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THE TOPOLOGY ON CERTAIN SPACES OF MULTIPLIERS OF TEMPERATE DISTRIBUTIONS

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ABSTRACT. In [2] and [3] the space $\mathcal{S}'$ of temperate distributions was characterized as a union of Hilbert spaces L_{-q}, $q \in N = \{0, 1, 2, \cdots\}$. Then spaces $\mathcal{O}_q$ were defined to consist of all functions u for which mappings $f \to uf : L_{-q} \to \mathcal{S}'$ are continuous, and it was proved that $\bigcap_{q \in N} \mathcal{O}_q = \mathcal{O}_M$, where $\mathcal{O}_M$ is Schwartz' space of multipliers on $\mathcal{S}'$, see [6]. In [4], two topologies for $\mathcal{O}_q$ were suggested. It is shown in this paper that those topologies coincide and each $\mathcal{O}_q$, equipped with this topology, is a bornological, complete, and reflexive space. In addition, bounded sets in $\mathcal{O}_q$ are characterized.

1. **Notation.** For $\alpha \in N^n$ and $x \in R^n$ we write $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, and $D^\alpha = \partial^{|\alpha|}(\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n})^{-1}$. Further, $C^\infty(R^n)$ is the space of all functions which have continuous derivatives of all orders on R^n, $C_0^\infty(R^n)$ is the space of all $f \in C^\infty(R^n)$ with compact support, $L^s(R^n)$, where $1 \leqq s < +\infty$ (resp. $s = +\infty$), is the space of all functions whose s-th power is integrable on R^n (resp. functions which are bounded almost everywhere on R^n),

$$\mathcal{S} = \{f \in C^\infty(R^n) : \sup_{x \in R^n} |x^\alpha D^\beta f(x)| < \infty, \alpha, \beta \in N^n\},$$

and $\mathcal{S}'$ is the strong dual of $\mathcal{S}$. It is convenient to introduce a weight function $W(x) = (1 + \|x\|^2)^{1/2}$, where $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$.

We use the so-called Sobolev (generalized, weak) derivatives. A function g is the Sobolev derivative of order α of a locally integrable function f if, for any $\phi \in C_0^\infty(R^n)$, we have

$$\int_{R^n} g\phi \, dx = (-1)^{|\alpha|} \int_{R^n} f D^\alpha \phi \, dx.$$

In [1] we defined for each $q \in N$ a Hilbert space

$$L_q = \left\{ f : R^n \to \mathbb{C} : \|f\|_q^2 = \sum_{|\alpha| \leqq q} \int_{R^n} |W^{q-|\alpha|} D^\alpha f|^2 \, dx < +\infty \right\}$$

and denoted by L_{-q} its strong dual. The derivatives in the definition of L_q are Sobolev. But it can be shown that each $f \in L_{q+r}$, where $r = 1 + [(1/2)n]$, has continuous classical derivatives of all orders $\leqq q$. The projective limit of $\bigcap_{q \in N} L_q$ equals $\mathcal{S}$. The set of all functionals from $\bigcup_{q \in N} L_{-q}$, restricted to $\mathcal{S}$, equals $\mathcal{S}'$. By [3] the inductive

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topology of $\bigcup_{q \in N} L_{-q}$ equals the strong topology $\beta(\mathcal{S}', \mathcal{S})$ of $\mathcal{S}'$.

For any $p, q \in N$, $\mathcal{O}_{p,q}$ is the set of all functions $u : R^n \to C$ for which the mapping $f \to uf : L_p \to L_q$ is continuous. Since $\mathcal{O}_{p,q}$ is a subspace of the Banach space $\mathcal{L}_b(L_p, L_q)$, of all continuous operators from L_p into L_q, it is a normed space. In fact, it is also Banach. The norm on $\mathcal{O}_{p,q}$ is denoted by $\| \cdot \|_{p,q}$. For any $q \in N$ the space $\bigcup_{p \in N} \mathcal{O}_{p,q}$, equipped with the inductive topology T_i, is denoted by $\mathcal{O}_q$.

