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ON A SET OF POLES AT THE WIENER BOUNDARY

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1. **Introduction.** On the boundary of a bounded plane region, polar sets have harmonic measure zero, and conversely, sets of harmonic measure zero, are polar. For an arbitrary Riemann surface however, it is well-known that this converse is not always valid. In this paper, we discuss the set of poles $\Phi(\Delta_1^M)$ at the Wiener ideal boundary of a hyperbolic Riemann surface R, and show that whenever $R \notin \mathcal{O}_{HB}^{\infty}$, the particular subset $\Phi(\Delta_1'')$ of $\Phi(\Delta_1^M)$ affords just such an example of a set of harmonic measure zero which is not polar. Whether or not this remains true for $R \in \mathcal{O}_{HB}^{\infty} - \mathcal{O}_{HP}^{\infty}$ is as yet unknown, although for $R \in \mathcal{O}_{HP}^{\infty}$, $\Pi(\Delta_1'')$ is shown to be a polar set.

2. **Preliminaries.** For an open Riemann surface R, we shall employ the following notation:

$R^W(R^M)$	:	the Wiener (Martin) compactification of R.
$\Delta^W(\Gamma^W)$	:	the Wiener ideal (harmonic) boundary.
Λ^W	:	$\Delta^W - \Gamma^W$.
Δ_1^M	:	the Martin minimal boundary.
K_ζ	:	the positive minimal harmonic function corresponding to $\zeta \in \Delta_1^M$.
$\hat{R}_u^E$	:	the balayage of u(superharmonic) relative to $E \subset R$.
$HB(R)$	:	the space of bounded harmonic functions on R.

For a discussion of the above topics refer to Brelot [1], the monographs of Constantinescu-Cornea [3], Sario-Nakai [7], and to Naim [6]. When R is hyperbolic, the Wiener harmonic boundary Γ^W is non-empty (cf. [7]).

The notion of poles was originally introduced by Brelot [1], subsequently developed by Naim [6] for a metrizable compactification, and for an arbitrary compactification (of a Riemann surface) by Ikegami [4] and Tanaka [9].

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DEFINITION. A point $p \in \Delta^W$ is a *pole* of $\zeta \in \Delta_1^M$ if for every neighborhood U of p in R^W, $U \cap R$ is not thin at ζ; i.e., $\hat{R}_{K_\zeta}^{U \cap R} = K_\zeta$.

For each $\zeta \in \Delta_1^M$, denote by $\Phi(\zeta)$ the set of poles of ζ on Δ^W. Then $\Phi(\zeta)$ is non-void and compact, and $\Phi(\Delta_1^M)$ is not polar (cf. [4]).

The Martin minimal boundary Δ_1^M can be divided into two significant subsets. Let

$$\Delta_1' = \{\zeta \in \Delta_1^M : K_\zeta \text{ is bounded}\},$$

and $\Delta_1'' = \Delta_1^M - \Delta_1'$. Topologically, Δ_1^M is a G_δ set and Δ_1' a $K_{\sigma\delta}$ set ([6]). Moreover, for $\zeta \in \Delta_1'$, $\Phi(\zeta)$ is a singleton (the converse also being true) ([9]), and for $\zeta \in \Delta_1''$, $\Phi(\zeta) \subset \Lambda^W$ ([4]).

3. **Main Results.** The basis for an identification of the poles of points in Δ_1' with the isolated points of Γ^W is contained in the works [4] and [9]. The following theorem (cf. Schiff [8]) completely characterizes the relationship between $\Phi(\Delta_1')$ and Γ^W. The proof given here, mutatis mutandis, is also valid in the theory of harmonic spaces of Brelot [2].

THEOREM 1. $\Phi(\Delta_1') =$ *isolated points of* Γ^W.

PROOF. Let p be an isolated point of Γ^W. Then there exists an HB-minimal function u on R such that $u(p) = 1$, $u(\Gamma^W - \{p\}) = 0$ (cf. [7]). Then $u = cK_\zeta$ for some $\zeta \in \Delta_1'$, $c > 0$. It is not difficult to see that ζ is unique.

Suppose there exists some neighborhood U of p in R^W such that $U \cap R$ is thin at ζ. We may assume that the points of $\partial(U \cap R)$ are regular, and hence $\hat{R}_{K_\zeta}^{U \cap R}$ is continuous, superharmonic on R. Since

$$\hat{R}_{K_\zeta}^{U \cap R} \neq K_\zeta,$$

it follows that $\hat{R}_{K_\zeta}^{U \cap R}$ is a continuous potential on R. Furthermore,

$$0 \leqq \hat{R}_{K_\zeta}^{U \cap R} \leqq K_\zeta$$

implies $\hat{R}_{K_\zeta}^{U \cap R}$ is also bounded, and therefore has a continuous extension to R^W (cf. [7]). Then

$$\lim_{R \ni z \to p} \hat{R}_{K_\zeta}^{U \cap R}(z) = \lim_{R \ni z \to p} K_\zeta(z) = K_\zeta(p) = 1/c > 0,$$

which contradicts the fact that $\hat{R}_{K_\zeta}^{U \cap R}$ is a potential. We conclude that p is a pole of ζ, and since $\Phi(\zeta)$ is a singleton, $\{p\} = \Phi(\zeta)$.

