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p-VALENT CLASSES RELATED TO FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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ABSTRACT. Let $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$ be analytic in $U:\{z : |z| < 1\}$. We say f belongs to the class $V_k^\lambda(p, q)$ $(k \geqq 2, |\lambda| < \pi/2, p \geqq q)$ if f has $(p - 1)$ critical points in U and for r sufficiently close to 1,

$$\overline{\lim_{r\to 1^-}} \int_0^{2\pi} \left| \operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right)\right\}\right| \leqq kp\pi \quad \cos \lambda.$$

For $k = 2$, we have the class of p-valent functions f for which zf' is λ-spiral-like in U. We obtain representation theorems for these classes which lead to distortion and rotation theorems. For $V_k^\lambda(p, p)$ bounds for $|a_{p+1}|$ and $|a_{p+2}|$ are determined. These results are sharp.

1. **Preliminaries.** Let A_q $(q \geqq 1)$ denote the class of functions $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$ which are analytic in $U : \{z : |z| < 1\}$.

For $f \in A_q$, we say that f belongs to the class $V_k^\lambda(p, q)(k \geqq 2, |\lambda| < \pi/2, p$ an integer, $p \geqq q)$ if for r sufficiently close to 1,

$$\int_0^{2\pi} \operatorname{Re} \left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta = 2p\pi \tag{1}$$

and

$$\overline{\lim_{r\to 1^-}} \int_0^{2\pi} \left| \operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right)\right\}\right| d\theta \leqq kp\pi \quad \cos \lambda. \tag{2}$$

Condition (1) implies that f has $(p - 1)$ critical points in U. From (2) with $k = 2$, we can show that $V_2^\lambda(p, q)$ is the class of p-valent functions satisfying

$$\operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \qquad (z \in U),$$

i.e., the class of functions f for which zf' is λ-spiral-like in U. For $p = 1$, this class was introduced by Robertson [11].

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From $V_k^\lambda(p, q)$, many other interesting subclasses can be obtained. For $p = q = 1, \lambda = 0$, $V_k^0(1, 1)$ is the class of functions of bounded boundary rotation that was introduced by Löwner [7] and Paatero [10]. For $p = q = 1$, $V_k^\lambda(1,1)$ is the class of functions investigated by Moulis [8]. The class $V_k^0(p, q)$ was recently studied by Leach [5].

Although the functions in the class $V_k^\lambda(p, q)$ need no longer have bounded boundary rotation, they have other interesting properties. In this paper, we initiate the investigation of some of these properties. We begin our study by obtaining representation theorems for the class $V_k^\lambda(p, q)$. Then we define a larger class, $\Phi_k^\lambda(p, q)$ which enables us to obtain distortion and rotation theorems.

2. **Representation Theorems for $V_k^\lambda(p, q)$.** We will need to use the functions

$$\zeta(z, \alpha_j) = [(1 - z/\alpha_j)(1 - \bar{\alpha}_j z)]^{e^{-i\lambda}\cos\lambda} \tag{3}$$

For $\lambda = 0, z^{-1}\zeta(z, \alpha_j)$ are the functions used by Bender [1], Goluzin [2] and Hummel [4].

Lemma 1. *Let $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \in V_k^\lambda(p, q)$ have non-zero critical points $\alpha_1, \cdots, \alpha_{p-q}$, counting multiplicities. If*

$$F'(z) = (p/q)f'(z)z^{p-q}\prod_{j=1}^{p-q}[\zeta(z, \alpha_j)]^{-1}, \tag{4}$$

then $F(z) \in V_k^\lambda(p, p)$.

