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MAXIMAL QHC-SPACES

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1. **Introduction and Background.** In 1924, Alexandroff and Urysohn [1] investigated Hausdorff spaces which are closed in any Hausdorff space in which they may be embedded; such spaces are called H-closed. In 1940, Katetov [6] proved that minimal Hausdorff spaces are precisely the Hausdorff spaces which are H-closed and semiregular. In 1941, Bourbaki [2] noted that the H-closed property is equivalent in a Hausdorff space to

H(i): 'Every open filter base has a cluster point
and that the minimal Hausdorff property is equivalent in a Hausdorff space to

H(ii): 'Every open filter base with unique cluster point converges.

Properties of H(i) spaces have been studied by Scarborough and Stone [10] and Porter and Thomas [9]. We shall adopt the terminology of Porter and Thomas [9] and call an H(i) topological space a quasi-H-closed space which we shall denote by QHC in the remainder of this paper and note that Hausdorff QHC spaces are H-closed.

Our investigation of QHC spaces was motivated in part by R. E. Larson [7] when he questioned the connection between minimal Hausdorff spaces and maximal H-compact spaces (a space is H-compact if it is H(i) and H(ii)). We showed that maximal H-compact, maximal H(i) and maximal H(ii) spaces are not necessarily minimal Hausdorff [4] but left unanswered the questions concerning characterizations and properties of these maximal spaces. Our purpose in this paper is to characterize maximal H(i) (that is, maximal QHC) spaces.

For a given topological property R and any set X, we shall let $R(X)$ denote the set of all topologies on X which have property R and note that $R(X)$ is partially ordered by set inclusion. A topological space on a set X with topology τ shall be denoted (X, τ) and is maximal R (respectively, minimal R) if τ is a maximal (respectively, minimal) element in $R(x)$. If τ and τ' are two topologies on the same set, we say that τ' is an expansion of τ if $\tau \subseteq \tau'$. If (X, τ) is a topological space, $A \subseteq X$, and $A \notin \tau$, then the topology with subbase $\tau \cup \{A\}$ is called the simple expansion of τ by A and is denoted $\tau(A)$.

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For a topological space (X, τ) the relative topology on a subset A of X is $\tau \mid A$, the collection of open neighborhoods for a point x is $\mathcal{V}_\tau(x)$, and the product of (X_α, τ_α) for $\alpha \in \mathcal{O}$ is $(\pi_{\mathcal{O}} X_\alpha, \pi_{\mathcal{O}} \tau_\alpha)$. We shall write cl A, int A and bd A for the closure, interior, and boundary respectively of a set A with respect to a topology τ. We shall subscript these sets if several different topologies are being discussed; for example, if τ and τ' are topologies on X and $A \subseteq X$, then $\mathrm{cl}_\tau A$ and $\mathrm{cl}_{\tau'} A$ are the closures of A with respect to τ and τ' respectively.

2. **QHC Spaces and Subspaces.** It has been observed [9] that unlike compactness and other covering axioms, a subset of a topological space (X, τ) may be QHC with respect to the topology of the space but not QHC with respect to its relative topology. This difference in structure is the main reason for a difference in the properties of maximal QHC spaces and maximal compact, maximal countably compact and maximal Lindelöf spaces [5].

The following characterization of QHC spaces is most useful in our discussion.

Definition 1. An open cover of the space (X, τ) has a *finite proximate subcover* of it has a finite subfamily whose closures form a cover of the space.

Theorem 1. *A topological space* (X, τ) *is* QHC *if and only if every open cover has a finite proximate subcover* [10].

Definition 2. A subset A of a topological space (X, τ) is QHC relative to (X, τ) if every τ-open cover of A has a finite subfamily whose τ-closures cover A.

Definition 3. A subset A of a topological space (X, τ) is QHC (τ-QHC if more than one topology on X is involved) if $(A, \tau \mid A)$ is QHC.

Theorem 2. *In a topological space* (X, τ), *a* QHC *subset is* QHC *relative to* (X, τ) *but not conversely* [9].

To facilitate our discussion of maximal QHC spaces we need to determine the properties of another type of QHC subset.

Definition 4. A subset A of a topological space (X, T) is *interiorly* QHC if every open cover of A has a finite subfamily whose closures cover int A.

In the case of interiorly QHC, the cover considered may be either a cover of τ open or $\tau \mid A$ open sets since in int A the two types of covers are essentially the same.

