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ON STRONG RIESZ AND STRONG GENERALIZED CESÀRO SUMMABILITY

F. P. CASS,[1] E. H. CHANG, AND D. C. RUSSELL[1]

1. **Introduction.** We suppose throughtout that $\lambda = \{\lambda_n\}$ is a given sequence satisfying

(1) $$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty.$$

For $\kappa \geqq 0$, $\mu > 0$, $p = 0, 1, 2, \cdots$, and a given (real or complex) number s, we define the following means of an infinite (real or complex) series $\sum a_n$ (here, and throughout, $\sum$ means $\sum_0^\infty$ unless otherwise specified). The (R, λ, κ)-mean:

(2) $$R^\kappa(\tau) = \sum_{\lambda_\nu < \tau} (1 - \lambda_\nu/\tau)^\kappa a_\nu \qquad (\tau > 0);$$

the $[R, \lambda, \kappa + 1]_\mu$-mean:

(3) $$F^{\kappa+1}(\omega) = \omega^{-1} \int_0^\omega |R^\kappa(\tau) - s|^\mu \, d\tau \qquad (\omega > 0);$$

the (C, λ, p)-mean:

(4) $$t_n^0 = s_n = \sum_{\nu=0}^n a_\nu,$$
$$t_n^p = \sum_{\nu=0}^n (1 - \lambda_\nu/\lambda_{n+1}) \cdots (1 - \lambda_\nu/\lambda_{n+p}) a_\nu \qquad (n = 0, 1, \cdots);$$

the $[C, \lambda, p + 1]_\mu$-mean:

(5) $$\sigma_m^{p+1} = \sum_{n=0}^m a_{mn} |t_n^p - s|^\mu \qquad (m = 0, 1, \cdots),$$

where $a_{mn} = (\lambda_{n+p+1} - \lambda_n) E_n^p / E_m^{p+1}$ $(0 \leqq n \leqq m)$, $a_{mn} = 0$ $(m > n)$, and $E_n^p = \lambda_{n+1} \cdots \lambda_{n+p}$ (with E_n^0 defined as 1).

Ordinary and strong Riesz summability (of real order) and ordinary, absolute, and strong generalized Cesàro summability (of integer

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order) are now defined respectively as follows.

$\sum a_n = s \quad (R, \lambda, \kappa)$	means	$R^\kappa(\tau) \to s$ as $\tau \to \infty$;
$\sum a_n = s \quad [R, \lambda, \kappa + 1]_\mu$	means	$F^{\kappa+1}(\omega) = o(1)$ as $\omega \to \infty$;
$\sum a_n = s \quad (C, \lambda, p)$	means	$t_n{}^p \to s$ as $n \to \infty$;
$\sum a_n = s \quad \lvert C, \lambda, p \rvert$	means	$t_n{}^p \to s$ and $\sum \lvert t_n{}^p - t_{n-1}^p \rvert < \infty$;
$\sum a_n = s \quad [C, \lambda, p + 1]_\mu$	means	$\sigma_m{}^{p+1} = o(1)$ as $m \to \infty$.

This definition of strong Riesz summability is due to P. Srivastava [10] and (independently, with slightly different notation) to Glatfeld [6]; earlier, Richert [8] had applied strong Riesz summability (with $\lambda_n = \log(n + 1)$, $\mu = 2$) to Dirichlet series, while Boyd and Hyslop [5] had examined the relation between strong Cesàro and strong Riesz summability with $\lambda_n = n$. Generalized Cesàro summability has been studied by a number of authors, and in Bosanquet and Russell [4] (where a comprehensive bibliography may be found), a definition of (C, λ, κ) summability is given, which is equivalent to (R, λ, κ) summability for all sequences λ satisfying (1) and for all $\kappa \geqq 0$. In the present paper (where $[C, \lambda, p + 1]_\mu$ summability appears for the first time) we consider only the case where κ is a non-negative integer, and our purpose is to prove some inclusion and equivalence relations between $[C, \lambda, p + 1]_\mu$ summability and the other summability methods defined above. For $\lambda_n = n$, the definition of $[C, n, p + 1]_\mu$ is equivalent to the usual definition of strong Cesàro summability $[C, p + 1]_\mu$; see Borwein and Cass [2, p. 98]. For some general properties of strong summability see also Borwein [1] and Borwein and Cass [3].