Proof. Logarithmic differentiation of (4) leads to

$$\begin{aligned} e^{i\lambda}\left(1 + \frac{zF''(z)}{F'(z)}\right) &= e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right) + (p - q)e^{i\lambda} \\ &\quad - \cos\lambda \sum_{j=1}^{p-q}\left(1 - \frac{\alpha_j - \bar{\alpha}_j z^2}{\alpha_j + \bar{\alpha}_j z^2 - (1 + |\alpha_j|^2)z}\right). \end{aligned} \tag{5}$$

For $z = e^{i\theta}, 0 \leqq \theta < 2\pi$, we have

$$\operatorname{Re}\left\{\frac{\alpha_j - \bar{\alpha}_j e^{i2\theta}}{\alpha_j + \bar{\alpha}_j e^{i2\theta} - (1 + |\alpha_j|^2)e^{i\theta}}\right\} = 0. \tag{6}$$

From (5) and (6), it follows that

$$\operatorname{Re}\left\{e^{i\lambda}\left(1 + \frac{zF''(z)}{F'(z)}\right)\right\} = \operatorname{Re}\left\{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \quad (|z| = 1).$$

Therefore, given $\epsilon > 0$ there exists $r_0, 0 < r_0 < 1$ such that for $r_0 < r < 1, |z| = r$,

$$\int_0^{2\pi} \left| \operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{zF''(z)}{F'(z)} \right) \right\} \right| d\theta$$

$$\leqq \int_0^{2\pi} \left| \operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| d\theta + \epsilon.$$

From (2), it follows that

$$\overline{\lim_{r\to 1^-}} \int_0^{2\pi} \left| \operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{zF''(z)}{F'(z)} \right) \right\} \right| d\theta \leqq pk\pi \cos\lambda + \epsilon.$$

Since ϵ was arbitrary, this proves the lemma.

As a straightforward application of Lemma 1, we have the following Representation Theorem.

THEOREM 1. *Let* $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \in V_k^{\lambda}(p, q)$ *have non-zero critical points* $\alpha_1, \cdots, \alpha_{p-q}$ *counting multiplicities. Then*

$$f'(z) = (q/p)F'(z)z^{q-p} \prod_{j=1}^{p-q} \zeta(z, \alpha_j) \tag{7}$$

where $F(z) \in V_k^{\lambda}(p, p)$.

Theorem 1 suggests that a closer examination of the classes $V_k^{\lambda}(p, p)$ would supply more information about $V_k^{\lambda}(p, q)$ $(p > q)$.

Our next goal is to obtain a representation theorem for the elements of $V_k^{\lambda}(p, p)$ $(p \geqq 1)$ in terms of functions of bounded variation.

DEFINITION. For an integer k, $k \geqq 2$, let M_k denote the class of real-valued functions m of bounded variation on $[0, 2\pi]$ which satisfy $\int_0^{2\pi} dm(t) = 2$ and $\int_0^{2\pi} |dm(t)| \leqq k$.

The class M_k was used by Paatero [10] to characterize the elements of $V_k^0(1, 1)$. Namely, he proved:

LEMMA A. *If* $g \in V_k^0(1,1)$, *then we can write*

$$g'(z) = \exp\left(-\int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t) \right)$$

for some $m \in M_k$.

The result of Lemma A was extended by Moulis [8] to the class $V_k^{\lambda}(1, 1)$ by use of:

LEMMA B. *The function* $h \in V_k^{\lambda}(1, 1)$ *if and only if there exists* $g \in V_k^0(1, 1)$ *such that*

$$h'(z) = [g'(z)]^{e^{-i\lambda}\cos\lambda}.$$

The final result necessary for our characterization of the elements of $V_k^\lambda(p, p)$ in terms of elements of M_k is:

LEMMA 2. *The function* $f \in V_k^\lambda(p, p)$, $p \geqq 1$, *if and only if* $f'(z) = pz^{p-1}[h'(z)]^p$ *for some* $h \in V_k^\lambda(1, 1)$.

PROOF. Let $f'(z) = pz^{p-1}[h'(z)]^p$ for $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in U$. By direct computation, we obtain

$$\int_0^{2\pi} \left| \operatorname{Re}\left\{ e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| d\theta$$

$$= \int_0^{2\pi} \left| \operatorname{Re}\left\{ e^{i\lambda}\left(1 + \frac{zh''(z)}{h'(z)}\right)\right\}\right| d\theta$$

and the result follows from (2).