THEOREM 3. *If a subset A of a space* (X, τ) *is* QHC *relative to* (X, τ), *then it is interiorly* QHC.

PROOF. If every open cover of A by τ open sets has a finite subfamily whose closures cover A, then the closures of that subfamily cover int A.

COROLLARY 1. *If a subset of a space is* QHC, *then it is interiorly* QHC.

EXAMPLE 1. This is an example of a set which is interiorly QHC but is not QHC relative to (X, τ) and therefore is not QHC.

Let (X, τ) be $[0, 1]$ with the usual topology; this space is H-closed since it is minimal Hausdorff. The set $A = [1/2, 1] \cup \{1/n \mid n \in N\}$ is not QHC relative to (X, τ) but is interiorly QHC since every open cover of A covers $[1/2, 1]$ which is QHC and int $A = (1/2, 1]$.

THEOREM 4. *If a subset A of a topological space* (X, τ) *is interiorly* QHC, *then* $C =$ cl(int A) *is* QHC.

PROOF. Let $\{G_\beta \mid \beta \in B\}$ be a τ open cover of C. Then $\{G_\beta \mid \beta \in B\} \cup \{X - C\}$ is a τ open cover of A and so there is a finite subset $B_1 \subseteq B$ such that $\{G_\beta \mid \beta \in B_1\}$ is a subfamily whose closures cover int A or equivalently $\{G_\beta \mid \beta \in B_1\}$ is dense in int A and so $\{G_\beta \cap C \mid \beta \in B_1\}$ is a subfamily of $\{G_\beta \cap C \mid \beta \in B\}$ whose union is dense in int A. For $x \in C$, if $x \notin \mathrm{cl}_{\tau|C}(G_\beta \cap C)$ for all $\beta \in B_1$, there is a neighborhood $U(x) \in \mathcal{V}_\tau(x)$ such that $U(x) \cap (G_\beta \cap \text{int } A) = \emptyset$ for all $\beta \in B_1$. Therefore for $y \in U(x) \cap \text{int } A$, $y \notin \mathrm{cl}\, G_\beta$ for $\beta \in B_1$. This is a contradiction and so for $x \in C$, $x \in \mathrm{cl}_{\tau|C}(G_\beta \cap C)$ for some $\beta \in B_1$. Therefore C is QHC.

COROLLARY 2. *If a subset A of a topological space is interiorly* QHC, *then* cl(int A) $\cap$ A *is* QHC.

COROLLARY 3. *For a topological space* (X, τ) *if A is interiorly* QHC, *then* cl(int A) *is* QHC *relative to* (X, τ).

COROLLARY 4. *In a regular space* (X, τ), *if A is interiorly* QHC, *then* cl(int A) *is compact.*

PROOF. A regular *QHC*-space is compact.

THEOREM 5. *A subset A of a topological space* (X, τ) *is interiorly* QHC *if and only if every collection of* int A *open sets with the finite intersection property has an adherent point in* cl(int A) $\cap$ A.

PROOF. Let $\{G_\beta \mid \beta \in B\}$ be a collection of int A open sets with the finite intersection property but no adherent point in cl(int A) $\cap$ A; that is, $\{G_\beta \mid \beta \in B\}$ is an int A open filter base with no adherent points in cl(int A) $\cap$ A. Then $\{X - \text{cl } G_\beta \mid \beta \in B\}$ is an open cover of A. Let $B_1 \subseteq B$ be a finite subset and consider $\cup\{\text{cl}(X - \text{cl } G_\beta) \mid \beta \in B_1\}$. This does not contain int A because

$$\text{int } A - \cup\{\text{cl}(X - \text{cl } G_\beta) \mid \beta \in B_1\} \supseteq \cap\{G_\beta \mid \beta \in B_1\} \neq \emptyset.$$

If the condition holds then if $\{G_\beta \mid \beta \in B\}$ is an open cover of A for which no finite subfamily is dense in int A, then $\{(X - \text{cl } G_\beta) \cap \text{int } A \mid \beta \in B\}$ has the finite intersection property but

$$\begin{aligned} A - \cap\{\text{cl}(X - \text{cl } G_\beta) \mid \beta \in B\} &= \cup\{A - \text{cl}(X - \text{cl } G_\beta) \mid \beta \in B\} \\ &\supseteq \cup\{G_\beta \mid \beta \in B\} \cap A. \end{aligned}$$

Therefore $(\cap\{\text{cl}(X - \text{cl } G_\beta) \mid \beta \in B\}) \cap A = \emptyset$ and so $\{(X - \text{cl } G_\beta) \cap \text{int } A \mid \beta \in B\}$ has no τ-adherent point in A and therefore not in cl(int A) $\cap$ A. This is a contradiction and therefore every open cover of A has a finite subfamily whose closures are a cover of int A and so A is interiorly QHC.