For any two summability methods P, Q, we write $P \Rightarrow Q$ if every series which is P-summable is also Q-summable to the same sum. P is *regular* if it sums every convergent series to its ordinary sum; P and Q are *equivalent*, written $P \Leftrightarrow Q$, if both $P \Rightarrow Q$ and $Q \Rightarrow P$. A few of the known relations between the methods defined above are as follows, all of them (except for (9)) with no restriction on λ other than (1).

(6) [10, Theorem 2] $(R, \lambda, \kappa) \Rightarrow [R, \lambda, \kappa + 1]_\mu$

$(\mu > 0, \kappa \geqq 0)$.

(7) [10, Theorems 1, 7] $[R, \lambda, \kappa + 1]_\mu \Rightarrow (R, \lambda, \kappa + 1)$

$(\mu \geqq 1, \kappa \geqq 0)$.

(8) [10, Th. 4], [6, Th. 1] $[R, \lambda, \kappa + 1]_{\mu_1} \Rightarrow [R, \lambda, \kappa + 1]_{\mu_2}$

$(\mu_1 > \mu_2 > 0, \kappa \geqq 0)$.

(9) [5, Theorem] $[R, n, \kappa + 1]_\mu \Rightarrow [C, \kappa + 1]_\mu$

$(\mu \geqq 1, \kappa \geqq 0)$.

(10) [7, Theorem] $(R, \lambda, p) \Rightarrow (C, \lambda, p)$

$(p = 0, 1, 2, \cdots)$.

(11) [9, Theorem 4] $(C, \lambda, p) \Rightarrow (R, \lambda, p)$

$(p = 0, 1, 2, \cdots)$.

There are also a number of counter-examples, particularly in [6].

2. **Elementary properties of $[C, \lambda, p + 1]_\mu$ summability.**

THEOREM 1.

(i) $(C, \lambda, p) \Rightarrow [C, \lambda, p + 1]_\mu \quad (\mu > 0, p = 0, 1, 2, \cdots)$.

(ii) $[C, \lambda, p + 1]_\mu \Rightarrow (C, \lambda, p + 1) \quad (\mu \geqq 1, p = 0, 2, 1, \cdots)$.

(iii) $[C, \lambda, p + 1]_{\mu_1} \Rightarrow [C, \lambda, p + 1]_{\mu_2} \quad (\mu_1 > \mu_2 > 0, p = 0, 1, 2, \cdots)$.

PROOF. The matrix (a_{mn}) in (5) is regular, so that $|t_n^p - s|^\mu \to 0$ implies $\sigma_m^{p+1} \to 0$, which gives (i). Moreover, $t_m^{p+1} - s = \sum_{n=0}^m a_{mn}(t_n^p - s)$, and (a_{mn}) is non-negative with row-sums 1, so that an application of Jensen's inequality (with $\mu \geqq 1$) then gives (ii). Finally, (iii) follows from a simple application of Hölder's inequality.

3. **Inclusion theorems between $[C, \lambda, p + 1]_\mu$ *and* $[R, \lambda, p + 1]_\mu$.**

LEMMA. *Let p be a positive integer and b_i $(i = 1, 2, \cdots, p)$ and d_j $(j = 0, 1, \cdots, p)$ be numbers such that*

(12) $|b_i| \leqq 1 \ (i = 1, 2, \cdots, p), \quad |d_j| \leqq 1 \quad (j = 0, 1, \cdots, p)$

and

(13) $|d_s - d_r| \geqq 1/H > 0 \qquad (r, s = 0, 1, \cdots, p; r \neq s)$.

Then there are numbers y_j $(j = 0, 1, \cdots, p)$ such that, for any number x,

$$\sum_{j=0}^{p} (\lambda_{n+p+1} - \lambda_n)|R^p(\theta_j)|^\mu \leqq (2p+2)\int_{\lambda_n}^{\lambda_{n+p+1}} |R^p(\tau)|^\mu\, d\tau. \tag{22}$$