2. Auxiliaries.

Lemma 1. *Let X, Y be Banach spaces, Y reflexive, $Y \subset X$, and the identity mapping $id : Y \to X$ continuous. Let $x_0 \in X \backslash Y$, $y_0 \in Y$, and let V be a bounded closed convex neighborhood of 0 in Y such that $y_0 \notin V$. Then there exists a bounded closed convex neighborhood U of 0 in X such that $V \subset U$ and $x_0 \notin U$, $y_0 \notin U$.*

Proof. We show first that V is closed in X. Let a be an element of the closure of V in X. There exists a net $\{a_\iota\}_{\iota \in I} \subset V$ such that $a_\iota \to a$ in X. Since V is weakly compact in Y (Alaoglu Theorem), there exists $K \subset I$ such that $\{a_\kappa\}_{\kappa \in K}$ weakly converges in Y to $b \in V$. Therefore for all $f \in X' \subset Y'$ we have $f(a) = \lim_{\kappa \in K} f(a_\kappa) = f(b)$, and by the Hahn-Banach theorem $a = b$.

Again, by the Hahn-Banach theroem, there exist $f, g \in X'$ such that $f(x_0) > 1$, $g(y_0) > 1$, and $f(x) < 1$, $g(x) < 1$ for all $x \in V$. Since V is bounded in X, we have $d = \sup\{\|x\|_X, x \in V\} < +\infty$, and the set $U = f^{-1}((-\infty, 1]) \cap g^{-1}((-\infty, 1]) \cap \{x \in X; \ \|x\|_X \leqq d\}$ has the required properties.

Lemma 2. *For any pair $p, q \in N$, the space $W^pL_q = \{u : R^n \to C; W^{-p}u \in L_q\}$ (with the norm of an element $u \in W^pL_q$ given by $\|W^{-p}u\|_q$) is a Hilbert space. The indictive limit $\bigcup_{p \in N} W^pL_q$ equals $(\mathcal{O}_q, T_i)$.*

Proof. Fix $p, q \in N$ and put $r = 1 + [(1/2)n]$. Since $W^{-p-r} \in L_p$ we have $W^{-p-r}u \in L_q$ for any $u \in \mathcal{O}_{p,q}$. Moreover, $\|W^{-p-r}u\|_q \leqq \|W^{-p-r}\|_p \cdot \|u\|_{p,q}$. Therefore, the identity mapping $id : \mathcal{O}_{p,q} \to W^{p+r}L_q$ is continuous, which implies the continuity of

$$id : \mathcal{O}_q = \bigcup_{p \in N} \mathcal{O}_{p,q} \to \bigcup_{p \in N} W^{p+r}L_q = \bigcup_{p \in N} W^pL_q.$$

On the other hand, choose $p, q \in N$. There is a constant $A > 0$ such that $|D^\gamma W^p(x)| \leqq AW^p(x)$ for all $x \in R^n$ and all $\gamma \in N^n$, $|\gamma| \leqq q$. It was shown in [1] that there exists another constant $B > 0$ such that $\|W^{p+q-|\delta|}D^\delta f\|_\infty \leqq B\|f\|_{p+q+r}$ for all $\delta \in N^n$, $|\delta| \leqq p + q$, and $f \in L_{p+q+r}$, where $r = 1 + [(1/2)n]$.

If $u \in W^p L_q$ and $f \in L_{q+q+r}$, then

$$\|uf\|_q = \|W^{-p} u W^p f\|_q$$

$$= \left(\sum_{|\alpha| \leqq q} \int_{R^n} |W^{q-|\alpha|} D^\alpha (W^{-p} u W^p f)|^2 \, dx \right)^{1/2}$$

$$\leqq \sum_{|\alpha| \leqq q} \sum_{\beta+\gamma+\delta=\alpha} \begin{bmatrix} \alpha \\ \beta, \gamma, \delta \end{bmatrix}$$

$$\left(\int_{R^n} |W^{q-|\alpha|} D^\beta (W^{-p} u) D^\gamma W^p D^\delta f|^2 \, dx \right)^{1/2}$$