COROLLARY 5. *If* cl(int A) $\subseteq$ A, *then A is interiorly* QHC *if and only if* cl(int A) *is* QHC.

COROLLARY 6. ***In a Hausdorff space, A is interiorly H-closed only if*** **cl(int A) $\subseteq$ A.**

We observe that in a Hausdorff space an open set which is not closed cannot be interiorly QHC; however in a T_1 space, open sets which are not closed may be interiorly QHC.

EXAMPLE 3. This is an example of an open subset of a T_1 space which is interiorly QHC.

Let X be infinite; $x_1, x_2 \in X$. The topology on X is generated by the following sets:

$\{x\}$ is open if $x \neq x_1$ and $x \neq x_2$;
$x_i \in U$ is open if $X - U$ is finite, $i = 1, 2$.

$X - \{x_1\}$ is open and is interiorly QHC since it is compact and Hausdorff in its relative topology.

THEOREM 6. *In a compact space* (X, τ) *if* cl(int A) $\subseteq$ A, *then A is interiorly* QHC.

PROOF. If cl(int A) $\subseteq$ A, then cl(int A) is QHC since it is compact and so A is interiorly QHC by Corollary 5.

COROLLARY 7. *In a compact Hausdorff space, A is interiorly* H-*closed if and only if* cl(int A) $\subseteq$ A.

PROOF. This follows from the preceding theorem and Corollary 6.

The following properties of QHC spaces are well known and so the proofs are omitted.

(QHC-1) The closure of a QHC subspace is QHC.

(QHC-2) In a QHC space, regular closed sets are QHC(A closed set C is regular closed if $C = \mathrm{cl}(\mathrm{int}\, C)$ or equivalently C is the closure of an open set).

(QHC-3) If A and B are QHC, so is $A \cup B$.

(QHC-4) QHC is preserved by continuous maps and therefore is *contractive.*

(A topological property is *contractive* if τ' has property R whenever τ has property R and τ is finer than τ').

The following result characterizes QHC spaces in terms of QHC subspaces.

THEOREM 7. *A topological space is* QHC *if and only if every proper regular closed space is* QHC.

PROOF. We already know that QHC is regular closed hereditary.

If every proper regular closed subset is QHC then let $\{U_\alpha \mid \alpha \in \mathcal{O}\}$ be an open cover of X and choose U_{α_0}, $\alpha_0 \in \mathcal{O}$. Let $B = \mathrm{cl}(\mathrm{int}(X - U_{\alpha_0}))$. B is regular closed and $\{U_\alpha \mid \alpha \in \mathcal{O}\}$ covers B and thus has a finite proximate subcover $\{U_\alpha \mid \alpha \in \mathcal{B}, |\mathcal{B}| < \aleph_0\}$. Then $\{U_{\alpha_0}\} \cup \{U_\alpha \mid \alpha \in \mathcal{B}\}$ is a finite proximate subcover of X.

An alternate characterization is given by the following.

THEOREM 8. (X, τ) *is* QHC *if and only if every closed subset is interiorly* QHC.

PROOF. If every closed set is interiorly QHC, the result follows from Theorems 4 and 7. If (X, τ) is QHC, the result follows from QHC-2 and Corollary 5.

3. Maximal QHC Spaces.

DEFINITION 5. An open subset U of a topological space (X, τ) is *regular open* if $\alpha(U) = \mathrm{int}(\mathrm{cl}\, U) = U$.

Since the collection of regular open sets in a topological space (X, τ) is closed under finite intersections and covers X, it is a base for a coarser topology τ_S. A topological space (X, τ) is *semiregular* if $\tau = \tau_s$.

THEOREM 9. *If* $U \in \tau$, *then* $\mathrm{cl}_\tau\, U = \mathrm{cl}_{\tau_S}\, U$ *and* $\alpha_\tau\, U = \alpha_{\tau_S}\, U$ [9].

Definition 6. Two topologies τ and τ' on X are *ro-equivalent* if $\tau_S = \tau_S{}'$.