Now take $d_j = (\theta_j - \lambda_n)/(\lambda_{n+p+1} - \lambda_n)$ and $b_i = (\lambda_{n+i} - \lambda_n)/(\lambda_{n+p+1} - \lambda_n)$ in the Lemma, so that $\{d_j\}$ and $\{b_i\}$ satisfy (12) and (13) with $H = 2p + 2$. Consequently, there are numbers $y_j = y_{nj}$ ($j = 0, 1, \cdots, p$) such that (14) and (15) hold. Setting $x = (\lambda_n - \lambda_\nu)/\lambda_{n+p+1} - \lambda_n)$ in (14) we obtain

$$\prod_{i=1}^{p} (1 - \lambda_\nu/\lambda_{n+i}) = \sum_{j=0}^{p} c_{nj}(1 - \lambda_\nu/\theta_{nj})^p, \tag{23}$$

where $c_{nj} = y_{nj}\,\theta_{nj}^p/E_n{}^p$. Since, by (15), $|y_{nj}| \leqq p!(2p+2)^{(1/2)p(p+1)} \equiv H_1$, we have

$$|c_{nj}| \leqq H_1(\lambda_{n+p+1}/\lambda_n)^p \leqq H_1 c^p. \tag{24}$$

Then, by (23), (24), and definitions (2) and (4), we obtain

$$|t_n{}^p|^\mu = \left| \sum_{j=0}^{p} c_{nj} \sum_{\nu=0}^{n} (1 - \lambda_\nu/\theta_{nj})^p a_\nu \right|^\mu$$

$$\leqq K \sum_{j=0}^{p} |R^p(\theta_{nj})|^\mu \quad (n \in M_2),$$

and hence

$$\Sigma_2 \equiv \sum_{M_2} (\lambda_{n+p+1} - \lambda_n) E_n{}^p |t_n{}^p|^\mu$$

$$\leqq KE_m{}^p \sum_{M_2} \sum_{j=0}^{p} (\lambda_{n+p+1} - \lambda_n)|R^p(\theta_{nj})|^\mu.$$

Thus, by (22) and hypothesis (17), we now have

$$(E_m{}^{p+1})^{-1}\, \Sigma_2 \leqq K_1(\lambda_{m+p+1})^{-1} \int_0^{\lambda_{m+p+1}} |R^p(\tau)|^\mu\, d\tau < \epsilon, \quad \text{for } m \geqq q'. \tag{25}$$

The combination of (21) and (25) yields (18), and the theorem is therefore proved.

REMARKS. (a) If $\lambda_{n+1} = O(\lambda_n)$, the theorem extends to $0 < \mu < 1$ because c can then be chosen so that the set M_1 is empty; consequently the use of property (7), which was the only feature of the proof which needed $\mu \geqq 1$, is not then required. (*b*) When $p = 0$, the theorem holds for all $\mu > 0$ without restriction on λ (other than (1)), by (19).

THEOREM 3. $[C, \lambda, p+1]_\mu \Rightarrow [R, \lambda, p+1]_\mu$ $(\mu > 0,\ p = 0, 1, 2, \cdots)$.

PROOF. We may assume without loss of generality that $\sum a_n = 0[C, \lambda, p+1]_\mu$, that is, $\sigma_m = \sigma_m^{p+1} = o(1)$. From [9, (34) and the proof of Theorem 4] we have $R^p(\tau) = \sum_{\nu=n-p}^{n} \alpha_\nu^p(\tau) t_\nu^p$, where $\alpha_\nu^p(\tau) \geqq 0$, $\sum_{\nu=n-p}^{n} \alpha_\nu^p(\tau) = 1, \lambda_n < \tau \leqq \lambda_{n+1}, n \geqq p$, and so, for all $\mu > 0$,

$$|R^p(\tau)|^\mu \leqq \sum_{\nu=n-p}^{n} |t_\nu^p|^\mu \qquad (\lambda_n < \tau \leqq \lambda_{n+1},\ n \geqq p). \tag{26}$$

Now inverting the summation in (5) (with $s = 0$) we obtain

$$\begin{aligned} |t_\nu^p|^\mu &= \frac{\lambda_{\nu+p+1}\sigma_\nu - \lambda_\nu \sigma_{\nu-1}}{\lambda_{\nu+p+1} - \lambda_\nu} \\ &\leqq \frac{(\lambda_{\nu+p+1} - \lambda_\nu)\sigma_\nu + \lambda_\nu \sigma_\nu - \lambda_{\nu-1}\sigma_{\nu-1}}{\lambda_{\nu+p+1} - \lambda_\nu}. \end{aligned} \tag{27}$$

