ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 1, Winter 1977

ST. VENANT'S COMPATIBILITY CONDITIONS AND BASIC PROBLEMS IN ELASTICITY*

TSUAN WU TING

1. **Introduction.** Let G be a bounded multiply connected domain in R^n with regular boundary, ∂G. Let $d \equiv (d_{ij})$ be a second-order symmetric tensor field on G. Our question is this. Under what conditions on d is there a vector field v such that d is the symmetric part of the gradient of v? That is, given d, when can one find a solution v of the system,

$$\frac{1}{2}(v_{i,j} + v_{j,i}) = d_{ij} \text{ in } G \text{ for } i, j = 1, 2, \cdots, n? \tag{1}$$

Here the commas stand for partial differentiations. Although (1) is an over-determined system for $v \equiv (v_i)$, the question has been partially answered by the well-known St. Venant compatibility conditions. However, the differentiability requirements in St. Venant's compatibility conditions are unnecessarily strong. Furthermore, the full set of compatibility conditions is still insufficient, if the domain G is multiply connected.

It was found in [11] that the above question leads to an orthogonal decomposition of the Hilbert space of square integrable second-order symmetric tensors. The objective here is to present a different proof which also shows how the orthogonal decomposition theorem is related to the existence problems in elasticity and the duality principle in convex analysis.

Although the above decomposition theorem is a recent one, the corresponding theorem for vector fields has been well developed and it plays a significant role in the mathematical treatment of fluid mechanics [6]. We refer to [11] for additional references. In the meantime, under the assumption of "ellipticity", similar decomposition theorems were established in [1] for which our results serve as a non-trivial example.

2. **The decomposition theorem.** A second-order symmetric tensor field $s = (s_{ij})$ is said to be solenoidal, if it is divergence free, i.e., if $s_{ij,j} = s_{ji,j} = 0$ in G. The class of all such tensors which are infinitely

Received by the editors on March 5, 1975, and in revised form on May 20, 1975.

*Work partially supported by N.S.F. Grant No. MPS75-07118.

AMS subject classification: Primary 7335, 7349, 7346; Secondary 4615, 5345.

smooth with compact support in G is denoted by $S_0^\infty(G)$. Let $L^2(G)$ be the Hilbert space of all square integrable second-order symmetric tensors. We denote the completion of $S_0^\infty(G)$ in $L^2(G)$ by $S(G)$. As usual, the orthogonal complement of $S(G)$ in $L^2(G)$ is denoted by $S^\perp(G)$.

In what follows, we denote by $H^k(G)$ the Hilbert space of functions which together with their distribution derivatives of order $\leqq k$ are square integrable over G. The usual k-fold Dirichlet norm on $H^k(G)$ is written as $\|\cdot\|_k$. We say that a vector or a tensor belongs to $H^k(G)$ if each of its components belongs to $H^k(G)$.

Let $D(G)$ be the closed subspace of $L^2(G)$ consisting of tensors d for which there exists a vector v in $H^1(G)$ such that the equations in (1) hold in the $\|\cdot\|_0$-norm.

THEOREM 1. *$L^2(G)$ is the direct sum of $D(G)$ and $S(G)$.*

3. **Proof of Theorem 1.** By the Riesz projection theorem, $L^2(G) = S(G) \oplus S^\perp(G)$. Hence, we need only to show that

$$D(G) = S^\perp(G).$$

Proof that $D(G) \subset S^\perp(G)$. Indeed, if $d \in D(G)$, then

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \equiv e_{ij}(v), \qquad i,j = 1, 2, \cdots, n, \tag{2}$$

for some v in $H^1(G)$. Hence, for all s in $S_0^\infty(G)$, we have from the symmetric and solenoidal character of s that

$$\int_G d_{ij}s_{ij}\,dx = \int_G \tfrac{1}{2}(v_{i,j} + v_{j,i})s_{i,j}\,dx = \int_G v_{i,j}s_{ij}\,dx$$
$$= -\int_G v_i s_{ij,j}\,dx = 0,$$

where we have adopted the summation convention over repeated indices and where the last equality follows from the divergence theorem. Since $S_0^\infty(G)$ is dense in $S(G)$ relative to $\|\cdot\|_0$-norm, the assertion follows.

