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PERIODIC SOLUTIONS TO NONLINEAR PARABOLIC DIFFERENTIAL EQUATIONS

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Introduction. We consider periodic solutions of the nonlinear parabolic equation

$$(1) \qquad u_t = f(t, x, u, u_x, u_{xx}) \qquad ((t, x) \in R \times (0, 1))$$

subject to

$$(2) \qquad u(t, 0) = u(t, 1) = 0 \qquad (t \in R).$$

Some of our results apply only to the special case of the quasilinear equation

$$(3) \qquad u_t = a(t, x, u, u_x)u_{xx} + b(t, x, u, u_x).$$

In § 1 we give general sufficient conditions for the existence of *a priori* bounds on T-periodic solutions $u(t, x)$ to (1) – (2). In § 2 we obtain bounds on derivatives of T-periodic solutions to (3) – (2). In § 3 we use degree theory in conjunction with results of Fife [2] for the linear case to obtain existence of T-periodic solutions to (3) – (2).

Periodic solutions of nonlinear parabolic equations have been studied by Bange [1], Fife [2], Prodi [6], Vaghi [7], Walter [9] and others. The results presented here apply to a wider class of equations with one space variable than those presented previously. Our *a priori* bound arguments do not require semi-linearity, monotonicity, or Lipschitz conditions of the nonlinear functions involved as in [2], [7], and [9], and do not require complicated analysis of Green's function representations as in [1] and [6].

1. Bounds for $u(t, x)$. We first present a general theorem which generalizes and simplifies the procedure developed by one of the authors (see [4]) for obtaining *a priori* bounds on solutions to nonlinear second order ordinary differential equations. In this section we assume:

(i) $f: R \times (0, 1) \times R^3 \to R$,
(ii) $f(t, x, z, p, r_1) \geqq f(t, x, z, p, r_2)$ for $r_1 \geqq r_2$, and
(iii) $f(t + T, x, z, p, r) = f(t, x, z, p, r)$ (T-periodic).

A solution will be a function $u(t, x)$ which is continuous on $R \times [0, 1]$

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with u_t, u_x and u_{xx} continuous in $R \times (0, 1)$, which is periodic of period T in t, and satisfying (1) and (2).

THEOREM 1.1 (A PRIORI BOUNDS). *Let $\sigma(t, x; \lambda)$ be continuous and T-periodic in $R \times [0, 1] \times (\lambda_0, \infty)$ with $\sigma(t, x, \lambda) \to +\infty$ as $\lambda \to +\infty$ uniformly in (t, x). Suppose that $\sigma(t, 0; \lambda) > 0$, $\sigma(t, 1; \lambda) > 0$, and*

(1.1) $\quad \sigma_t > f(t, x, \sigma, \sigma_x, \sigma_{xx}) \qquad ((t, x, \lambda) \in R \times (0, 1) \times (\lambda_0, \infty)).$

Then

$$u(t, x) \leqq \inf\{\sigma(t, x, \lambda) : \lambda_0 < \lambda < +\infty\}$$

for any T-periodic solution $u(t, x)$ to (1) – (2). Similarly, if $\rho(t, x; \lambda)$ is continuous and T-periodic in $R \times [0, 1] \times (\lambda_0, +\infty)$ with $\rho(t, x; \lambda) \to -\infty$ as $\lambda \to +\infty$ uniformly in (t, x), $\rho(t, 0; \lambda) < 0$, $\rho(t, 1, \lambda) < 0$ and

(1.2) $\quad \rho_t < f(t, x, \rho, \rho_x, \rho_{xx}) \qquad ((t, x, \lambda) \in R \times (0, 1) \times (\lambda_0, +\infty)),$

then

$$u(t, x) \geqq \sup\{\rho(t, x, \lambda) : \lambda_0 < \lambda < +\infty\}.$$

PROOF. Suppose not. Then there exist λ_1 and (t_1, x_1) such that $u(t_1, x_1) = \sigma(t_1, x_1, \lambda_1)$ and $\sigma(t, x, \lambda_1) \geqq u(t, x)$ for $(t, x) \in R \times (0, 1)$; i.e., $\sigma(t, x; \lambda_1)$ "touches" $u(t, x)$ at (t_1, x_1). Since $\sigma(t, 0; \lambda_1) > 0$ and $\sigma(t, 1; \lambda_1) > 0$ and u satisfies (2), (t_1, x_1) is an interior point and we have $\sigma_x(t_1, x_1; \lambda_1) = u_x(t_1, x_1; \lambda_1)$, $\sigma_t(t_1, x_1; \lambda_1) = u_t(t_1, x_1)$, and $\sigma_{xx}(t_1, x_1; \lambda_1) \geqq u_{xx}(t_1, x_1)$. Thus by the monotonicity assumption on f with respect to r, at (t_1, x_1) we have

$$\begin{aligned} \sigma_t = u_t &= f(t_1, x_1, u, u_x, u_{xx}) \\ &\leqq f(t_1, x_1, \sigma, \sigma_x, \sigma_{xx}). \end{aligned}$$

This contradicts (1.1). The proof for the lower bound is similar.

If we seek a family of functions $\sigma(x; \lambda)$ independent of t and satisfying the conditions of the above theorem, then (1.1) and (1.2) become ordinary differential inequalities, and we might expect to construct $\sigma(x; \lambda)$ by solving appropriate ordinary differential equations. This procedure is described in the following corollary.