THEOREM 2. *If* $f \in V_k^\lambda(p, p)$ $(p \geqq 1)$, *then we can write*

$$f'(z) = pz^{p-1}\exp\left\{ -pe^{-i\lambda}\cos\lambda \int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t) \right\} \tag{8}$$

for some $m \in M_k$.

PROOF. For $f \in V_k^\lambda(p, p)$ let $h \in V_k^\lambda(1, 1)$, $g \in V_k^0(1,1)$ and $m \in M_k$ be the functions given by Lemma 2, Lemma B and Lemma A, respectively. The result follows upon direct substitution.

An immediate application of Lemma 2, is:

LEMMA 3. *The function* $f \in V_k^\lambda(p, p)$ $(p \geqq 1)$ *if and only if there are two starlike functions* s_1 *and* s_2 *normalized by* $s_j(0) = 0$, $s_j'(0) = 1$ $(j = 1, 2)$ *such that*

$$f'(z) = pz^{p-1}\left\{ \frac{[s_1(z)/z]^{(k+2)/4}}{[s_2(z)/z]^{(k-2)/4}} \right\}^{pe^{-i\lambda}\cos\lambda} \tag{9}$$

PROOF. It is known [8], that $h \in V_k^\lambda(1, 1)$ if and only if

$$h'(z) = \left\{ \frac{[s_1(z)/z]^{(k+2)/4}}{[s_2(z)/z]^{(k-2)/4}} \right\}^{e^{-i\lambda}\cos\lambda}$$

The result follows from Lemma 2.

A consequence of Theorems 1, 2, and Lemma 3 is the following representation theorem for $V_k^\lambda(p, q)$.

THEOREM 3. *Let* $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \in V_k^\lambda(p, q)$, $(p > q)$, *have non-zero critical points* $\alpha_1, \cdots, \alpha_{p-q}$ *counting multiplicities. Then*

(i) *for some* $m \in M_k$, *we can write*

$$f'(z) = qz^{q-1} \prod_{j=1}^{p-q} \zeta(z, \alpha_j) \cdot \exp\left\{ -pe^{-i\lambda}\cos\lambda \int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t) \right\}, \tag{10}$$ *and*

(ii) *there are two starlike functions* s_1 *and* s_2 *normalized by* $s_j(0) = 0, s_j'(0) = 1$ $(j = 1, 2)$ *such that*

$$f'(z) = qz^{q-1} \prod_{j=1}^{p-q} \zeta(z, \alpha_j) \left\{ \frac{[s_1(z)/z]^{(k+2)/4}}{[s_2(z)/z]^{(k-2)/4}} \right\}^{pe^{-i\lambda}\cos\lambda}. \tag{11}$$

PROOF. From Theorem 1, we may write

$$f'(z) = (q/p)F'(z)z^{q-p} \prod_{j=1}^{p-q} \zeta(z, \alpha_j), \tag{12}$$

where $F \in V_k^\lambda(p, p)$. Applying Theorem 2 to $F \in V_k^\lambda(p, p)$, there exists an $m \in M_k$ such that

$$F'(z) = pz^{p-1}\exp\left\{ -pe^{-i\lambda}\cos\lambda \int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t) \right\}. \tag{13}$$

We obtain (9) by substituting (13) into (12). Equation (11) follows from (12) and (9).

The determination of the coefficient bounds for the class $V_k^\lambda(p, q)$ is an open problem. However, we can obtain the bounds on the modulus of the second and third coefficients for functions in $V_k^\lambda(p, p)$. To do this, we recall that for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_k^\lambda(1, 1)$, it is known [12] that $|a_2| \leqq (k/2)\cos\lambda$. Further, by a slight modification of an argument used by Lehto [6] we can show that $|a_3] \leqq (1/6)\{(k^2 + 4)\cos^2\lambda + k|\sin\lambda|\cos\lambda\}$.