Proof that $D(G) \supset S^\perp(G)$. To show this we show that if $d \in S^\perp(G)$ then there exists a vector v in $H^1(G)$ such that the equations in (2) hold. To this end, we first prove it for all d in $S^\perp(G) \cap H^1(G)$ and then apply a limiting process. Now, for $d \in S^\perp(G) \cap H^1(G)$, we define

$$f_i = d_{ij,j},\ i = 1, 2, \cdots, n. \tag{3}$$

Then, the vector f is square integrable over G. For every point x on ∂G, let $n(x)$ be the unit outward normal to ∂G at x and define

(4) $$t_i = d_{ij}n_j, \quad i = 1, 2, \cdots, n.$$

According to the trace theorem [2], the vector t belongs to $L^2(\partial G)$. Consider now the

Dual problem. Find a symmetric tensor t in $H^1(G)$ which minimizes the functional

(5) $$I_D(t) = \int_G t_{ij}t_{ij}\, dx$$

over the closed convex set,

(6) $$T = \{t \mid t \in H^1(G),\ t_{ij,j} - f_i = 0 \quad \text{a.e. in } G,\\ t_{ij}n_j = t_i \text{ in } L^2(\partial G)\}.$$

We claim that the given tensor d is a minimizer. In fact, for all t in the admissible class T in (6), the tensor, $t - d$, belongs to $S(G) \cap H^1(G)$. Hence,

(7) $$\int_G d_{ij}(t_{ij} - d_{ij})\, dx = 0 \quad \text{for all } t \text{ in } T.$$

This is, in view of the convexity of $I_D(t)$ in t, precisely the necessary and sufficient condition for d being a minimizer. Note that it is here we have used the fact that $d \in S^{\perp}(G)$.

Next, we observe that if t and t' are any two minimizers of the dual problem, then $\|t - t'\|_0 = 0$. Indeed, the tensor, $(t + t')/2$, belongs to the admissible class T. Hence, we have

$$I_D(\tfrac{1}{2}(t + t')) \geqq I_D(t) = I_D(t').$$

This inequality can be written as $I_D(t - t') \leqq 0$ which is possible only when $\|t - t'\|_0 = 0$.

Motivated by the second-boundary value problem in elasticity, we also consider the

Primary problem. Find a vector v in $H^2(G)$ so as to minimize the functional,

(8) $$I_P(w) = \int_G [e_{ij}(w)e_{ij}(w) + f_iw_i]\, dx - \int_{\partial G} t_iw_i\, ds,$$

over the closed subspace

(9) $$W = \left\{ w \mid w \in H^2(G),\ \int_G w_i\, dx = 0,\ \int_G (w_{i,j} - w_{j,i})\, dx = 0 \right\}.$$

Here, f_i and t_i are given by equations in (3) and (4), respectively.

Also, $e_{ij}(w)$ is defined in (2). Needless to say, the conditions

$$\int_G w_i\,dx = 0, \quad \int_G [w_{i,j} - w_{j,i}]\,dx = 0, \quad i,j = 1, 2, \cdots, n,$$

were introduced so as to assure the uniqueness of the minimizer. Thanks to Korn's inequality, [2, 3, 8, 11], the primary problem has, indeed, a unique solution v in $H^2(G)$.

We assert that if v is the unique minimizer of the primary problem, then $e_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})$ is a minimizer of the dual problem. To show this, we note that for all w in W,

$$\int_G [e_{ij}(v)e_{ij}(w - v) + f_i \cdot (w_i - v_i)]\,dx$$
$$- \int_{\partial G} t_i \cdot (w_i - v_i)\,ds = 0$$

as a necessary consequence of the minimizing character of v. Upon applying the divergence theorem, this variational equality can be written as

$$\int_G [f_i \cdot (w_i - v_i) - e_{ij,j}(v) \cdot (w_i - v_i)]\,dx$$
$$+ \int_{\partial G} [e_{ij}(v)n_j - t_i] \cdot (w_i - v_i)\,ds = 0.$$

Since the vector, $w - v$, is sufficiently arbitrary, we conclude that

(10) $$e_{ij,j}(v) = f_i \quad \text{a.i. in } G \qquad \text{for } i = 1, 2, \cdots, n.$$

(11) $$e_{ij}(v)n_j = t_i \quad \text{in } L^2(\partial G) \qquad \text{for } i = 1, 2, \cdots, n.$$

The results in (10) and (11) assure us that $(e_{ij}(v))$ belongs to the admissible class T in (6). Finally, for all t in T, the tensor, $e(v) - t$, belongs to $S(G)$. Hence

$$\int_G e_{ij}(v) \cdot (t_{ij} - e_{ij}(v))\,dx$$
$$= \int_G v_i \cdot (t_{ij} - e_{ij}(v))n_j\,ds - \int_G (t_{ij,j} - e_{ij,j}(v)) \cdot v_i\,dx$$

which ensures that $e_{ij}(v)$ is also a minimizer of the dual problem. Since the minimizer of the dual problem is uniquely determined in the $\|\cdot\|_0$-norm, it is necessary that

(12) $$e_{ij}(v) = d_{ij} \text{ in } L^2(G) \text{ for } i,j = 1, 2, \cdots, n.$$

The assertion is now proved if d belongs to $S^{\perp}(G) \cap H^1(G)$.