COROLLARY 1.2. (BOUNDS FROM ORDINARY DIFFERENTIAL EQUATIONS). *Let $g(z, p)$ be continuous in R^2. Suppose the initial value problem*

(1.3) $\quad y'' + g(|y|, |y'|) + \epsilon = 0, y(1/2) = \lambda, y'(1/2) = 0$

has a unique solution $\phi(x; \lambda, \epsilon)$ defined on $[0, 1]$ for each $\lambda \geqq \lambda_0$ and $|\epsilon| \leqq \epsilon_0$. Suppose $\phi(x; \lambda, 0) > 0$ for $\lambda > \lambda_0$ and $\phi(x; \lambda, 0) \to +\infty$ as $\lambda \to +\infty$ uniformly in x. If there is a positive function $P(t, x, z, p, r)$ such that

(1.4) $\quad \operatorname{sgn} z \cdot f(x, z, p, r) \leqq P(t, x, z, p, r) \; [g(|z|, |p|) + r \operatorname{sgn} z]$,

then $|u(t, x)| \leqq \phi(x; \lambda_0, 0)$ *for any T-periodic solution* $u(t, x)$ *to* (1) – (2).

PROOF. By continuous dependence, there exists a continuous function $\epsilon(\lambda) > 0$ for $\lambda > \lambda_0$ such that the solution $\sigma(x; \lambda)$ of

$$y'' + g(y, y') + \epsilon(\lambda) = 0, y(1/2) = \lambda, y'(1/2) = 0$$

is also positive in $[0, 1]$ for $\lambda > \lambda_0$ and $\sigma(x, \lambda) \to +\infty$ as $\lambda \to +\infty$ uniformly in x. We have

$$\begin{aligned} \sigma_t = 0 > -\epsilon(\lambda)P &= P[g(|\sigma|, |\sigma_x|) + \sigma_{xx}] \\ &\geqq f(t, x, \sigma, \sigma_x, \sigma_{xx}). \end{aligned}$$

Thus, the hypotheses of the theorem are satisfied and we conclude that $u(t, x) \leqq \sigma(x; \lambda)$ for $\lambda > \lambda_0$. Since $\epsilon(\lambda_0)$ can be taken arbitrarily small we can assert that $u(t, x) \leqq \phi(x; \lambda, 0)$ for $\lambda > \lambda_0$, and hence $u(t, x) \leqq \phi(x; \lambda_0, 0)$. By similar arguments, $u(t, x) \geqq -\phi(x; \lambda_0, 0)$.

REMARK. For the quasilinear equation (3), if we take $P = a$ (assuming $a > 0$), than (1.4) becomes

$$\operatorname{sgn} z \cdot \frac{b(t, x, z, p)}{a(t, x, z, p)} \leqq g(z, p).$$

As illustrations of the use of Corollary 2.2, we give some special conditions under which the bounding function $\phi(x; \lambda_0, 0)$ in the corollary exists.

EXAMPLE 1. Suppose

$$\operatorname{sgn} z \cdot f(t, x, z, p, r) \leqq P(t, x, z, p, r)[\psi(|p|) + r \operatorname{sgn} z]$$

where P is a positive function, $\psi(s)$ is positive and of class C^1 in $[0, \infty)$, and

$$\int_0^\infty \frac{ds}{\psi(s)} > 1/2. \tag{1.5}$$

Let $g(z, p) = \psi(|p|)$ in the corollary. Uniqueness of solutions to (1.3) is assured since $\psi(|p|)$ is locally Lipschitz continuous. We have

$$\phi(x; \lambda, \epsilon) = \lambda + \int_{1/2}^{x} w(s; \epsilon)\, ds$$

where $w(x; \epsilon)$ is the unique solution to

$$w' = -[\psi(|w|) + \epsilon]$$
$$w(1/2; \epsilon) = 0$$

and is thus independent of λ. The inequality (1.5) is a well-known condition to guarantee that $w(x; \epsilon)$ and hence $\phi(x; \lambda, \epsilon)$ is extendible to $[0, 1]$ for small ϵ. It is immediate that $\phi(x; \lambda, 0) > 0$ for $\lambda > \lambda_0 = \int_{1/2}^{1} w(s; 0)\, ds$ and $\phi(x; \lambda, 0) \to +\infty$ uniformly as $\lambda \to +\infty$.

EXAMPLE 2. Suppose

$$\operatorname{sgn} z \cdot f(t, x, z, p, r) \leqq P(t, x, z, p, r)[A + B|p| + C|z| + r \operatorname{sgn} z]$$

where P is positive and A, B, C are positive constants such that $\Gamma(B, C) > 1/2$, where

$$\Gamma(B, C) = \begin{cases} 2D^{-1/2} \tanh^{-1}(\sqrt{D}/B), & \text{for } D = B^2 - 4C > 0 \\ 2(-D)^{-1/2} \tan^{-1}(\sqrt{-D}/B), & \text{for } D < 0 \\ 1/2B, & \text{for } D = 0. \end{cases}$$

If we let $g(z, p) = A + B|p| + C|z|$ in the corollary, uniqueness of solutions $\phi(x; \lambda, \epsilon)$ and their existence on $[0, 1]$ for all λ and ϵ again follows from well-known theorems. The other hypotheses on $\phi(x; \lambda, 0)$ can be verified by explicitly solving (1.3); cf. [4, Theorem 3.8]. The resulting bound $\phi(x; \lambda_0, 0)$ depends only on A, B, C.