Suppose $\omega > \lambda_{2p+1}$ and choose m so that $\lambda_m < \omega \leqq \lambda_{m+1}$. Then, by (26),

$$\begin{aligned} &\omega^{-1} \int_{\lambda_p}^{\omega} |R^p(\tau)|^\mu \, d\tau \\ &= \omega^{-1} \sum_{n=p}^{m-1} \int_{\lambda n}^{\lambda n+1} |R^p(\tau)|^\mu \, d\tau + \omega^{-1} \int_{\lambda m}^{\omega} |R^p(\tau)|^\mu \, d\tau \\ &\leqq (1/\lambda_m) \sum_{n=p}^{m-1} (\lambda_{n+1} - \lambda_n) \sum_{\nu=n-p}^{n} |t_\nu^p|^\mu \\ &\quad + (1 - \lambda_m/\omega) \sum_{\nu=m-p}^{m} |t_\nu^p|^\mu \\ &\leqq (1/\lambda_m) \sum_{n=p}^{m-p-1} \sum_{\nu=n-p}^{n} (\lambda_{\nu+p+1} - \lambda_\nu)|t_\nu^p|^\mu \end{aligned} \tag{28}$$

$$+ \sum_{n=m-p}^{m} (1 - \lambda_n/\lambda_{n+1}) \sum_{\nu=n-p}^{n} |t_\nu^p|^\mu$$

$$\equiv \Sigma_1 + \Sigma_2.$$

By (27), we have

$$\Sigma_1 \leqq (1/\lambda_m) \sum_{n=p}^{m-p-1} \sum_{\nu=n-p}^{n} (\lambda_{\nu+p+1} - \lambda_\nu)\sigma_\nu$$

$$+ (1/\lambda_m) \sum_{n=p}^{m-p-1} \sum_{\nu=n-p}^{n} (\lambda_\nu \sigma_\nu - \lambda_{\nu-1}\sigma_{\nu-1})$$

$$= (1/\lambda_m) \sum_{n=p}^{m-p-1} \sum_{r=0}^{p} (\lambda_{n-r+p+1} - \lambda_{n-r})\sigma_{n-r}$$

$$+ (1/\lambda_m) \sum_{r=p+1}^{2p+1} \lambda_{m-r}\sigma_{m-r}$$

$$\leqq \sum_{r=0}^{p} \rho_{m-r} + \sum_{r=p+1}^{2p+1} \sigma_{m-r},$$

$$\text{where } \rho_j = (1/\lambda_j) \sum_{i=0}^{j-p-1} (\lambda_{i+p+1} - \lambda_i)\sigma_i.$$

By hypothesis, $\sigma_{m-r} \to 0$ as $m \to \infty$, for each r, and since the transformation from $\{\sigma_i\}$ to $\{\rho_j\}$ is regular for null sequences, it follows also that $\rho_{m-r} \to 0$ as $m \to \infty$, for each r. Thus $\Sigma_1 = o(1)$.

Now, by hypothesis, $\sum_{i=0}^{\nu} (\lambda_{i+p+1} - \lambda_i)\ E_i^p |t_i^p|^\mu = o(E_\nu^{p+1})$, and selecting only the term $i = \nu$ on the left, we obtain

$$|t_\nu^p|^\mu = o(\lambda_{\nu+p+1}/(\lambda_{\nu+p+1} - \lambda_\nu))$$

$$= o(\lambda_{n+1}/(\lambda_{n+1} - \lambda_n)) \qquad (n \to \infty, n - p \leqq \nu \leqq n).$$

Hence

$$\Sigma_2 = \sum_{n=m-p}^{m} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \cdot o\left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}\right)$$

$$= o(1) \text{ as } m \to \infty.$$

Thus, from (28), we now have $\omega^{-1} \int_{\lambda_p}^{\omega} |R^p(\tau)|^\mu\, d\tau = o(1)$ as $\omega \to \infty$, which is equivalent to the required conclusion.

4. Relation between absolute and strong generalized Cesàro summability.

THEOREM 4.

$$|C, \lambda, p+1| \Rightarrow [C, \lambda, p+1]_1 \qquad (p = 0, 1, 2, \cdots).$$

PROOF. Write $T_n \equiv \sum_{\nu=0}^{n} |t_\nu^{p+1} - t_{\nu-1}^{p+1}|$ and observe that, from the definitions in §1,

$$(\lambda_{n+p+1} - \lambda_n)t_n^p = (\lambda_{n+p+1} - \lambda_n)t_n^{p+1} - \lambda_n(t_n^{p+1} - t_{n-1}^{p+1}).$$