THEOREM 4. *If* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in V_k^\lambda(p, p)$, *then*

$$(p + 1)|a_{p+1}| \leqq p^2 k\cos\lambda,$$

$$(p + 2)|a_{p+2}| \leqq ((1/2)pk^2 + 2)p^2\cos^2\lambda + (k/2)p^2|\sin\lambda|\cos\lambda.$$

These results are sharp with equality for $f'(z) = pz^{p-1}[F'(z)]^p$, *where*

$$F'(z) = \left\{ \frac{(1+\epsilon z)^{k/2-1}}{(1-\epsilon z)^{k/2+1}} \right\}^{e^{-i\lambda}\cos\lambda}, \quad |\epsilon| = 1.$$

PROOF. By Lemma 2, there exists an $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k^\lambda(1, 1)$ such that

$$\begin{aligned} f'(z) &= pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1} \\ &= pz^{p-1} \left[1 + \sum_{n=2}^{\infty} nb_n z^{n-1}\right]^p. \end{aligned} \tag{14}$$

Expanding the right hand side of (14), we obtain

$$f'(z) = pz^{p-1} + 2p^2 b_2 z^p + p(3pb_3 + 2p(p-1)b_2^2)z^{p+1} + \cdots. \tag{15}$$

Equating coefficients from (14) and (15), we have

$$(p+1)a_{p+1} = 2p^2 b_2,$$
$$(p+2)a_{p+2} = p(3pb_3 + 2p(p-1)b_2^2).$$

The result follows from the known bounds on $|b_2|$ and $|b_3|$.

3. **The Classes $\Phi_k^\lambda(p, q)$.** The importance of the class M_k to $V_k^\lambda(p, q)$ $(p \geqq q)$, demonstrated by Theorems 2 and 3, motivates the following:

DEFINITION. We say $\varphi(z) = z^q + \sum_{n=q+1}^{\infty} b_n z^n \in \Phi_k^\lambda(p, q)$ if φ' has a representation in U given by

$$\varphi'(z) = \begin{cases} pz^{p-1}\exp\left\{-pe^{-i\lambda}\cos\lambda \int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t)\right\}, \\ \qquad\qquad \text{for } p = q, \\ qz^{q-1} \prod_{j=1}^{p-q} \zeta(z, \alpha_j)\exp\left\{-pe^{-i\lambda}\cos\lambda \right. \\ \qquad \left. \cdot \int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t)\right\}, \text{for } p > q, \end{cases}$$

where $0 < |\alpha_j| < 1$, $\varphi'(\alpha_j) = 0$, $j = 1, 2, \cdots, (p-q)$ and $m \in M_k$.

To summarize the relationship between $V_k^\lambda(p, q)$ and $\Phi_k^\lambda(p, q)$, we have

THEOREM 5. *The classes* $V_k^\lambda(p, p)$ *and* $\Phi_k^\lambda(p, p)$ *are equivalent for all positive integers* p. *Further,* $V_k^\lambda(p, q) \subset \Phi_k^\lambda(p, q)$, *and if* $f \in \Phi_k^\lambda(p, q)$, *then* $f(z)$ *has* $(p - 1)$ *critical points in* $|z| < 1$ *and* $f(z)$ *is the limit of a sequence of functions from* $V_k^\lambda(p, q)$ *where the convergence is uniform on the interior of* U.

PROOF. In view of Theorems 2 and 3, only the final statement remains to be proved. Suppose $f \in \Phi_k^\lambda(p, q)$ satisfies

$$f'(z) = qz^{q-1} \prod_{j=1}^{p-q} \zeta(z, \alpha_j)\exp \left\{ - \mathrm{pe}^{-i\lambda}\cos\lambda \cdot \int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t) \right\}$$

for $m \in M_k$. Let

$$F'(z) = pz^{p-1}\exp\left\{-pe^{i\lambda}\cos\lambda \int_0^{2\pi} \log(1 - e^{-it}z)\, dm(t)\right\}. \tag{16}$$

Consider the sequence of functions $\{f_n\}$ defined by

$$f_n'(z) = (q/p)F'(z/t)z^{q-p} \prod_{j=1}^{p-q} \zeta(z, \alpha_j) \quad (t = 1 + 1/n),$$

where $F_t(z) = F(z/t)$ is given by (16). For sufficiently large n, $F_t \in V_k^\lambda(p, p)$ and $f_n \in V_k^\lambda(p, q)$. Since f_n converges uniformly in the interior of U to f, it follows [9, p. 146] that f has $(p - 1)$ critical points.