EXAMPLE 3. Suppose

$$\operatorname{sgn} z \cdot f(t, x, z, p, r) \leqq P(t, x, z, p, r)[A + B|p| + C(|z|)|z| + r \operatorname{sgn} z]$$

where P is positive, A and B are positive constants, and $C(s)$ is a positive function of class $C^1[0, \infty)$ with $C'(s) < 0$ and $\lim_{s \to +\infty} C(s) = 0$. With $g(z, p) = A + B|p| + C(|z|)|z|$ the uniqueness of the solutions $\phi(x; \lambda, \epsilon)$ and their existence on $[0, 1]$ for all λ and ϵ are again immediate. The other hypotheses on ϕ are verified by analysis of the representation

$$\phi(x; \lambda, 0) = \lambda - \int_{1/2}^{x} e^{B\sigma} \int_{1/2}^{\sigma} [A + C(\phi(s; \lambda, 0))\phi(s; \lambda, 0)]\, e^{-Bs}\, ds$$

for $\phi > 0$ and $x \geqq 1/2$. The resulting bound $\phi(x; \lambda_0, 0)$ depends only on A, B, and $C(s)$.

EXAMPLE 4. Suppose $f = f(t, x, p, r)$; i.e., f is independent of z, and $f(t, x, p, r)/r$ is a positive continuous function. In this case (1.4) becomes

$$|f(t, x, p, r) - rP(t, x, p, r)| \leqq g(p)P(t, x, p, r).$$

If we take $P = f/r$ and $g = 0$, then the hypotheses of the corollary are clearly satisfied with $\phi(x; \lambda, \epsilon) = -\epsilon(x - 1/2)^2/2 + \lambda$ and $\phi(x; 0, 0) = 0$. Thus $u(t, x) = 0$ is the unique periodic solution.

EXAMPLE 5. Suppose $f = a(t, x, z, p)r$; i.e., (1) has the form $u_t = a(t, x, u, u_x)u_{xx}$, where $a > 0$. By the above remark we may take $g = 0$ again. Again $u(t, x) = 0$ is the only periodic solution.

EXAMPLE 6. Suppose $f = f(r)$; i.e., (1) has the form $u_t = f(u_{xx})$. Let

$$f_e(r) = [f(r) + f(-r)]/2$$

and

$$f_0(r) = [f(r) - f(-r)]/2.$$

Suppose $f_0(r)/r > 0$ for $|r| \geqq 1$ and $|f_e(r)| \leqq Cf_0(r)/r$ for $r \geqq 1$. We take $g(z, p) = C$ and define $P(r)$ so that P is positive and continuous and $P(r) = f_0(r)/r$ for $|r| \geqq 1$. Then (1.4) becomes

$$|f(r) - rP(r)| = |f(r) - f_0(r)| = |f_e(r)| \leqq Cf_0(r)/r = CP.$$

The remaining hypotheses of the corollary are immediate.

REMARK. If we assume in Corollary 1.2 that $g(z, p)$ is non-negative, then $\phi(x; \lambda, 0)$ is concave down and symmetric with respect to $x = 1/2$. It follows that there exists λ_0 such that $\phi(x; \lambda_0, 0) \geqq 0$ on $[0, 1]$ and $\phi(0; \lambda_0, 0) = \phi(1; \lambda_0, 0) = 0$. (Note that the $g(z, p)$ in each of the examples is non-negative.) Since $|u(t, x)| \leqq \phi(x; \lambda_0, 0)$ on $[0, 1]$ we obtain $|u_x(t, 0)| \leqq \phi'(0; \lambda_0, 0)$ and $|u_x(t, 1)| \leqq -\phi'(1; \lambda_0, 0)$. The existence of bounds on $u(t, x)$, $u_x(t, 0)$, and $u_x(t, 1)$ will be important in the next section.

2. **Bounds for u_x.** In this section we restrict our attention to the quasilinear equation

$$u_t = a(t, x, u, u_x)u_{xx} + b(t, x, u, u_x). \tag{3}$$

We assume throughout this section that:

(i) $a, b \in C^1(R \times (0, 1) \times R^2)$,

(ii) $a > 0$, and

(iii) a and b are T-periodic.

LEMMA 2.1. *Let u be a T-periodic solution to* (3) *with* $|u(t, x)| \leqq U$ in $R \times [0, 1]$ *and u_x continuous on* $R \times [0, 1]$. *Suppose* $|b(t, x, z, p)| \leqq Lap^2$ *for* $|z| \leqq U$ *and* $|p| \geqq M$. *Then there exists an* $N > 0$ *depending only on U, L, and M such that*

$$|u_x(t, 0)|, |u_x(t, 1)| \leqq N$$

on R.

PROOF. We first find an upper bound for $u_x(t, 0)$. Let $w = e^{Lu}$. Then

$$w_t = Lu_t e^{Lu},$$
$$w_x = Lu_x e^{Lu},$$
$$w_{xx} = Lu_{xx} e^{Lu} + L^2(u_x)^2 e^{Lu},$$

and we have

$$\begin{aligned} w_t &= Le^{Lu} u_t \\ &= Le^{Lu}(au_{xx} + b) \\ &= aw_{xx} + Le^{Lu}(b - aL(u_x)^2). \end{aligned}$$

Let $y = w + Ce^{-x}$, where $C = LMe^{1+LU}$. Then

$$\begin{aligned} y_t - ay_{xx} &= w_t - aw_{xx} - aCe^{-x} \\ &= Le^{Lu}(b - aLu_x^{\,2}) - aCe^{-x}. \end{aligned}$$