Thus

$$\sigma_m^{p+1} \equiv \sum_{n=0}^{m} a_{mn}|t_n^p - s| \leqq S_m + S_m{}', \tag{29}$$

where

$$S_m = \sum_{n=0}^{m} a_{mn}|t_n^{p+1} - s|$$

and

$$E_m^{p+1}S_m{}' = \sum_{n=0}^{m} \lambda_n E_n^p|t_n^{p+1} - t_{n-1}^{p+1}|$$

$$= \sum_{n=0}^{m} E_{n-1}^{p+1}(T_n - T_{n-1}).$$

A partial summation now gives $S_m{}' = T_m - \sum_{n=0}^{m} a_{mn}T_n$. Since (a_{mn}) is regular, and the hypothesis $\sum a_n = s|C, \lambda, p+1|$ means that $t_n^{p+1} - s \to 0$ and $T_n \to s'$ (say), it follows that $S_m \to 0$ and $S_m{}' \to s' - s' = 0$. Hence, by (29), $\sigma_m^{p+1} \to 0$, so that $\sum a_n = s[C, \lambda, p+1]_1$.

5. Strict inclusion. We conclude with some remarks relating to the strictness of the inclusions in Theorem 1(i) and (ii). It is a consequence of Borwein and Cass [3, Corollary 2] that

$$\liminf_{n\to\infty} (\lambda_{n+p+1}/\lambda_n) = 1 \tag{30}$$

is necessary and sufficient for the existence of a series summable $[C, \lambda, p+1]_\mu$ but not summable (C, λ, p). We now show, however, that it is possible for (30) to hold with $p = 0$, and to have $[C, \lambda, 1]_1 \Leftrightarrow (C, \lambda, 1)$.

Let $p_0 > 0$, $p_n \geqq 0$, $P_n = p_0 + \cdots + p_n$, $c_n = p_n/P_n$, and for a sequence (s_n) define $u_n = P_n^{-1}\sum_{\nu=0}^{n} p_\nu s_\nu$, so that $u_n - (1 - c_n)u_{n-1}$

$= c_n s_n$. Now choose $c_{2n} = 1 - (n+1)^{-2}$ and $c_{2n+1} = (n+2)^{-1}$, for $n \geqq 1$. A routine argument then shows that, with this choice of c_n, we have $u_n \to s$ if and only if $s_{2n} \to s$ and $s_{2n+1} = o(n)$. It is now easy to check that

$$P_n^{-1} \sum_{\nu=0}^{n} p_\nu |s_\nu - s| = o(1)$$

(31)

$$\Leftrightarrow u_n - s \equiv P_n^{-1} \sum_{\nu=0}^{n} p_\nu (s_\nu - s) = o(1).$$

If we now define $\lambda_0 = 0$ and $\lambda_n = P_{n-1}$ for $n \geqq 1$, then $\{\lambda_n\}$ satisfies (1), $\liminf(\lambda_{n+1}/\lambda_n) = \liminf(1 - c_n)^{-1} = 1$, and (31) becomes $[C, \lambda, 1]_1 \Leftrightarrow (C, \lambda, 1)$. On the other hand, using the same λ, with $\mu_1 > \mu_2 > 0$, and choosing $s_{2n} = 0$, $s_{2n+1} = n^{1/\mu_1}$, we have $s_n \to 0[C, \lambda, 1]_{\mu_2}$ but $\{s_n\}$ is not summable $[C, \lambda, 1]_{\mu_1}$.

Since $(C, \lambda, 1) \Leftrightarrow (R, \lambda, 1)$ and $[C, \lambda, 1]_\mu \Leftrightarrow [R, \lambda, 1]_\mu$ $(\mu > 0)$, the above choice of λ furnishes an example of a Riesz method which is not equivalent to convergence, for which the inclusion $[R, \lambda, 1]_{\mu_1} \Rightarrow [R, \lambda, 1]_{\mu_2}$ $(\mu_1 > \mu_2 > 0)$ is strict, but for which $[R, \lambda, 1]_1 \Leftrightarrow (R, \lambda, 1)$.

Incidentally, (31) shows that if, in [3, Theorem 12], the condition $\lim p_{nn} = 0$ is replaced by $\liminf p_{nn} = 0$, then the conclusion of that theorem fails.

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The University of Western Ontario, London, Ontario, Canada N6A 3K7.
York University, Downsview, Ontario, Canada M3J 1P3.