Theorem 10. *An expansion τ' of τ is ro-equivalent to τ if and only if* $\mathrm{cl}_{\tau'} U = \mathrm{cl}_\tau U$ *for all* $U \in \tau'$ [8].

Theorem 11. *A topological space (X, τ) is* QHC *if and only if (X, τ_S) is* QHC.

Proof. If (X, τ) is QHC then (X, τ_S) is QHC since QHC is contractive.

If (X, τ_S) is QHC and $\{G_\alpha \mid \alpha \in \mathcal{O}\}$ is a τ-open cover, then $\{\alpha_\tau G_\alpha \mid \alpha \in \mathcal{O}\}$ is a τ_S-open cover since $\alpha_\tau G_\alpha \supseteq G_\alpha$ and is regular open. Thus there is a finite proximate subcover $\{\alpha_\tau G_{\alpha_i} \mid i = 1, \cdots, n\}$. Since $\mathrm{cl}_{\tau_S} \alpha_\tau G_{\alpha_i} = \mathrm{cl}_\tau G_{\alpha_i}$, $\{G_{\alpha_i} \mid i = 1, \cdots, n\}$ is a finite proximate subcover of $\{G_\alpha \mid \alpha \in \mathcal{O}\}$.

Definition 7. For a given semiregular topology τ_0, the set $E(\tau_0)$ of all topologies τ on X such that $\tau_S = \tau_0$ is partially ordered by set containment. A maximal element τ' of $E(\tau_0)$ is called a *submaximal topology* and (X, τ') is a submaximal space.

The following properties of submaximal topologies are well-known [3].

S-1: Every topology of $E(\tau_0)$ is coarser than some submaximal topology.

S-2: A topology τ on X is submaximal if and only if every subset of X which is τ-dense is open.

S-3: Every subspace of a submaximal space is submaximal.

S-4: Every subspace of a submaximal space is locally closed (that is it is the intersection of an open set and a closed set).

S-5: If A is a subset of a submaximal space X, then $\mathrm{bd}\, A = \mathrm{cl}\, A - \mathrm{int}\, A$ is discrete.

From the properties of submaximal spaces and Theorem 11, we have

M-1: Every maximal QHC is submaximal.

Theorem 12. *Every maximal* QHC *space is T_1.*

Proof. Let (X, τ) be QHC such that $\{x_0\}$ is *not* closed for some $x_0 \in X$ and $y_0 \in \mathrm{cl}\{x_0\}$, $y_0 \neq x_0$. Define τ^* on X by

$$G \in \tau^* \text{ for } G \in \tau,\ y_0 \notin G,$$

$$G \in \tau^* \text{ if } G \cup \{x_0\} \in \tau,\ y_0 \in G, \text{ and } G - \{y_0\} \in \tau.$$

Let $\mathcal{G} = \{G_\alpha \mid \alpha \in \mathcal{O}\}$ be a τ^* open cover of X. Define

$$\mathcal{G}^* = \{G_\alpha^* \mid G_\alpha^* = G_\alpha, \text{ if } G_\alpha \in \tau;\ G_\alpha^* = G_\alpha \cup \{x_0\} \text{ if } G_\alpha \notin \tau\}.$$

$\mathcal{G}^*$ is a τ-open cover and thus has a finite proximate subcover $\{G_i^* \mid i = 1, 2, \cdots, n\}$. For $x \neq y_0$ if $x \in X - \mathrm{cl}_\tau\{x_0\}$, then for $x \in \mathrm{cl}_\tau G_i^*$, $x \in \mathrm{cl}_{\tau*} G_i$. Now there is $G_{x_0} \in \mathcal{G}$ such that $x_0 \in G_x$ and for $x \in \mathrm{cl}_\tau\{x_0\}$, $x \neq y_0$, $x \in \mathrm{cl}_{\tau*} G_{x_0}$, and there is $G_{y_0} \in \mathcal{G}$ such that $y_0 \in G_{y_0}$. Therefore $\{G_i \mid i = 1, 2, \cdots, n\} \cup \{G_{x_0}, G_{y_0}\}$ is a τ^* finite proximate subcover of $\mathcal{G}$ and so (X, τ^*) is QHC.

Corollary 8. *A* QHC *space (X, τ) is strongly* QHC *only if $\tau \vee \mathcal{F}$ is* QHC *where $\mathcal{F}$ is the topology of finite complements.*

The following example of a submaximal QHC space for which $\tau \vee \mathcal{F}$ is not QHC shows that every QHC space is not strongly QHC; however every Hausdorff QHC space (that is, every H-closed space) is strongly QHC and every submaximal Hausdorff QHC space is maximal QHC [8].