REMARK. The containment in Theorem 5 is proper. Consider $m \in M_k$, a piecewise constant function having t distinct jumps equal to $1/t$ at $\theta_j \in [0, 2\pi]$. Let

$$\varphi_t'(z) = qz^{q-1} \prod_{j=1}^{p-q} \zeta(z, \alpha_j) \prod_{j=1}^{t} (1 - e^{-i\theta_j}z)^{-(pe^{-i\lambda}\cos\lambda)/t}$$

where $p > q \geqq 1$ and $0 < |\alpha_j| < 1$. The functions $\varphi_t(z)$ $(t = 1, 2, \cdots)$ are in the class $\Phi_k^\lambda(p, q)$ but are not in $V_k^\lambda(p, q)$ for carefully selected α_j and $\theta_j, j = 1, 2, \cdots t$.

The main advantage afforded to us by the class $\Phi_k^\lambda(p, q)$ is that its elements have explicit representations in terms of analytic functions that contain a Stieltjes integral of the form

$$\int_0^{2\pi} g(z, t)\, dm(t), \tag{17}$$

where $m \in M_k$. It has been shown [3] that extremal problems for classes having representations (17) with $m \in M_2$ may be solved by determining the m corresponding to the extremum. In a previous paper, the author [12] used Goluzin's variation technique on the class $V_k^\lambda(1, 1)$ to obtain

LEMMA C. *Let $\zeta \neq 0$ be a given point in U, and let $F(x_1, x_2, \cdots, x_{n+1})$ be analytic in a neighborhood of each point $F(f'(\zeta), \cdots, f^n(\zeta), \zeta)$, $f \in \Phi_k^\lambda(1, 1)$. The functional $J(f') = \operatorname{Re} F(f'(\zeta), \cdots, f^n(\zeta), \zeta)$ attains its maximum (minimum) in $\Phi_k^\lambda(1, 1)$ only for a function of the form*

$$f'(z) = \prod_{j=1}^{M} (1 - \epsilon_j z)^{\gamma_j e^{-i\lambda}\cos\lambda} \prod_{j=1}^{N} (1 - e_j z)^{-\beta_j e^{-i\lambda}\cos\lambda},$$

where $M \leqq n, N \leqq n, |\epsilon_j| = |e_j| = 1$, $\sum_{j=1}^{M} \gamma_j \leqq k/2 - 1$ and $\sum_{j=1}^{N} \beta_j \leqq k/2 + 1$.

As a direct application of Lemmas 2 and C, we have

THEOREM 6. *Let $\zeta \neq 0$ be a point in U and let $F(x_1, x_2, \cdots, x_{n+1})$ be analytic in a neighborhood of each point $F(\varphi'(\zeta), \varphi''(\zeta), \cdots, \varphi^n(\zeta), \zeta)$, $\varphi \in \Phi_k^\lambda(p, p)$. The functional $J(\varphi') = \operatorname{Re} F(\varphi'(\zeta), \cdots, \varphi^n(\zeta), \zeta)$ attains its maximum (minimum) in $\Phi_k^\lambda(p, p)$ only for a function of the form*

$$\varphi'(z) = pz^{p-1} \prod_{j=1}^{M} (1 - \epsilon_j z)^{\gamma_j pe^{-i\lambda}\cos\lambda} \prod_{j=1}^{N} (1 - e_j z)^{-\beta_j pe^{-i\lambda}\cos\lambda},$$

where $M \leqq n, N \leqq n, |\epsilon_j| = |e_j| = 1, \sum_{j=1}^{N} \gamma_j \leqq k/2 - 1$, and $\sum_{j=1}^{N} \beta_j \leqq k/2 + 1$.