If y has its maximum on $[0, T] \times [0, 1]$ in the parabolic interior at (t_0, x_0), then $y_t(t_0, x_0) - ay_{xx}(t_0, x_0) \geqq 0$. On the other hand,

$$y_x(t_0, x_0) = Lu_x(t_0, x_0)e^{Lu(t_0, x_0)} - Ce^{-x_0} = 0,$$

hence $u_x(t_0, x_0) \geqq M$. But then, $b - aLu_x^{\,2} \leqq 0$ at (t_0, x_0) and $y_T(t_0, x_0) - ay_{xx}(t_0, x_0) < 0$. This contradiction implies that the maximum of $y(t, x)$ occurs at the parabolic boundary of the rectangle $[0, T] \times [0, 1]$. Since y is T-periodic in t, the maximum must in fact occur on $x = 0$ or $x = 1$. We have $y(t, 0) = 1 + C$ and $y(t, 1) = 1 + Ce^{-1}$. Thus the maximum is assumed at every point of $x = 0$. But then

$$\begin{aligned} 0 \geqq y_x(t, 0) &= w_x - C \\ &= Lu_x(t, 0)e^{Lu(t,0)} - C. \end{aligned}$$

Thus $u_x(t, 0) \leqq CL^{-1}$. The other estimates may be obtained by similar arguments.

We now seek bounds on u_x on $R \times [0, 1]$. We begin by obtaining bounds assuming the continuity of u_{xxx}. In this case we can derive a differential equation satisfied by a function involving u_x. In a second step we remove the assumption of additional smoothness by considering the finite difference of u in the x direction. We will employ the following growth restrictions:

Assumption (A). There exist positive constants L, M and μ such that for $|z| \leqq U$ and $|p| \geqq M$,

$$|b|, |b_x| \leqq aLp^2$$
$$|b_z|, |a_x + b_p| \leqq aL|p|$$
$$ap^2 \geqq \mu$$
$$|a_z| \leqq aL$$
$$|a_p| \leqq a\beta, \quad \beta = e^{-2U(3L+2)}/(3L+2).$$

THEOREM 2.2. *Let u be a T-periodic solution to* (3) *with $|u| \leqq U$ in $R \times [0, 1]$. Suppose u_x is continuous in $R \times [0, 1]$ and u_{xxx} is continuous in $R \times (0, 1)$. If the functions a and b satisfy Assumption* (*A*), *then there exists a constant P* (*depending only on U, L, M, and N*) *such that $|u_x| \leqq P$ on $R \times [0, 1]$.*

PROOF. Let $w = u_x + \gamma(u)$ where $\gamma(s)$ is a smooth function to be determined later in the proof. Then w satisfies

$$w_t = aw_{xx} + Aw_x + Bu_x^2 + Cu_x + D, \tag{2.1}$$

where

$$A = a_x + a_z u_x + b_p + a_p(u_{xx} - \gamma' u_x)$$
$$B = a_p(\gamma')^2 - \gamma' a_z - \gamma'' a$$
$$C = b_z - \gamma' a_x - \gamma' b_p$$
$$D = b_x + \gamma' b.$$

Suppose the maximum of w occurs at an interior point (t_m, x_m). Then $w_t = w_x = 0$ and $w_{xx} \leqq 0$ at (t_m, x_m) and

$$0 \leqq -aw_{xx} = Bu_x^2 + Cu_x + D. \tag{2.2}$$

From Assumption (A), for $p > M$,

$$Bp^2 + Cp + D \leqq ap^2[\beta(\gamma')^2 + 3L\gamma' + 2L - \gamma'']. \tag{2.3}$$

If $\gamma(z) = \exp(3L + 2)(z + u)$ and β is chosen as in Assumption (A), then

$$\beta(\gamma')^2 + 3L\gamma' + 2L - \gamma'' < -2$$

for $p > M$. This leads to a contradiction unless we conclude $u_x \leqq M$ at (t_m, x_m). Then

$$w(t, x) \leqq w(t_m, x_m) \leqq M + \gamma(U).$$

If the maximum of w occurs at a boundary point (t_m, x_m), then

$$\begin{aligned} w(t, x) &\leqq w(t_m, x_m) \\ &= u_x(t_m, x_m) + \gamma(u(t_m, x_m)) \\ &\leqq N + \gamma(U) \end{aligned}$$

where N is given by Lemma 2.1. Thus we have an upper bound Q for w on $R \times [0, 1]$. But then $u_x + \gamma(u) \leqq Q$ implies that

$$u_x \leqq Q - \gamma(u) \leqq Q - \min \gamma(u) \leqq Q - \gamma(-U).$$

A lower bound is obtained by observing that $\bar{u} = -u$ satisfies $\bar{u}_t = \bar{a}\bar{u}_{xx} + \bar{b}$ where $\bar{a} = a(t, x, -z, -p)$ and $\bar{b} = -b(t, x, -z, -p)$.

REMARK. The proof of Theorem 2.2 used only $a > 0$ instead of $ap^2 \geqq \mu$.

THEOREM 2.3. *Let u be a T-periodic solution to* (3) *with* $|u| \leqq U$ *in* $R \times [0, 1]$. Suppose u_x, and u_{xx} *are continuous in* $R \times [0, 1]$. *If the functions a and b satisfy Assumption* (A), *then there exists a constant P (depending only on U, L, M, and N) such that* $|u_x| \leqq P$ *on* $R \times [0, 1]$.