$$\leqq A \sum_{|\alpha| \leqq q} \sum_{\beta+\gamma+\delta=\alpha} \begin{bmatrix} \alpha \\ \beta, \gamma, \delta \end{bmatrix}$$

$$\left(\int_{R^n} |W^{p+q-|\alpha|} D^\beta (W^{-p} u) D^\delta f|^2 \, dx \right)^{1/2}$$

$$\leqq AB \|f\|_{p+q+r} \sum_{|\alpha| \leqq q} \sum_{\beta+\gamma+\delta=\alpha} \begin{bmatrix} \alpha \\ \beta, \gamma, \delta \end{bmatrix}$$

$$\left(\int_{R^n} |D^\beta (W^{-p} u)|^2 \, dx \right)^{1/2}$$

$$\leqq C \|f\|_{p+q+r} \cdot \|W^{-p} u\|_q$$

for a sufficiently large constant C. It implies $u \in \mathcal{O}_{p+q+r,q}$ and $\|u\|_{p+q+r,q} \leqq C \|W^{-p} u\|_q$, i.e., the identity mapping $id : W^p L_q \to \mathcal{O}_{p+q+r,q}$ is continuous and the mapping $id : \bigcup_{p \in N} W^p L_q \to \bigcup_{p \in N} \mathcal{O}_{p+q+r,q} = \bigcup_{p \in N} \mathcal{O}_{p,q} = \mathcal{O}_q$ is continuous, too.

LEMMA 3. *The space $\mathcal{O}_q$ is the strong dual of the Fréchet space $\bigcap_{p \in N} W^{-p} L_{-q}$ with the duality form*

$$(u, v) \to \langle u, v \rangle = (W^p u)(W^{-p} \bar{v}) : \bigcap_{p \in N} W^{-p} L_{-q} \times \bigcup_{p \in N} W^p L_q \to C.$$

Proof follows from [5], Chapter IV, Theorem 4.4.

LEMMA 4. *Let $B \subset \bigcap_{p \in N} W^{-p} L_{-q}$ be bounded and Δ the unit ball in L_{-q}. Then there exists $\phi \in \mathcal{S}$ such that $B \subset \phi \Delta$.*

PROOF. For each $p \in N$ there exists $C_p > 0$ such that $B \subset C_p^{-1} W^{-p} \Delta$. Let B°, resp. Δ°, be the polar of B, resp. Δ. Let $v \in C_p W^p \Delta^\circ = (C_p^{-1} W^{-p} \Delta)^\circ$. Then for each $u \in C_p^{-1} W^p \Delta$ we have $C_p W^p u \in \Delta$, $C_p^{-1} W^{-p} v \in \Delta^\circ$, and $|\langle u, v\rangle| = |(C_p W^p u)(C_p^{-1} W^{-p} \bar{v})| \leqq 1$. Since $B \subset C_p^{-1} W^{-p} \Delta$, it follows that $|\langle W^p u, W^{-p} v\rangle| = |\langle u, v\rangle| \leqq 1$ for all $u \in B$, i.e., $v \in B^\circ$ and $C_p W^p \Delta^\circ \subset B^\circ$.

Take $\psi \in C^\infty(R)$ such that $0 \leqq \psi(t) \leqq 1$ for $t \in R$, $\psi(t) = 0$ for $t < 0$ and $\psi(t) = 1$ for $t > 1$. Then

$$\lim_{d\to\infty} \sup_{h\in\Delta^\circ} \|\psi(\|\cdot\| - d) W^{-1} h\|_q$$

$$\leqq \lim_{d\to\infty} \sup_{h\in\Delta^\circ} \sum_{|\alpha|\leqq q} \sum_{\beta\leqq\alpha} \binom{\alpha}{\beta}$$

$$\left(\int_{R^n} |W^{q-|\alpha|} D^\beta h D^{\alpha-\beta}(\psi(\|x\| - d) W^{-1})|^2 \, dx \right)^{1/2}$$

$$\leqq \sum_{|\alpha|\leqq q} \sum_{\beta\leqq\alpha} \binom{\alpha}{\beta} \lim_{d\to\infty} \|D^{\alpha-\beta}(\psi(\|\cdot\| - d) W^{-1})\|_\infty = 0,$$

where $\|\cdot\|$ is the norm in R^n and $\|\cdot\|_\infty$ is the norm in $L^\infty(R^n)$. Therefore for each $p \in N$, there exists d_p such that

$$\sup_{h\in\Delta^\circ} \|\psi(\|\cdot\| - d_p) W^{-1} h\|_q \leqq C_{p+1} 2^{-1-p}.$$