For arbitrary d in $S^{\perp}(G)$, there is a sequence $\{D^{(\nu)}\}$ in $S^{\perp}(G) \cap H^1(G)$ such that $\|d - d^{\nu}\|_0 \to 0$ as $\nu \to \infty$. Now, for each ν, there is a vector $v^{(\nu)}$ in $H^1(G)$ such that

$$\int v_i^{(\nu)}\, dx = 0, \qquad \int [v_{i,j}^{(\nu)} - v_{j,i}^{(\nu)}]\, dx = 0. \tag{13}$$

$$d_{ij}^{(\nu)} = e_{kj}(v^{(\nu)}) \text{ in the } \|\cdot\|_0\text{-norm for } i,j = 1, 2, \cdots, n. \tag{14}$$

Moreover, the conditions in (11) enable us to apply Korn's inequality to conclude that $\{\nu^{(\nu)}\}$ is a Cauchy sequence in $H^1(G)$. Denote by v the $\|\cdot\|_1$-limit of $v^{(\nu)}$. Then $d_{ij} = e_{ij}(v)$ in $\|\cdot\|_0$-norm for $i, j = 1, 2, \cdots, n$. The proof is now compete.

4. **The Beltrami-Michell compatibility conditions.** Having established the decomposition theorem, we now apply it to derive the Beltrami-Michell compatibility conditions for the minimizer of the dual problem. According to the sharp form of the trace theorem, [7], if g is a function in $L^2(G)$, then its restriction to ∂G belongs to $H^{-1/2}(\partial G)$. Let f and t be given vectors in $L^2(G)$ and $H^{-1/2}(\partial G)$, respectively, and let $a_{ijhk}(x)$ be a given smooth tensor field on G, which is symmetric in the first two and last two indices and $a_{ijhk} = a_{hkij}$ such that for all second-order symmetric tensors t in $L^2(G)$, $a_{ijhk}t_{ij}t_{hk} \geqq$ const. $t_{ij}t_{ij}$ on G. In its general form, the dual problem is to find a second-order symmetric tensor t in $L^2(G)$ such that it minimizes the functional

$$I_D(t) \equiv \int_G a_{ijhk}(x) t_{ij} t_{hk}\, dx, \tag{15}$$

over the closed convex set

$$T^* \equiv \{t \mid t \in L^2(G),\ t_{ij,j} - f_i = 0 \quad \text{a.e. in } G,$$
$$t_{ij}n_j = t_i \ in\ H^{-1/2}(\partial G)\}$$

in $L^2(G)$. If T^* is non-empty, then it is immediate that the problem has a unique solution t. But then for all t^* in T^*,

$$\int_G a_{ijhk}(x) t_{hk} \cdot (t_{ij}^* - t_{ij})\, dx = 0. \tag{16}$$

Since the tensor, $t^* - t$, *belongs to* $S(G)$, we have from the variational equality in (16) and Theorem 1, THEOREM 2. *If t is the minimizer of the dual problem in* (15), *then the tensor,* $a_{ijhk}t_{hk}$, *belongs to* $S^{\perp}(G)$. *That is, the Beltami-Michell compatibility conditions are satisfied by t and hence the existence of a corresponding displacement field is assured.*

REFERENCES

1. M. Berger and D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian Manifold,* J. Diff. Geometry 3 (1969), pp. 379–392.

2. G. Fichera, *Existence theorems in elasticity,* Handbuch der physik, VI a/2, pp. 347–387, Springer-Verlag, New York, 1972.

3. K. O. Fridriechs, *On the boundary value problems of the theory of elasticity and Korn's inequality,* Ann. Math. **48** (1947), pp. 441–471.

4. P. R. Garabedian, *Partial differential equations,* John Wiley and Sons, New York, 1964.

5. R. J. Knops and L. E. Payne, *Uniqueness theorems in elasticity,* Springer Tracts in Natural Philosophy, Vol 19., New York, 1971.

6. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow,* Gordon and Breach, New York, 1963.

7. J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications,* II, Springer-Verlag, New York, 1972, Chapt. I.

8. G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique,* Dunod, Paris, 1972.

9. A. E. H. Love, *A treatise on the mathematical theory of elasticity,* 4th Ed., Dover, New York, 1944.

10. S. G. Mikhlin, *The problem of the minimum of a quadratic functional,* Holden-Day Inc., San Francisco, 1965.

11. T. W. Ting, *Generalized Korn's inequalities,* Tensor, N. S. **25** (1972), pp. 295–302; St Venant's compatibility conditions ibid. **28** (1974), pp. 5–12.

UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801