In considering the class $\Phi_k^\lambda(p, q), p > q$, we are restricted to the subclasses having certain critical points.

THEOREM 7. *Let $\zeta \neq 0$ be a given point in U, and let $F(x_1, x_2 \cdots, x_{n+1})$ be analytic in a neighborhood of each point $F(\varphi'(\zeta), \varphi''(\zeta), \cdots, \varphi^n(\zeta), \zeta), \varphi \in \Phi_k^\lambda(p, q)$. Then the functional $J(\varphi') = \operatorname{Re} F(\varphi'(\zeta), \cdots, \varphi^n(\zeta), \zeta)$ attains its maximum (minimum) in the subclass of $\Phi_k^\lambda(p, q)$ of functions having non-zero critical points $\alpha_1, \cdots, \alpha_{p-q}$ counting multiplicities only for a function of the form*

$$\varphi'(z) = qz^{q-1} \prod_{j=1}^{p-q} \zeta(z, \alpha_j) \prod_{j=1}^{M} (1 - \epsilon_j z)^{\gamma_j pe^{-i\lambda} \cos\lambda}$$

$$\cdot \prod_{j=1}^{N} (1 - e_j z)^{-\beta_j pe^{-i\lambda} \cos\lambda} ,$$

where $p > q \geqq 1$, $\zeta(z, \alpha_j)$ $(j = 1, \cdots, p - q)$ *are given by* (3), $M \leqq n$, $N \leqq n$, $|\epsilon_j| = |e_j| = 1$, $\sum_{j=1}^{M} \gamma_j \leqq k/2 - 1$ *and* $\sum_{j=1}^{N} \beta_j \leqq k/2 + 1$.

The functional $J(\varphi') = |\varphi'|$ can be used in Theorems 6 and 7 to obtain distortion theorems for $\Phi_k^\lambda(p, q)$ $(p \geqq q)$. The result for $\Phi_k^\lambda(p, p)$ is analogous to that obtained in [12], while for $\Phi_k^\lambda(p, q)(p > q)$ the result is less explicit but comparable to that stated by Leach [5] for $V_k^0(p, q)$. More explicit formulas for the bounds on $|\arg \varphi'|$, $\varphi \in \Phi_k^\lambda(p, q)$ can be proved and are thus of greater interest to us.

THEOREM 8. *Let* $\varphi \in \Phi_k^\lambda(p, q)$ $(p \geqq q)$ *and for* $p > q$, *let* $\alpha_1, \cdots, \alpha_{p-q}$ *be the non-zero critical points counting multiplicites of* φ, *then*

$$|\arg \varphi'(re^{i\theta})| \leqq \begin{cases} (p-1)|\theta| + pk\cos\lambda \arcsin r, \; p = q, \\ (q-1)|\theta| + \cos\lambda \displaystyle\sum_{j=1}^{p-q} \psi(r, \alpha_j) \\ \qquad + \mathrm{pk} \cos\lambda \; \arcsin r, \quad p > q \geqq 1, \end{cases}$$

where $\psi(r, \alpha_j) = \arcsin r/|\alpha_j| + \arcsin|\alpha_j| r$.

PROOF. Let $J(\varphi') = \operatorname{Re} \pm i \log \varphi'(z) = \mp \arg \varphi'(z)$ and take $n = 1$ in Theorem 6. Then for $\varphi \in \Phi_k^\lambda(p, p)$,

$$\begin{aligned} |J(\varphi') = |\arg \varphi'(z)| &= |(p-1)\arg z + pe^{-i\lambda} \cos\lambda \\ &\quad \cdot \{(k/2 - 1)\arg(1 - \epsilon z) - (k/2 + 1)\arg(1 - ez)\}| \\ &\leqq (p-1)|\theta| + p\cos\lambda\{(k/2 - 1)|\arg(1 - \epsilon z)| \\ &\qquad + (k/2 + 1)|\arg(1 - ez)|\}. \end{aligned}$$

The result follows from the well-known fact that

$$|\arg(1 - \mathrm{r}e^{i\theta})| \leqq \arcsin r \tag{18}$$

for $0 < r < 1, 0 \leqq \theta \leqq 2\pi$.