PROOF. We consider $w(t, x) = \delta u + \gamma(u)$ where $\delta u = (u^h - u)/h$, $u^h = u(t, x + h)$ and $0 < h < 1$. For $(t, x) \in R \times (0, 1 - h)$, we have

$$\begin{aligned} w_x &= \delta u_x + \gamma' u_x, \\ w_{xx} &= \delta u_{xx} + \gamma' u_{xx} + \gamma'' u_x^2, \end{aligned}$$

and

$$w_t = \delta u_t + \gamma' u_t.$$

Thus

$$w_t = a\delta u_{xx} + u_{xx}^h \delta a + \delta b + \gamma'(a u_{xx} + b).$$

Making substitutions and rearranging,

$$\begin{aligned} w_t = a w_{xx} &+ A_1 w_x + B_1 u_x^2 + B_2 u_x \delta u \\ &+ C_1 u_x + C_2 \delta u + D_1 E_1 (u_{xx}^h - \delta u_x), \end{aligned}$$

where

$$\begin{aligned} A_1 &= \tilde{a}_p u_{xx}^h + \tilde{b}_p + \tilde{a}_x + \tilde{a}_z \delta u - \gamma' a_p u_x \\ B_1 &= -a\gamma'' + \tilde{a}_p (\gamma')^2, B_2 = -\gamma' \tilde{a}_z \\ C_1 &= -\gamma'(\tilde{a}_x + \tilde{b}_p), C_2 = \tilde{b}_z \end{aligned}$$

$$D_1 = \gamma' b + \tilde{b}_x$$

$$E_1 = \tilde{a}_x + \tilde{a}_z \delta u - \gamma' \tilde{a}_p u_x.$$

The superscript h indicates that the function is evaluated at $(t, x + h)$, while $\tilde{g}$ indicates that the function g is evaluated at a mean value between (t, x, u, u_x) and $(t, x + h, u^h, u_x{}^h)$.

If w has its maximum at an interior point (t_m, x_m), then

$$0 \leqq B_1 u_x{}^2 + B_2 u_x \delta u + C_1 u_x + C_2 \delta u + D_1 + E_1(u^h_{xx} - \delta u_x). \tag{2.4}$$

Let $\epsilon > 0$ be arbitrarily given. For h sufficiently small, the expression on the right is within ϵ of $Bu_x{}^2 + Cu_x + D$ appearing in (2.2). But from (2.3),

$$Bp^2 + Cp + D \leqq -2ap^2 < -2a_0 < 0$$

for $|p| \geqq N$ and $|z| \leqq U$. Thus for h sufficiently small, the expression on the right side of (2.4) is negative if $u_x(t_m, x_m) > P$. Thus we conclude that $u_x(t_m, x_m) \leqq P$. Arguing as in the proof of Theorem 2.2 we obtain a bound on the difference δu independent of h, and hence a bound on u_x.

REMARK. In the proofs of Theorem 2.2 and 2.3 we only needed to have $a, b,$ and $\bar{a}, \bar{b}$ satisfy one of the following one-sided growth restrictions in place of Assumption (A).

Assumption (B). $(|z| \leqq U, p \geqq N)$

$$b, b_x \leqq aLp^2$$

$$b_z, -(a_x + b_p) \leqq aLp$$

$$-a_z \leqq aL$$

$$a_p \leqq ae^{-2U(3L+2)}/(3L + 2).$$

Assumption (C). Same as (B) but with $-a_z, -(ax + b_p), b$ replaced by $a_z, a_x + b_p, -b$.

The same techniques can be used to obtain bounds on u_t and u_{xx}.

THEOREM 2.4. *Let u be a T-periodic solution to* (3) *with $|u| \leqq U$ and $|u_x| \leqq P$ in $R \times [0, 1]$. Suppose $u, u_t, u_x,$ and u_{xx} are continuous on $R \times [0, 1]$. Then there exists a constant V (depending only on $U, P, \mu,$ and M, where M is a bound on a and b and their first derivatives on* $R \times [0, 1] \times [-U, U] \times [-P, P]$) *such that $|u_t|, |u_{xx}| \leqq V$ on $R \times [0, 1]$.*

PROOF. Again the computations are clearer if we begin by assuming the existence of continuous higher order derivatives. If we let $w(t, x) = u_t + \gamma(u_x)$, then w satisfies

$$w_t = aw_{xx} + Aw_x + Bu_{xx}^2 + Cu_{xx} + D,$$

where

$$\begin{aligned} A &= a_p u_{xx} + b_p \\ B &= a(a_z - \gamma'') \\ C &= a_t + a_z b + b_z a + \gamma'(a_x + a_z u_x) \\ D &= b_t + b_z b + \gamma'(b_x + b_z u_x). \end{aligned} \tag{2.5}$$

Suppose w attains its maximum at (t_m, x_m). At (t_m, x_m),

$$0 \leqq Bu_{xx}^2 + Cu_{xx} + D. \tag{2.6}$$

We have

$$Br^2 + Cr + D = a(a_z - \gamma'')r^2 + (a_t + a_z b + b_z a)r + \gamma'(a_x + a_z u_z)r.$$

If we take $\gamma(p) = Mp^2$, then

$$a(a_z - \gamma'') = a(a_z - 2M) < -\mu M. \tag{2.7}$$

Thus for r sufficiently large $Br^2 + Cr + D < 0$. Thus (2.6) implies that $u_{xx}(t_m, x_m) \leqq V_0$. Then $u_t(t_m, x_m) \leqq au_{xx}(t_m, x_m) + b \leqq MV_0 + M$. Thus $w(t_m, x_m) \leqq MV_0 + M + \gamma(P)$.