Choose $\{d_p\}_{p\in N}$ so that $d_{p+1} \geqq 1 + d_p$, $p \in N$. For any $p \in N$ and any $h \in \Delta^\circ$ we have

$$\psi(\|\cdot\| - d_p) W^p h = W^{p+1} \psi(\|\cdot\| - d_p) W^{-1} h \in 2^{-1-p} C_{p+1} W^{p+1} \Delta^\circ.$$

Put $\phi^{-1}(x) = (1/2) C_1 + \sum_{p=1}^\infty \psi(\|x\| - d_p) W^p(x)$, $x \in R^n$. Then $\phi \in \mathcal{S}$ and, for any $h \in \Delta^\circ$,

$$\phi^{-1} h = (1/2) C_1 h + \sum_{p=1}^\infty \psi(\|\cdot\| - d_p) W^p h \in (1/2) C_1 \Delta^\circ$$

$$+ \sum_{p=1}^\infty 2^{-1-p} C_{p+1} W^{p+1} \Delta^\circ \subset \sum_{p=1}^\infty 2^{-p} C_p W^p \Delta^\circ$$

$$\subset \sum_{p=1}^\infty 2^{-p} B^\circ = B^\circ, \text{ i.e., } \phi^{-1} \Delta^\circ \subset B^\circ.$$

Further, $\|\phi W^p\|_\infty < \infty$ for each $p \in N$. Therefore $\phi W^p L_q \subset L_q$ and $B^\circ \subset \bigcup_{p\in N} W^p L_q \subset \phi^{-1} L_q$. Take $u \in B$. Then $|\langle u, v\rangle| = |(W^{+p} u)(W^{-p} \bar{v})| \leqq 1$ for each $p \in N$ and $v \in B^\circ \cap W^p L_q$. Finally, $\phi^{-1} u \in \mathfrak{D}'$ and $\phi^{-1} \Delta^\circ \subset B^\circ$. Hence for any $\omega \in \mathfrak{D} \cap \Delta^\circ \subset \mathfrak{D} \cap \phi B^\circ$, we have $|(\phi^{-1}u)\omega| = |u(\phi^{-1}\omega)| = |\langle u, \phi^{-1}\bar{\omega}\rangle| \leqq 1$. Since $\mathcal{D}$ is dense in L_q, this implies $\phi^{-1}u \in \Delta$, i.e., $B \subset \phi\,\Delta$.

Topology of $\mathcal{O}_q$. We have already introduced the inductive topology T_i of $\mathcal{O}_q = \bigcup_{p \in N} \mathcal{O}_{p,q}$. In [4] for each $\phi \in \mathcal{S}$ a seminorm $\|u\|_\phi = \|\phi u\|_q$ was defined on $\mathcal{O}_q$. Let T be the topology of $\mathcal{O}_q$ generated by the system $\{\|\cdot\|_\phi; \phi \in \mathcal{S}\}$. Evidently the topology T is for any s, $1 \leqq s \leqq +\infty$, generated by the seminorms $\|\phi D^\alpha u\|_{L^s}$, $\phi \in \mathcal{S}$, $\alpha \in N^n$, $|\alpha| \leqq q$, where $\|\cdot\|_{L^s}$ is the norm in $L^s(R^n)$.

Theorem. $(\mathcal{O}_q, T_i) = (\mathcal{O}_q, T)$.