For $p > q \geqq 1$, the proof follows similarly. Consider $J(\varphi') = \mp \arg \varphi'(z)$ and take $n = 1$ in Theorem 7. Then for $\alpha_1, \cdots, \alpha_{p-q}$ the non-zero critical points counting multiplicities of φ in U, we have

$$|\arg \varphi'(z)| \leqq (q-1)|\theta| + \cos\lambda \sum_{j=1}^{p-q} \{(|\arg(1 - z/\alpha_j)| + |\arg(1-\alpha_j z)|\}$$
$$+ kp \cos\lambda \arcsin r$$

$$\leqq (q-1)|\theta| + \cos\lambda \sum_{j=1}^{p-q} \{\arcsin r/|\alpha_j| + \arcsin|\alpha_j| r\}$$
$$+ \text{kp} \cos\lambda \ \arcsin r$$

where the last inequality follows from (18).

In conclusion we observe that following a procedure similar to that used in [12], we may define the class $\tilde{V}_k{}^\lambda(p,q)$ as consisting of the functions $f \in A_q$ for which $\int_0^z (f(\zeta)/\zeta)\, d\zeta \in V_k{}^\lambda(p,q)$. For these classes, the results analogous to those obtained for $V_k{}^\lambda(p,q)$ follow from this very familiar relationship.

The author has further been able to solve various extremal problems for $V_k{}^\lambda(p,q)$ using a $V_k{}^\lambda(p,q)$-preserving transformation analogous to that employed by Moulis [8].

References

1. John Bender, *Some Extremal Theorems for Multivalently Starlike Functions,* Duke Math J. **29** (1962), 101-106.

2. G. M. Goluzin, *On the Coefficients of a Class of Functions, Regular in a Disk and Having an Integral Representation in it,* J. of Soviet Math. (2) **6** (1974), 606-617.

3. ——, *Geometric Theory of Functions of a Complex Variable,* (Amer. Math. Soc. Transl. Math. Mono., Providence, R.I., 1969).

4. J. A. Hummel, *Extremal properties of weakly starlike p-valent starlike functions,* Trans. Amer. Math. Soc. **130** (1968), 544-551.

5. Ronald Leach, *Multivalent and Meromorphic Functions of Bounded Boundary Rotation,* Can. J. Math (1) **27** (1975), 186-199.

6. O. Lehto, *On the Distortion of Conformal Mappings with Bounded Boundary Rotation,* Ann. Acad. Sci. Fenn. Ser. A, 1, **124** (1952), 14 pp.

7. K. Lōwner, *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises* $|z| < 1$, *die durch Funktionen mit nicht verschwindender Ableitung gellefert werden,* Ber. Konigl. Sachs. Ges Wiss. Leipzig **69** (1917), 89-106.

8. Edward Moulis, Jr., *A Generalization of Univalent Functions with Bounded Boundary Rotation,* Trans. Amer. Math. Soc. **174** (1972), 369-381.

9. Z. Nehari, *Conformal Mapping,* Dover, New York, 1952.

10. V. Paatero, *Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind,* Ann. Acad. Sci. Fenn. Ser A, 33 (1931), 1-79.

11. M. S. Robertson, *Univalent Functions* $f(z)$ for which $zf'(z)$ *is Spirallike,* Michigan Math. J. **16** (1969), 97-101.

12. E. M. Silvia, *A Variation Method on Certain Classes of Functions,* Rev. Roum. Math. Pures and Appl. **21** (1976), 549-557.

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