Since $u_t(t, 0) = u_t(t, 1) = 0$, we also have a bound for w at boundary points. Thus we have an upper bound Q for $w(t, x)$ on $R \times [0, 1]$. Thus

$$u_t = w - \gamma(u_x) \leqq Q - \gamma(0) = Q.$$

A lower bound for u_t may be obtained using $\gamma(P) = -Mp^2$ and considering the minimum of w. Bounds on u_{xx} are obtained from (3).

To remove the assumption of the existence of higher order derivatives we consider $w = \delta_t u + \gamma(\delta u)$ where $\delta_t u = (u^k - u)/k, u^k = u(t + k, x)$, and δu is defined as before. We have for $(t, x) \in R \times [0, 1 - h]$,

$$\begin{aligned} w_x &= \delta_t u_x + \gamma'\delta u_x \\ w_{xx} &= \delta_t u_{xx} + \gamma'\delta u_{xx} + \gamma''(\delta u_x)^2 \\ w_t &= \delta_t u_t + \gamma'\delta u_t = \delta_t(au_{xx} + b) + \gamma'\delta(au_{xx} + b). \end{aligned}$$

Using the mean value theorem we obtain

$$\delta_t(au_{xx}) = a\delta_t u_{xx} + \overset{k}{u}_{xx}\,\delta_t a,$$

$$\delta_t a = a_t{}' + a_z{}'\delta_t u + a_p{}'\delta_t u_x,$$

$$\delta_t b = b_t{}' + b_z{}'\delta_t u + b_p{}'\delta_t u_x,$$

where the prime denotes evaluation of the function at a point between (t, x, u, u_x) and $(t + k, x, u^k, u_x{}^k)$. Similarly,

$$\delta(au_{xx}) = a\delta u_{xx} + u_{xx}(a_x{}'' + a_z{}''\delta u + a_p{}''\delta u_x)$$

where the two primes denote evaluation between (t, x, u, u_x) and $(t, x + h, u^h, u_x{}^h)$. Moreover, $\delta_t u = \tilde{a}\tilde{u}_{xx} + \tilde{b} = \tilde{u}_t$, where the $\tilde{}$ denotes evaluation at a point between (t, x) and $(t + k, x)$. Making use of these equations we can show that

$$w_t = aw_{xx} + A'w_x + Bu_{xx}^2 + Cu_x{}^2 + D + O(1)$$

as $h \to 0$ where $A' = a_p{}'u_{xx}^k + b_p{}'$ and B, C, and D are defined in (2.5). The proof can then be completed by the same argument as in the first part of the proof.

As a consequence of Theorem 2.4 we have the following standard result which we state for later reference.

THEOREM 2.5. *Under the hypotheses of Theorem* 2.4, *there exists* $H > 0$ *depending only on* U, P, μ, *and* M *such that*

$$|u_x(t, x) - u_x(t', x')| \leqq H[|x - x'| + |t - t'|^{1/2}]$$

for (t, x) *and* (t', x') *in* $R \times [0, 1]$.

PROOF. This bound follows directly from the bounds on u_{xx} and u_t. For a proof see [5] (Lemma 3.1).

3. **Existence.** For $0 < \alpha < 1$, let

$$|u|_0{}^S = \sup_S |u(t, x)|$$

$$|u|_\alpha{}^S = |u|_0{}^S + \sup_{(t,x),(t',x')\in S} \frac{|u(t, x) - u(t', x')|}{[|x - x'|^2 + |t - t'|]^{\alpha/2}}$$

$$|u|_{1+\alpha}^S = |u|_0{}^S + |u_x|_\alpha{}^S$$

$$|u|_{2+\alpha}^S = |u|_0{}^S + |u_{xx}|_\alpha{}^S + |u_t|_\alpha{}^S.$$

Then $C^0(S)$, $C^\alpha(S)$, $C^{1+\alpha}(S)$, $C^{2+\alpha}(S)$ are spaces of continuous functions on S for which the corresponding norms are finite.

We will need some results of Fife [2] for the linear equation

$$u_t = a(t,x)u_{xx} + b_1(t,x)u_x + c(t,x)u + b(t,x) \tag{3.1}$$

which we state in a version appropriate for our purposes.

THEOREM 3.1. (FIFE) *Let* $S = R \times [0,1]$. *Suppose*

(i) $a, b_1, c, b \in C^\alpha(S)$,
(ii) $a \geqq \mu > 0$, (μ *constant*),
(iii) a, b_1, c, b *are T-periodic, and*
(iv) $c \leqq 0$.

There is a unique T-periodic function $u(t,x) \in C^{2+\beta}(S)(\beta < \alpha)$ *satisfying* (3.1) *on* $R \times (0,1)$ *and* (2). *Furthermore* $|u|^S_{2+\beta} \leqq C|b|_\alpha{}^S$ *where C depends only on* μ *and* $K = \max[|a|_\alpha{}^S, |b_1|_\alpha{}^S, |c|_\alpha{}^S]$.