Proof. It was shown in [4] that T_i is stronger than T. To prove the other inclusion, choose $U \in T_i$, $0 \in U$. By Lemma 3, there exists a bounded set B in $\bigcap_{p \in N} W^{-p}L_{-q}$ such that the neighborhood $V = \{u \in \mathcal{O}_q : |\langle u, v\rangle| < 1 \text{ for all } v \in B\}$ is contained in U. The inner product $\langle\cdot,\cdot\rangle$ was introduced in Lemma 3. By Lemma 4, there exists $\phi \in \mathcal{S}$ such that $B \subset \phi\,\Delta$, where Δ is the unit ball in L_{-q}. Since for any $u \in \mathcal{O}_q$ we have $\sup\{|\langle u, v\rangle| : v \in B\} \leqq \sup\{|\langle u, \phi w\rangle|; w \in \Delta\} = \sup\{|\bar{w}(\phi u)| : w \in \Delta\} \leqq \|\phi u\|_q = \|u\|_\phi$, the T_i-neighborhood V of 0 contains the T-neighborhood $\{u \in \mathcal{O}_q : \|u\|_\phi < 1\}$ of 0. Hence $V \in T$.

3. **Bounded sets in $\mathcal{O}_q$.**

Proposition. *Let $B \subset (\mathcal{O}_q, T_i)$ be bounded. Then there exists $p \in N$ such that $B \subset W^pL_q$.*

Proof. Assume that $B \backslash W^pL_q \neq \emptyset$ for every $p \in N$. Then there exist sequences $p_1 < p_2 < \cdots$ and $\{f_k\}_{k \in N} \subset B$ such that $f_k \in W^{p_k}L_q \backslash W^{p_{k-1}}L_q$. Choose a closed bounded convex neighborhood V_1 of 0 in $W^{p_1}L_q$ such that $f_1 \notin V_1$. When $V_1, V_2, \cdots, V_{k-1}$ are chosen, we choose, according to Lemma 1, a closed bounded convex neighborhood V_k of 0 in $W^{p_k}L_q$ so that $V_{k-1} \subset V_k$ and $s^{-1}f_s \notin V_k$ for all $s = 1, 2, \cdots, k$. Then $\bigcup_{k \in N} V_k$ is a convex neighborhood of 0 in $\mathcal{O}_q$ which does not absorb the bounded set $\{f_k\}_{k \in N}$, which is a contradiction.

Theorem. *Let $B \subset (\mathcal{O}_q, T_i)$ be bounded. Then there exists $p \in N$ such that B is bounded in W^pL_q.*

Proof. There exists $p \in N$ such that $B \subset W^pL_q$. Assume that B is not bounded in any W^sL_q, $s \geqq p$. Put $V_p = \{x \in W^pL_q : \|W^{-p}x\|_q \leqq 1\}$ and take $f_p \in B \backslash pV_p$. For the induction, assume that $V_p \subset V_{p+1} \subset \cdots \subset V_{k-1}$ and $f_p, f_{p+1}, \cdots, f_{k-1}$, are chosen. Since $\sup\{\|W^{-s}f\|_q : f \in B\} = +\infty$ for any $s \geqq p$, there exists $f_k \in B$

such that $f_k \notin kV_{k-1}$. By Lemma 1 there exists a closed bounded convex neighborhood V_k of 0 in W^kL_q such that $V_{k-1} \subset V_k$ and $f_s \notin sV_k$ for $s = p, p+1, \cdots, k$. Therefore $\bigcup_{k \in N} V_k$ is a convex neighborhood of 0 in $\mathcal{O}_q$ which does not absorb the bounded set $\{f_k\}_{k \geqq p} \subset B$, which is a contradiction.

MISECLLANEA. *$\mathcal{O}_q$ is a reflexive, complete, bornological space.*

PROOF. The Fréchet space $F = \bigcap_{p \in N} W^{-p}L_{-q}$ is barreled. By [5], IV, 5.8, it is semi-reflexive. Hence, by [5], IV, 5.6, it is reflexive. Finally, the strong dual $\mathcal{O}_q$ of a reflexive Fréchet space F is reflexive, complete, and bornological.

$\mathcal{S}$ is dense in $\mathcal{O}'_q$.