Example 4. Let X be infinite; $x_0 \in X$. τ is the topology on X such that if $U \in \tau$, then $x_0 \in U$. This space is QHC since $\mathrm{cl}\{x_0\} = X$; also every set is either open or closed, and $\tau \vee \mathcal{F}$ is the discrete topology.

Theorem 13. *A* QHC *space (X, τ) is maximal* QHC *if and only if (X, τ) is submaximal and if, for any $A \subset X$, both X-int A and A are* QHC *subspaces then A is closed.*

Proof. If (X, τ) is submaximal and not maximal QHC, then by Theorem 12, there is finer QHC topology τ' such that $\tau_s' \neq \tau_s$. By Theorem 11, there is an $U \in \tau'$ such that $\mathrm{cl}_\tau U \supset \mathrm{cl}_{\tau'} U$; so, $\mathrm{cl}_{\tau'} U$ is not τ-closed. However, by QHC-2 and QHC-4, both $\mathrm{cl}_{\tau'} U$ and $\mathrm{cl}_{\tau'}(X - \mathrm{cl}_{\tau'} U)$ are τ-QHC subspaces. By QHC-1, $\mathrm{cl}_\tau(cl_{\tau'}(X - \mathrm{cl}_{\tau'} U)) = \mathrm{cl}_\tau(X - \mathrm{cl}_{\tau'} U) = X\text{-int}_\tau(\mathrm{cl}_{\tau'} U)$ is τ-QHC subspace. Conversely, suppose (X, τ) is maximal QHC. Assume there is a nonclosed subset $A \subset X$ such that both A and X-$\mathrm{int}_\tau A$ are τ-QHC subspaces. It suffices to show that $\tau' = \tau(X - A)$ is QHC. Since (X, τ) is submaximal by M-1, then $X - A \cup \mathrm{int}_\tau A$ is τ-open implying $\tau \mid B = \tau' \mid B$ where $B = \mathrm{cl}_\tau(X - A)$. Also, $\tau \mid A = \tau' \mid A$; so, both A and B are τ'-QHC subspaces. Since $X = A \cup B$, then by QHC-3, (X, τ') is QHC.

Corollary 9. *A submaximal* QHC *space in which every* QHC *subspace is closed is maximal* QHC.

Definition. A QHC space is *strongly QHC* if it has a finer maximal QHC topology.

Corollary 10. *A* QHC *space* (X, τ) *is maximal* QHC *if and only if* (X, τ) *is submaximal and if, for any* $A \subset X$, *A is interiorly* QHC *and* $X - \operatorname{int} A$ *is a* QHC *subspace, then A is closed.*

Proof. Most of the proof of Theorem 13 works here. Modify the conclusion of the "converse" part of the proof as follows: Since $\tau \mid A = \tau' \mid A$ and for any $\tau \mid A$-open set U, $\mathrm{cl}_\tau\, U \cap A = \mathrm{cl}_{\tau'}\, U \cap A$, it follows that A is τ'-interiorly QHC. Hence, by Theorem 4, $\mathrm{cl}_{\tau'}(\mathrm{int}_\tau A)$ is an τ'-QHC subspace. Since $\mathrm{int}_{\tau'} A = \mathrm{int}_\tau A$, then $X = B \cup \mathrm{cl}_{\tau'}(\mathrm{int}_{\tau'} A)$. By QHC-3, (X, τ') is QHC.

This example was first used by H. Tong [11] as an example of a maximal compact space which is not Hausdorff and has recently been shown to be maximal QHC [4].

Example 5. Let $X = \{a, b\} \cup N \times N$ where N is the set of natural numbers, E be the set of even natural numbers and $O = N - E$. The topology τ on X consists of those sets U such that one of the following is true:

(1) $a \notin U$ and $b \notin U$,

(2) $a \in U$ implies there is a family of finite subsets $A_m \subseteq N$ for $m \in E$ such that $\bigcup\{\{m\} \times (N - A_m) : m \in E\} \subseteq U$,

and

(3) $b \in U$ implies there is a family of finite subsets $B_m \subseteq N$ for $m \in O$ such that $\bigcup\{\{m\} \times (N - B_m) : m \in 0\} \subseteq U$ and there is a finite subset $B \subseteq N$ such that $(N - B) \times N \subseteq U$.