PROOF. Let a, b_1, c, b be extended to $\bar a, \bar b_1, \bar c, \bar b$ on $S(\delta) = R \times (-\delta, 1+\delta)$ in such a way that $\bar a, \bar b_1, \bar c$, and $\bar b$ are T-periodic; $\bar c \leqq 0$, and $|\bar a|_\alpha{}^{S(\delta)}$; $|\bar b_1|_\alpha{}^{S(\delta)}$, $|\bar c|_\alpha{}^{S(\delta)}$, and $|\bar b|_\alpha{}^{S(\delta)}$ are equal to the corresponding norms on S. Let $\{\delta_n\}$ be a sequence such that $0 < \delta_n < \delta$ and $\delta_n \to 0$. By Theorem 5.2 in [2], for each n there is a unique T-periodic solution $u_n \in C^{2+\alpha}(S(\delta_n))$ of

$$u_t = \bar a u_{xx} + \bar b_1 u_x + \bar c u + \bar b$$

satisfying $u_n(t, -\delta_n) = u_n(t, 1+\delta_n) = 0$. Moreover,

$$|u_n|_{2+\alpha}^{S(\delta_n)} \leqq \bar C |b|_\alpha{}^{S(\delta_n)},$$

where the constant $\bar C$ can be chosen independent of n. If $\beta < \alpha$, the sequence $\{u_n\}$ has a subsequence converging to a function $u_0(t,x) \in C^{2+\beta}(S)$. It can be shown that $u_0(t,x)$ is a solution to (3.1) on $R \times (0,1)$ by standard arguments. Moreover, it is easily seen that $u_0(t,x)$ is T-periodic.

We have

$$|u_n(t,0)| = |u_n(t,-\delta_n) + u_{nx}(t,\xi_n)\delta_n| \leqq \bar C \delta_n.$$

Thus $u_n(t,0) \to 0$ and we must have $u_0(t,0) = 0$. Similarly, $u_0(t,1) = 0$.

Uniqueness follows from Theorem 5.1 in [2].

LEMMA 3.2. *Suppose* $S = R \times [0,1]$. *Let* $T: C^{1+\alpha}(S) \to C^\alpha(S)$ *be defined by* $Tv = f(t,x,v,v_x)$ *where* $f: C^1(R \times [0,1] \times R^2)$. *Then T is continuous.*

PROOF. We have

$$
\begin{aligned}
&[f(t_1, x_1, v_1(t_1, x_1), v_{1x}(t_1, x_1)) - f(t_1, x_1, v_2(t_1, x_1), v_{2x}(t_1, x_1))] \\
&- [f(t_2, x_2, v_1(t_2, x_2), v_{1x}(t_2, x_2)) - f(t_2, x_2, v_2(t_2, x_2), v_{2x}(t_2, x_2))] \\
&= \int_0^1 [f_t(P_1(\theta))(t_1 - t_2) + f_x(P_1(\theta))(x_1 - x_2) + f_z(P_1(\theta))(v_1(t_1, x_1) \\
&- v_1(t_2, x_2)) + f_p(P_1(\theta))(v_{1x}(t_1, x_1) - v_{1x}(t_2, x_2))]\, d\theta \\
&- \int_0^1 [f_t(P_2(\theta))(t_1 - t_2) + f_x(P_2(\theta))(x_1 - x_2) + f_z(P_2(\theta))(v_2(t_1, x_1) \\
&- v_2(t_2, x_2)) + f_p(P_2(\theta))(v_{2x}(t_1, x_1)) - v_{2x}(t_2, x_2))]\, d\theta,
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
P_1(\theta) = (&\theta t_1 + (1 - \theta)t_2, \theta x_1 + (1 - \theta)x_2, \\
&\theta v_1(t_1, x_1) + (1 - \theta)v_1(t_2, x_2), \\
&\theta v_{1x}(t_1, x_1) + (1 - \theta)v_{1x}(t_2, x_2)),
\end{aligned}
$$

and

$$
\begin{aligned}
P_2(\theta) = (&\theta t_1 + (1 - \theta)t_2, \theta x_1 + (1 - \theta)x_2, \\
&\theta v_2(t_1, x_1) + (1 - \theta)v_2(t_2, x_2), \\
&\theta v_{2x}(t_1, x_1) + (1 - \theta)v_{2x}(t_2, x_2)).
\end{aligned}
$$

Note that as $\|v_1 - v_2\|_{1+\alpha} \to 0$, $P_1(\theta) \to P_2(\theta)$ uniformly. The difference on the right-hand side of (3.2) may be rewritten as

$$
\begin{aligned}
&\int_0^1 (f_t(P_1(\theta)) - f_t(P_2(\theta)))\, d\theta (t_1 - t_2) \\
&+ \int_0^1 (f_x(P_1(\theta)) - f_x(P_2(\theta)))\, d\theta (x_1 - x_2) \\
&+ \int_0^1 f_z(P_1(\theta))\, d\theta [v_1(t_1, x_1) - v_2(t_1, x_1) - (v_1(t_2, x_2) - v_2(t_2, x_2))] \\
&+ \int_0^1 f_p(P_1(\theta))\, d\theta [v_{1x}(t_1, x_1) - v_{2x}(t_1, x_1) - (v_{1x}(t_2, x_2) - v_{2x}(t_2, x_2))] \\
&+ \int_0^1 (f_z(P_1(\theta)) - f_z(P_2(\theta)))\, d\theta (v_2(t_1, x_1) - v_2(t_2, x_2)) \\
&+ \int_0^1 (f_p(P_1(\theta)) - f_p(P_2(\theta)))\, d\theta (v_{2x}(t_1, x_1) - v_{2x}(t_2, x_2)).
\end{aligned}
$$

This entire sum may be bounded by

$$M(v_1, v_2)[|t_1 - t_2| + |x_1 - x_2|^2]^{\alpha/2},$$

where $M(v_1, v_2) \to 0$ as $\|v_1 - v_2\|_{1+\alpha} \to 0$. Thus it can be shown that

$$\|f(t, x, v_1, v_{1x}) - f(t, x, v_2, v_{2x})\|_\alpha \to 0$$

as $\|v_1 - v_2\|_{1+\alpha} \to 0$.