Conversely, let $\zeta \in \Delta_1'$. Then K_ζ is an HB-minimal function on R and there exists an isolated point $p \in \Gamma^W$ such that $K_\zeta(p) > 0$,

$K_\zeta(\Gamma^W - \{p\}) = 0$ (cf. [7]). Using the argument above, we find that p is a pole of ζ and $\Phi(\zeta) = \{p\}$.

Henceforth, let the isolated points of Γ^W be denoted by I. It is well-known that $\dim HB(R) = n$ if and only if Γ^W consists of n points $(1 \leqq n < \infty)$. The class $\mathcal{O}_{HB}^n$ represents those Riemann surfaces R for which $\dim HB(R) \leqq n$, and the Riemann surfaces which have $\bar{I} = \Gamma^W$ belong to the class $\mathcal{O}_{HB}^\infty$. These classes are related by the inclusion $\bigcup_{n=1}^\infty \mathcal{O}_{HB}^n \subsetneqq \mathcal{O}_{HB}^\infty$ ([7]).

We quote the following result due to Ikegami [4] which will be useful in the sequel.

MAXIMUM PRINCIPLE. *Let u be a superharmonic function on R, bounded from below. If*

$$\liminf_{R \ni z \to p} u(z) \geqq 0$$

for all $p \in \Phi(\Delta_1^M)$, then $u \geqq 0$ on R.

We turn our attention now to the question of the "size" of the set $\Phi(\Delta_1'')$. Although $\Phi(\Delta_1'')$ has (Wiener) harmonic measure zero, it may or may not be polar.

THEOREM 2. *If $R \notin \mathcal{O}_{HB}^\infty \cup \mathcal{O}_G$, then $\Phi(\Delta_1'')$ is not polar.*

PROOF. We first consider the case $I \neq \emptyset$. Assuming $\Phi(\Delta_1'')$ is a polar set, there exists a positive superharmonic function s on R such that $\lim_{R \ni z \to p} s(z) = \infty$ for each $p \in \Phi(\Delta_1'')$.

Suppose that for a bounded from below superharmonic function u on R,

$$\liminf_{R \ni z \to p} u(z) \geqq 0$$

for all isolated points $p \in \Gamma^W$. Then for any $\epsilon > 0$,

$$\liminf_{R \ni z \to p} (u + \epsilon s)(z) \geqq 0,$$

for all points $q \in \Phi(\Delta_1^M) = \Phi(\Delta_1') \cup \Phi(\Delta_1'')$ by Theorem 1. From the preceding maximum principle, it follows that $u + \epsilon s \geqq 0$ on R, and since ϵ was arbitrary, that $u \geqq 0$ on R. Thus, any $u \in HB(R)$ attains its maximum (and minimum) on the set of isolated points I, in Γ^W.

Since $\bar{I} \subsetneqq \Gamma^W$, choose a point $p \in \Gamma^W - \bar{I}$. Then there exists a function $f \in C(\Gamma^W)$ such that $0 \leqq f \leqq 1$, on Γ^W, $f(p) = 1$, $f \mid \bar{I} = 0$. The function $u_f \in HB(R)$ such that $u_f \mid \Gamma^W = f$, contradicts the fact that u_f must attain its maximum on I. Hence $\Phi(\Delta_1'')$ is not polar.

To treat the case $I = \emptyset$ (cf. also [4]), the assumption that $\Phi(\Delta_1'')$ is polar together with a slight modification of the above argument, yields the contradiction $HB(R) = \{0\}$. This completes the proof of the theorem.

For emphasis we reiterate:

COROLLARY. *If* $R \notin \mathcal{O}_{HB}^{\infty} \cup \mathcal{O}_G$, $\Phi(\Delta_1'')$ *is not polar, but has zero harmonic measure.*

For Reimann surfaces $R \in \mathcal{O}_{HB}^{\infty} - \mathcal{O}_{HP}^{\infty}$, whether or not $\Phi(\Delta_1'')$ is polar remains an open question. However, the matter is easily settled for the remaining class of surfaces by the following:

THEOREM 3. *If* $R \in \mathcal{O}_{HP}^{\infty} - \mathcal{O}_G$, $\Phi(\Delta_1'')$ *is a polar set.*

PROOF. $R \in \mathcal{O}_{HP}^{\infty} - \mathcal{O}_G$ implies dim HP(R) is at most countable. Hence Δ_1^M is a countable set and the same must be true for Δ_1''. Setting $\Delta_1'' = \{\zeta_n\}_{n=1}^{\infty}$, then $\Phi(\zeta_n)$ is a compact subset of Λ^W and is therefore polar. It follows that $\Phi(\Delta_1'') = \bigcup_{n=1}^{\infty} \Phi(\zeta_n)$ is a polar set.

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