). \end{aligned}$$

Now it is easy to check that

$$\ker L = \{x \in X : x(t) = a_0 + (t/1!)a_1 + (t^2/2!)a_2 + \cdots + (t^{m-1}/(m-1)!)a_{m-1}, a_0 \in R^n, \cdots, a_{m-1} \in 1$$

so that

$$\dim \ker L = mn$$

and

$$\operatorname{Im} L = C(I, R^n) \times \{0\}.$$

Thus Im L is closed and

$$\operatorname{codim} \operatorname{Im} L = q.$$

Therefore L is a Fredholm mapping of index mn $- q$ and also, Arzela-Ascoli's theorem, L has compact right inverses. Thus N is compact on bounded sets of X. We shall denote by $\Gamma : R^{mn} \to$ ke the isomorphism

$$(a_0, a_1, \cdots, a_{m-1}) \mapsto \xi(\cdot; a_0, a_1, \cdots, a_{m-1})$$

where

$$\xi(t; a_0, \cdots, a_{m-1}) = \sum_{j=0}^{m-1} (t^j/j!)a_j \qquad (t \in R),$$

and we shall define the mapping γ by

$$(3.4) \qquad \gamma(u) = \frac{g\Gamma(Ru)}{|g\Gamma(Ru)|},$$

for all points where $g\Gamma(Ru) \neq 0$.

We then have the following continuation theorem.

THEOREM 3.1. *Assume that the following conditions hold.*

a. *There exist $M > 0$ such that, for all $(t, x^1, \cdots, x^m) \in I \times R^n \cdots \times R^n$, one has*

$$|f(t, x^1, \cdots, x^m)| \leqq M.$$

b. *There exists* $R > 0$ *such that, for all* $x \in C^m(I, R^n)$ *for which*

$$g(x) = 0$$

and

$$|x^{(m)}|_0 \leqq M,$$

one has

$$|x| \neq R.$$

c. *The mapping* $\gamma : S^{mn-1} \to S^{q-1}$ *defined by* (3.4) *has nontrivial stable homotopy.*

Then the boundary value problem (3.1) *has at least one solution* x *such that* $|x| \leqq R$.

PROOF. We shall apply theorem 2.1 to the equivalent problem (3.2) with L and N defined in (3.3). Equation

$$Lx = \lambda Nx$$

for $\lambda \in [0, 1]$ is clearly equivalent to

$$x^{(m)} = \lambda f(t, x, x', \cdots, x^{(m-1)})$$
$$0 = g(x)$$

and hence, by conditions (a) and (b), assumption (1) of Theorem 2.1 is verified. Now

$$QNx = (0, g(x))$$

and hence by (b) applied to the elements of ker L, condition (2) of Theorem 2.1 holds. Now assumption (c) clearly corresponds to condition (3) of Theorem 2.1 and the proof is complete.

4. A class of nonlinear two point boundary value problems. Let $h : R^{mn} \to R^q$ be continuous and let α_{ij} $(i = 0, 1, \cdots, m-1; j = 1, \cdots, n)$ denote 0 or 1. We shall consider in this section the special case of (3.1) where

$$g(x) = h(x_1(\alpha_{01}), x_1'(\alpha_{11}), \cdots, x_1^{(m-1)}(\alpha_{m-1,1}), x_2(\alpha_{02}), \cdots, x_n^{(m-1)}(\alpha_{m-1,n})). \tag{4.1}$$

THEOREM 4.1. *Assume that conditions* (a) *of Theorem* 3.1 *as well as the following assumptions hold.*

b′. *There exists* $S > 0$ *such that each solution b of*

$$h(b) = 0$$

is such that

$$|b| < S.$$

c′. *The mapping* γ *defined by* (3.4) *with g given by* (4.1) *and*

$$R = n^{1/2}(mS + M)$$

has nontrivial stable homotopy.

Then problem (3.1) *has at least one solution.*

PROOF. We shall apply theorem 3.1. *Let* $x \in C^m(I, R^n)$ be such that

$$h(x_1(\alpha_{01}), \cdots, x_n^{(m-1)}(\alpha_{m-1,n})) = 0,$$

and

$$|x^{(m)}|_0 \leqq M.$$

Then, by assumption (b'), necessarily,

$$|x_k^{(j)}(\alpha_{jk})| < S \quad (j = 0, 1, \cdots, m-1; k = 1, \cdots, n)$$

and hence, using the relations

$$x_k^{(j-1)}(t) = x_k^{(j-1)}(\alpha_{j-1,k}) + \int_{\alpha_{j-1,k}}^{t} x_k^{(j)}(s)\, ds,$$

$$(j = 1, 2, \cdots, m; k = 1, \cdots, n),$$

one gets successively

$$|x_k^{(m-1)}|_0 < S + M,$$

$$|x_k^{(m-2)}|_0 < S + S + M,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|x_k|_0 < mS + M,$$

and hence

$$|x| < n^{1/2}(mS + M).$$

Putting $R = n^{1/2}(mS + M)$ achieves the proof.