THEOREM 3.3. *Assume*

(i) $a, b \in C^1(R \times [0,1] \times R^2)$

(ii) $a > 0$ *on* $R \times [0,1] \times R^2$, *and*

(iii) a, b *are periodic in* t *of period* T.

Suppose there exist constants U *and* P *such that for any* $0 \leqq \lambda \leqq 1$, *if* $u \in C^{2+\beta}(S)$ $(S = R \times [0,1], 0 < \beta < 1)$ *is a* T*-periodic function which satisfies* (2) *and*

$$u_t = a(t, x, \lambda u, \lambda u_x)u_{xx} + b(t, x, \lambda u, \lambda u_x) \tag{3λ}$$

on $R \times (0,1)$, *then* $|u|_0^S \leqq U$ *and* $|u_x|_0^S \leqq P$. *Then there is at least one* T*-periodic solution* $u \in C^{2+\beta}(S)$ *to* (3) − (2).

PROOF. The proof is by standard Leray-Schauder arguments. Let

$$D_{1+\alpha} = \{v \in C^{1+\alpha}(S) : v(t + T, x) = v(t, x)\},$$

where $0 < \beta < \alpha < 1$. Define $\psi_\lambda : D_{1+\alpha} \to D_{1+\alpha}$ for $0 \leqq \lambda \leqq 1$ by $\psi_\lambda v = u$, where u is the unique T-periodic solution to

$$u_t = a(t, x, \lambda v, \lambda v_x)u_{xx} + b(t, x, \lambda v, \lambda v_x)$$

satisfying (2) whose existence is guaranteed by Theorem 3.1. Moreover,

$$|u|_{2+\beta}^S \leqq C|b(t, x, \lambda v, \lambda v_x)|_\alpha, \tag{3.3}$$

where C depends on the α norm of $a(t, x, \lambda v, \lambda v_x)$ and

$$\mu = \min_{[0,T] \times [0,1]} a(t, x, \lambda v, \lambda v_x).$$

The inequality (3.3) implies that ψ_λ is compact for each λ.

To show that ψ_λ is continuous on $D_{1+\alpha}$ for fixed λ, we see that $w = \psi_\lambda v_1 - \psi_\lambda v_2$ satisfies

$$\begin{aligned} a(t, x, \lambda v_1, \lambda v_{1x})w_{xx} - w_t & \\ &= -[b(t, x, \lambda v_1, \lambda v_{1x}) - b(t, x, \lambda v_2, \lambda v_{2x})] \\ &\quad - [a(t, x, \lambda v_1, \lambda v_{1x}) - a(t, x, \lambda v_2, \lambda v_{2x})]\, u_{2xx} \\ &\equiv g(v_1, v_2, x, t). \end{aligned}$$

We have

$$|g(v_1, v_2, x, t)|_\beta \leqq |b(t, x, \lambda v_1, \lambda v_{1x}) - b(t, x, \lambda v_2, \lambda v_{2x})|_\beta$$
$$+ |a(t, x, \lambda v_1, \lambda v_{1x}) - a(t, x, \lambda v_2, \lambda v_{2x})|_\beta \cdot |u_{2xx}|_\beta.$$

If $|v_1 - v_2|_{1+\alpha} \to 0$, then $|v_1 - v_2|_{1+\beta} \to 0$. By Lemma 3.1, $|g|_\beta \to 0$. By Theorem 3.1, for any $0 < \alpha_1 < \beta$, $|w|_{2+\alpha_1} \leqq C|g|_\beta$ which implies $|w|_{2+\alpha_1} \to 0$ as $|v_1 - v_2|_{1+\alpha} \to 0$. But then $|w|_{1+\alpha} \to 0$ as $|v_1 - v_2|_{1+\alpha} \to 0$.

The continuity of ψ_λ in λ (uniformly on bounded subsets of $S_{1+\alpha}$) can be proved by similar arguments.

By our hypotheses if u is a fixed point of ψ_λ, then $|u|_0^S \leqq U$ and $|u_x|_0^S \leqq P$. By Theorem 2.4 and 2.5 we have a bound on $|u|_{1+1/2}^S$. But then by Theorem 3.1 again, we have a bound on $|u|_{2+\beta_1}^S$ for $\beta_1 < 1/2$. Hence we have a bound on $|u|_{1+\alpha}^S$ independently of λ.

For $\lambda = 0$, $\psi_0 v = u_0$ where u_0 is the unique T-periodic solution to

$$a(x, t, 0, 0)u_{xx} - u_t = -b(x, t, 0, 0).$$

Thus $\deg[I - \psi_0, B, 0] \neq 0$, where B is an appropriate ball centered at 0. By Leray-Schauder degree theory, ψ_1 has at least one fixed point.

Combining the results of § 1 and § 2 with Theorem 3.3 we have the following existence theorem.

THEOREM 3.4. *Suppose*
(i) $a, b \in C^1(R \times [0, 1] \times R^2)$,
(ii) $a > 0$ *on* $R \times [0, 1] \times R^2$,
(iii) a, b *are* T-*periodic*,
(iv) $\operatorname{sgn} z \cdot b/a \leqq g(|z|, |p|)$, *where* $g(z, p)$ *satisfies the hypotheses of Corollary* 1.2, *and*
(v) *Assumption* (A) *is satisfied.*

Then there is at least one T-*periodic solution* $u \in C^{2+\beta}(R \times [0, 1])$ *to* (3) – (2).

REMARK 1. Recall that (iv) may be replaced by any of the more concrete hypotheses in Examples 1–6 in § 1.

REMARK 2. Recall that Assumption (A) may be replaced by the one-sided conditions (B) or (C).

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