PROOF. Since $\mathcal{O}_q$ is reflexive, the strong topology on $\mathcal{O}'_q$ is the Mackey topology of the duality $\langle \mathcal{O}'_q, \mathcal{O}_q \rangle$. Hence convex subsets of $\mathcal{O}'_q$ have the same closures in both the weak* and norm topologies, see [5], IV, 3.3. Thus, we need only to show that $\mathcal{S}$ is weakly dense in $\mathcal{O}'_q$.

If $g \in \mathcal{O}_q$, $g \neq 0$, there exists $f \in \mathcal{S}$ such that $\langle f, g \rangle = \int f\bar{g}\, d\mu \neq 0$, where μ is the Lebesgue measure on R^n, i.e., $\mathcal{S}$ separates points of $\mathcal{O}_q$ with respect to the duality and by [5], IV, 1.3, $\mathcal{S}$ is weakly dense in $\mathcal{O}'_q$.

EXAMPLE. $\mathcal{O}_q$ is neither Montel nor nuclear. Let $\Omega(r) = \{x \in R^n; \|x\| \leqq r\}$. The set $B = \{u \in L_q;\ \|u\|_q \leqq 1 \text{ and } \operatorname{supp} u \subset \Omega(1)\}$ is bounded and closed in $\mathcal{O}_q$. To show that it is not compact, take $u \in B$, $u \neq 0$, and put $u_k(x) = u(kx)$, $x \in R^n$, $v_k = u_k\|u_k\|_q^{-1}$, $k = 1, 2, \cdots$. Then $v_k \in B$ and for any $\omega \in \mathcal{S}$ we have

$$
\begin{aligned}
|\langle v_k, \omega\rangle| &= \left| \int_{R^n} v_k\bar{\omega}\, dx \right| \\
&= \left| \int_{\Omega(k^{-1})} v_k\bar{\omega}\, dx \right| \\
&\leqq \left(\int_{\Omega(k^{-1})} |v_k\omega|^2\, dx \right)^{1/2} \\
&\quad \cdot \left(\int_{\Omega(k^{-1})} dx \right)^{1/2} \\
&\leqq \|\omega\|_\infty \cdot \|v_k\|_q \cdot \mu(\Omega(k^{-1})) \\
&\leqq \|\omega\|_\infty \cdot \mu(\Omega(1)) \cdot k^{-(1/2)n} \to 0
\end{aligned}
$$

as $k \to \infty$. Since $\mathcal{S}$ is dense in $\mathcal{O}'_q$, $\{v_k\}$ converges weakly to 0 in $\mathcal{O}_q$.

On the other hand, take $\phi \in \mathcal{S}$ such that $\phi(x) = 1$ for $x \in \Omega(1)$. Then the seminorm $\|v_k\|_\phi = \|\phi v_k\|_q = 1$ for any $k = 1, 2, \cdots$. Therefore, neither $\{v_k\}$ nor any of its subsequences converge to 0 in $\mathcal{O}_q$. Hence $\mathcal{O}_q$ is not Montel.

Since $\mathcal{O}_q$ is complete and B is closed in $\mathcal{O}_q$, B is not precompact and $\mathcal{O}_q$ is not nuclear.

PROPOSITION. *The inclusion $\mathcal{O}_q \subset \mathcal{L}(\mathcal{S}, L_q)$ holds. The bounded and the pointwise topology of $\mathcal{L}(\mathcal{S}, L_q)$ coincide on $\mathcal{O}_q$.*

PROOF. Since the topology T of $\mathcal{O}_q$ is the pointwise topology, it remains to show that T is stronger than the topology of $\mathcal{O}_q$ relative to $\mathcal{L}_b(\mathcal{S}, L_q)$.

The identity mapping $id : \mathcal{O}_{p,q} \to \mathcal{L}_b(L_p, L_q)$ is continuous. Therefore $id : \mathcal{O}_{p,q} \to \mathcal{L}_b(\mathcal{S}, L_q)$ and $id : (\mathcal{O}_q, T_i) \to \mathcal{L}_b(\mathcal{S}, L_q)$ are continuous too. Since $T = T_i$, the proof is complete.

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