As an example let us consider the case where $m = n = 2$, $q = 3$ and

$$h = h(x(0), x'(1)) = (x_1^2(0) + x_1'^2(1) - x_2^2(0) - x_2'^2(1) - c_1,$$

$$2(x_1(0)x_2(0) + x_1'(1)x_2'(1)) - c_2, 2(x_1'(1)x_2(0) - x_1(0)x_2'(1)) - c_3),$$

with c_1, c_2, c_3 real constants. If we write

$$w = x_1(0) + ix_1{}'(1), v = x_2(0) + ix_2{}'(1),$$

then

$$h(x(0), x'(1)) = 0 \tag{4.2}$$

can be written

$$|w|^2 - |v|^2 - c_1 = 0$$
$$2 \operatorname{Re} w\bar{v} - c_2 = 0$$
$$2 \operatorname{Im} w\bar{v} - c_3 = 0,$$

and hence each solution $(x(0), x'(1))$ of (4.2) is necessarily such that

$$|w|^2 - |v|^2 - c_1 = 0$$
$$|w|^2|v|^2 = 4^{-1}(c_2{}^2 + c_3{}^2),$$

and therefore such that

$$|v|^4 \leqq |c_1|\,|v|^2 + 4^{-1}(c_2{}^2 + c_3{}^2),$$

which implies that

$$|v|^2 < d_1{}^2$$

with $d_1{}^2$ any number strictly greater than the positive root of the equation

$$z^2 - |c_1|z - 4^{-1}(c_2{}^2 + c_3{}^2) = 0.$$

Consequently,

$$|w|^2 < |c_1| + |d_1|^2 = d_2{}^2,$$

and condition (b') of Theorem 4.1 holds with

$$S = (d_1{}^2 + d_2{}^2)^{1/2}$$

and it still holds if c_1, c_2 and c_3 are replaced by $\lambda c_1, \lambda c_2, \lambda c_3$ for any $\lambda \in [0, 1]$. Now, if $u = (u_1, u_2, u_3, u_4) \in S^3$, i.e., if

$$u_1{}^2 + u_2{}^2 + u_3{}^2 + u_4{}^2 = 1,$$

then, if we define Γ by

$$\Gamma(a, b, c, d) = (a, c) + t(b, d),$$
$$\gamma(u) = |g\Gamma(Ru)|^{-1}g\Gamma(Ru)$$

with

$$g\Gamma(Ru) = (R^2(u_1{}^2 + u_2{}^2 - u_3{}^2 - u_4{}^2) - c_1, 2R^2(u_1u_3 + u_2u_4) - c_2,$$
$$2R^2(u_3u_2 - u_1u_4) - c_3).$$

If

$$R > (d_1{}^2 + d_2{}^2)$$

and if

$$G(u,\lambda) = (R^2(u_1{}^2 + u_2{}^2 - u_3{}^2 - u_4{}^2) - \lambda c_1, 2R^2(u_1u_3 + u_2u_4) - \lambda c_2,$$
$$2R^2(u_3u_2 - u_1u_4) - \lambda c_3),$$

then

$$G(u, 1) = g\Gamma(Ru),$$
$$G(u,\lambda) \neq 0 \quad \text{for any} \quad u \in S^3 \text{ and } \lambda \in [0, 1],$$

which implies that $|g\Gamma(Ru)|^{-1}\, g\Gamma(Ru)$ is homotopic to the Hopf map $j : S^3 \to S^2$ defined by

$$j : (u_1, u_2, u_3, u_4) \mapsto (u_1{}^2 + u_2{}^2 - u_3{}^2 - u_4{}^2, 2(u_1u_3 + u_2u_4),$$
$$2(u_3u_2 - u_1u_4)).$$

But the suspensions $\sum^k j$ of the Hopf map ($k = 1, 2, \cdots$) are the generators of the homotopy groups $\pi_{3+k}(S^{2+k})$ which are cyclic of order two and hence j has nontrivial stable homotopy. The existence result for the example then follows from Theorem 4.1.

References

1. R. E. Gaines and J. Mawhin, *Ordinary differential equations with nonlinear boundary conditions*, J. Differential Equations, to appear.

2. A. Granas, *The theory of compact vector fields and some of its applications to topology of functional spaces* (I), Rozprawy Mat. **20** (1962), 1–93.

3. J. Mawhin, *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J. Differential Equations **12** (1972), 610–636.

4. ———, *Topology and nonlinear boundary value problems*, Dynamical Systems: An International Symposium, vol. I, Academic Press, New York, 1976, 51–82.

5. L. Nirenberg, *An application of generalized degree to a class of nonlinear problems*, Troisième Colloque CBRM d'analyse fonctionnelle, Vander, Louvain, 1971, 57–74.

6. ———, *Topics in Nonlinear Fucntional Analysis*, New York University Lecture Notes, 1973–74.

Universite De Louvain, Institut Mathematique, B-1348 Louvain-La-Neuve, Belgium