The following space first introduced by P. Urysohn [12] is maximal H-closed and, being minimal Hausdorff, is minimal H-closed.

Example 6. Let $X = \{a, b\} \cup N \times I$ where I is the set of integers. The neighborhood bases for τ consist of the following sets:

$\{(m, n)\}$ for $m \in N, n \in I \times \{0\}$;

$U_r((m, 0)) = \{(m, 0)\} \cap \{(m, n)\ |n| > r\}$ for $r \in N$;

$U_r(a) = \{a\} \cup \{(m, n) \mid m \geqq r, n > 0\}$ for $r \in N$;

$U_r(b) = \{b\} \cup \{(m, n) \mid m \geqq r, n < 0\}$ for $r \in N$.

The following is an example of a minimal Hausdorff space which is not maximal H-closed.

Example 7. Let $Y = \{a\} \cup \mathrm{N} \times I^+$ where I^+ is the set of non-negative integers and τ^* the relative topology on Y as a subspace of (X, τ). (Y, τ^*) is H-closed since Y is regularly closed (cf. QHC-2) in the topology of (X, τ), and since Y is submaximal it is maximal H-closed.

The space $(Y, (\tau^*)_s)$ where (Y, τ^*) is the subspace in Example 6 is Hausdorff, compact, and strictly weaker than (Y, τ^*). Since it is compact and Hausdorff, it is minimal Hausdorff and since $\tau^* \supset \tau_s^*$, it is not maximal H-closed. We may also observe that this example shows that regular, normal, or completely regular Hausdorff spaces are not necessarily maximal H-closed.

THEOREM 14. *If* $(\prod_{\mathcal{O}} X_\alpha, \prod_{\mathcal{O}} \tau_\alpha)$ *is maximal* QHC, *then* (X_α, τ_α) *is maximal* QHC *for each* $\alpha \in \mathcal{O}$.

PROOF. C. T. Scarborough and A. H. Stone [10] have shown that the product of QHC spaces is QHC. Therefore if (X_β, τ_β) is not maximal QHC for some $\beta \in A$, then there is $\tau_\beta{}' \supset \tau_\beta$ such that $(X_\beta, \tau_\beta{}')$ is QHC and $(\prod_{\mathcal{O}} X_\alpha, \prod_{\mathcal{O}} \tau_\alpha{}')$ is QHC where $\tau_\alpha{}' = \tau_\alpha$ for $\alpha \neq \beta$ and $\tau_\alpha{}' = \tau_\beta{}'$ for $\alpha = \beta$. This is a contradiction since $\prod_{\mathcal{O}} \tau_\alpha{}' \supseteq \prod_{\mathcal{O}} \tau_\alpha$ and therefore (X_α, τ_α) is maximal QHC.

Infinite products of non-trivial maximal QHC spaces are never maximal QHC. For each (X_α, τ_α) select $G_\alpha \in \tau_\alpha, G_\alpha \neq X_\alpha, \mathrm{cl}_\tau G = X_\alpha$; then $\prod_{\mathcal{O}} G_\alpha$ is dense in $\prod_{\mathcal{O}} X_\alpha$ but is not open. Thus $(\prod_{\mathcal{O}} X_\alpha, \prod_{\mathcal{O}} \tau_\alpha)$ is not submaximal.

Also if there exists a maximal QHC space (X, τ) which is not Hausdorff and which has no isolated points, then $(X \times X, \tau \times \tau)$ is not submaximal since $\Delta = \{(x, x) \mid x \in X\}$ is not closed but $X \times X - \Delta$ is dense.

We have not been able to answer some questions which are important enough to merit consideration. One of these is stated in the preceding discussion:

Does there exist a maximal QHC space which is not Hausdorff and which has no isolated points? [If (X, τ) is an H-closed space without isolated points (e.g., the unit interval with the usual topology) and τ' is a finer submaximal topology on X (cf. S-1), then (X, τ') is maximal H-closed and has no isolated points.]

For $R \in$ {compact, countably compact, Lindelöf} an R-space is maximal R if and only if the R subsets are precisely the closed ones [5]. Is it true in maximal QHC spaces that the interiorly QHC subspaces are precisely the closed subsets? [One might ask, "In a maximal QHC space, are the QHC subspaces precisely the regular closed ones?" However, if that was the case, then singleton points (being QHC) would be open and the space discrete and would be QHC only if it was finite.]

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