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A CLASS OF MODELS FOR OPERATORS

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ABSTRACT. A class of Hilbert space operators is defined and each member of the class is shown to be a model for a large class of operators.

1. The purpose of this paper is to introduce a class of operator models. The symbols $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{G}$ will be reserved for Hilbert spaces and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ will denote the Banach space of (bounded linear) operators from $\mathcal{H}$ into $\mathcal{K}$. An operator S in $\mathcal{K}$ is called a model (see [1]) for an operator A in $\mathcal{H}$ provided there is a bicontinuous member ϕ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\phi A \phi^{-1} = S \mid \phi(\mathcal{H})$. In this case Lat A, the lattice of invariant subspaces of A, is isomorphic to Lat $S \mid \phi(\mathcal{H})$, and if A and S are invertible then $\phi A^{-1} \phi^{-1} = S^{-1} \mid \phi(\mathcal{H})$, i.e., S^{-1} is a model for A^{-1}. It is clear that if S is a model for A and V is a bicontinuous operator mapping $\mathcal{K}$ onto $\mathcal{G}$ then VSV^{-1} is also a model for A.

2. Let $|\cdot|$ denote the norm on $\mathcal{B}(\mathcal{H})$ and let $\mathcal{K}$ be the product space $\times_{-\infty}^{\infty} H$. If c is a nonnegative real number define the member S_c of $\mathcal{B}(\mathcal{K})$ as follows: if $0 \leqq c \leqq 1$ let

$$[S_c\{x_p\}_{-\infty}^{\infty}]_n = \begin{cases} x_{n+1} & \text{if } n \geqq 0 \\ cx_{n+1} & \text{if } n < 0 \end{cases},$$

and if $1 \leqq c$ let

$$[S_c\{x_p\}_{-\infty}^{\infty}]_n = \begin{cases} cx_{n+1} & \text{if } n \geqq 0 \\ x_{n+1} & \text{if } n < 0 \end{cases}.$$

Let $\mathcal{O}_0 = \{A : A \text{ belongs to } \mathcal{B}(\mathcal{H}) \text{ and } |A| < 1\}$, if $0 < c < 1$ let $\mathcal{O}_c = \{A : A \text{ belongs to } \mathcal{O}_0, A \text{ is invertible, and } |A^{-1}| < c^{-1}\}$ and if $c > 1$ let $\mathcal{O}_c = \{A^{-1} : A \text{ belongs to } \mathcal{O}_{c^{-1}}\}$. Rota has proven in [1] that S_0 is a model for each member of $\mathcal{O}_0$. The following theorems indicate that the operators S_c, $0 < c < 1$ or $1 < c$, may be considered in place of S_0 as an approach to the invariant subspace problem.

THEOREM 1. *If* $0 < c < 1$, *then*
(a) S_c *is a model for each member of* $\mathcal{O}_c$, *and*
(b) S_c^{-1} *is a model for each member of* $\mathcal{O}_{c^{-1}}$.

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PROOF. Conslusion (b) follows from (a). To prove (a), suppose A belongs to $\mathcal{O}_c$ and define ϕ from $\mathcal{H}$ into $\mathcal{K}$ by

$$[\phi u]_n = \begin{cases} A^n u & \text{if } n \geqq 0 \\ c^{-n}A^n u & \text{if } n < 0 \end{cases}.$$

The function ϕ is linear, $|cA^{-1}|^2/(1 - |cA^{-1}|^2) + 1/(1 - |A|^2)$ is a bound for ϕ, and since $[\phi u]_0 = u$, 1 is a bound for ϕ^{-1}; hence ϕ is a bicontinuous operator. A simple computation shows that $[\phi Au]_n = [S_c\phi u]_n$, i.e., $\phi A\phi^{-1} = S_c \mid \phi(\mathcal{H})$.

THEOREM 2. *If* $1 < c$, *then* (a) *and* (b) *of theorem* 1 *hold true.*

PROOF. Define the unitary operator U in $\mathcal{K}$ by the formula $[Ux]_n = x_{-n}$. A straightforward computation shows that $[US_cx]_n = [S^{-1}_{c^{-1}}Ux]_n$; thus $S_c = US^{-1}_{c^{-1}}U$. Part (a) now follows from theorem 1, and (a) implies (b).

THEOREM 3. *Suppose A belongs to* $\mathcal{B}(\mathcal{H})$. *Then*

(a) *there is a number c,* $0 < c < 1$, *and a member C of* $\mathcal{O}_c$ *such that* Lat A = Lat C, *and*

(b) *there is a number d,* $1 < d$, *and a member D of* $\mathcal{O}_d$ *such that* Lat A = Lat D.

PROOF. We shall prove (a) only. If $A = 0$ let $c = 1/3$ and $C = 1/2$; otherwise let $C = (1/2)|1 - (1/2)|A|^{-1}A|^{-1}[1 - (1/2|A|^{-1}A]$ and let $c = (1/2)|C^{-1}|^{-1}$, and (a) follows.

The lattices Lat S_c, $c \neq 0$, may prove more approachable than Lat S_0. It is well known that $S_0{}^2$ is unitarily equivalent to S_0; a similar relationship fails to hold for S_c if $c \neq 0$ or 1.

REFERENCES

1. G.-C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. 13 (1960), 469-472.

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