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PRODUCT INTEGRALS AND CONTINUITY

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ABSTRACT. Suppose R is the set of real numbers and G is a function from $R \times R$ to R.

One type of continuity for G is to require that $\int_a^b G^2 = 0$ for G to be continuous on $[a, b]$. Jon C. Helton, the author, and others have investigated product integrals and sum integrals of interval functions and their relationship to this type of continuity. This paper examines interval functions which may not have this type of continuity. In an earlier paper [2] it is shown that if G has this type of continuity and $\int_a^b G$ exists, then $\prod_a^b (1 + G)$ exists and is $\exp(\int_a^b G)$. Using several conditions on G including the condition that $\int_a^b G$ exists but not the condition that $\int_a^b G = 0$, the main theorem (Theorem 5) of this paper shows that $\prod_a^b (1 + G)$ exists and gives an evaluation of $\prod_a^b (1 + G)$. A stronger continuity requirement is to require $\prod_a^b (1 + G^2) = 1$ and in Theorems 3 and 4 a relationship between these two types of continuity is established.

1. Definitions, Notations, and Basic Theorems.

All functions are from $R \times R$ to R, where R denotes the set of real numbers and all integrals are of the subdivision-refinement type. $[a, b]$ and (a, b) will be used to denote the set to which p belongs if and only if $a \leqq p \leqq b$ or $a < p < b$, respectively. $D = \{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$ means (1) D is a finite subset of $[a, b]$ and (2) $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. D' is a refinement of the subdivision D of $[a, b]$ means (1) D' is a subdivision of $[a, b]$ and (2) D is a subset of D'.

If $D' = \{x_i\}_{i=0}^n$ is a refinement of the subdivision $D = \{p_j\}_{j=0}^m$ of $[a, b]$ and G is a function from $R \times R$ to R then the following notations will be used when no misinterpretation is likely:

(1) $$\sum_{D'} G_i = \sum_{\substack{i=1 \\ D'}}^{n} G(x_{i-1}, x_i)$$

and $$\prod_{D'} (1 + G_i) = \prod_{\substack{i=1 \\ D'}}^{n} [1 + G(x_{i-1}, x_i)];$$

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(2) $$\sum_{D'\cdot[p_{j-1},p_j]} G_i = \sum_{\substack{k=1\\Q}}^{z} G(x_{k-1}, x_k)$$

and $$\prod_{D'\cdot[p_{j-1},p_j]} (1 + G_i) = \prod_{\substack{k=1\\Q}}^{z} [1 + G(x_{k-1}, x_k)],$$

where $Q = \{x_k\}_{k=0}^{z}$ denotes the subdivision of $[p_{j-1}, p_j]$ for some $0 < j \leqq m$ consisting of the numbers of D' in $[p_{j-1}, p_j]$,

(3) $$G(p, p^+) = \lim_{x\to p^+} G(p, x)$$

and $$G(p^-, p) = \lim_{x\to p^-} G(x, p).$$

G is:

(a) bounded on $[a, b]$ means there is a subdivision D of $[a, b]$ and a number M such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D and $0 < i \leqq n$, then $|G(x_{i-1}, x_i)| < M$.

(b) product bounded on $[a, b]$ means there is a number M and a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D and $0 < p < q \leqq n$, then $|\prod_{i=p}^{q} (1 + G_i)| < M$.

(c) of bounded variation on $[a, b]$ means there is a number M and a subdivision D of $[a, b]$ such that if D' is a refinement of D, then $\sum_{D'} |G_i| < M$.

It should be noted that the set of functions having property (c) constitute a proper subset of the set of functions having property (b).

The sum integral of G exists on $[a, b]$ means there is a number A such that if $c > 0$ then there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D, then $|\sum_{D'} G_i - A| < c$. A will be denoted by $\int_a^b G$.

The product integral of G exists on $[a, b]$ means there is a number A such that if $c > 0$ then there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D, then $|\prod_{D'} [1 + G_i] - A| < c$. A will be denoted by $\prod_a^b (1 + G)$.

Definitions of words, phrases, or symbols used, but not defined, may be found in [3].

The following theorems are used later and are stated here for convenience.

A. [8, p. 151]. *If n is an integer greater than* 1 *and each of $\{a_i\}_{i=1}^{n}$ and $\{b_i\}_{i=1}^{n}$ is a sequence of numbers, then*

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} b_j\right)(a_i - b_i)\left(\prod_{k=i+1}^{n} a_k\right),$$

where

$$\prod_{j=1}^{0} b_j = \prod_{k=n+1}^{n} a_k = 1.$$

B. [3, Theorem 4.1]. *Suppose G is a function from $R \times R$ to R, $\int_a^b G$ exists, and for each $a \leqq x < y \leqq b$, $H(x, y) = |G(x, y) - \int_x^y G|$. Then, $\int_a^b H$ exists and is 0.*

C. [2, Theorem 3]. *If G is a function from $R \times R$ to R and $\int_a^b G^2 = 0$ then the following two statements are equivalent:*
(1) *$\int_a^b G$ exists and*
(2) *$\prod_a^b (1 + G)$ exists and is not zero.*
Furthermore, if either (1) *or* (2) *is true, then $\int_a^b G = \ln \prod_a^b (1 + G)$.*

D. *If $\prod_a^b (1 + G)$ exists and $\prod_b^c (1 + G)$ exists, then $\prod_a^c (1 + G)$ exists and is $\prod_a^b (1 + G) \cdot \prod_b^c (1 + G)$.*

The following theorem follows directly from Theorem B.

E. *If G is a function from $R \times R$ to R, $\int_a^b G$ exists and $\epsilon > 0$, then there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D, $0 < i \leqq n$, and $D_i = \{p_j\}_{j=0}^m$ is a subdivision of $[x_{i-1}, x_i]$, then $|\sum_{D_i} G_j - G(x_{i-1}, x_i)| < \epsilon$.*

F. [3, Theorem 4.2]. *If G is a function from $R \times R$ to R such that $\prod_a^b (1 + G)$ exists and for each $x < y$*

$$H(x, y) = \left|[1 + G(x, y)] - \prod_x^y (1 + G)\right|,$$

then $\int_a^b H$ exists and is 0.

2. **Theorems.** Theorems 1–4 establish relationships between sum integrals and product integrals using the two similar continuity conditions $\prod_a^b (1 + G^2) = 1$ and $\int_a^b G^2 = 0$.

THEOREM 1. *If $\prod_a^b (1 + G^2) = 1$ and $\epsilon > 0$, then there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D then for each $0 < i \leqq n$, $|G(x_{i-1}, x_i)| < \epsilon$.*

PROOF. Since $\prod_a^b (1 + G^2) = 1$ and $\epsilon^2 > 0$ there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D, then

$$\begin{aligned}
\epsilon^2 &> \left| \prod_{D'} (1 + G_i^2) - 1 \right| \\
&= \left| \prod_{D'} (1 + G_i^2) - \prod_{D'} 1 \right| \\
&= \left| \sum_{D'} \left[\prod_{j=1}^{i-1} 1 \right] [1 + G_i^2 - 1] \left[\prod_{j=i+1}^{n} (1 + G_i^2) \right] \right| \\
&= \sum_{D'} G_i^2 \left(\prod_{j=i+1}^{n} (1 + G_i^2) \right) \\
&\geqq \sum_{D'} |G_i|^2.
\end{aligned}$$

Hence for $0 < i \leqq n, \epsilon^2 > |G_i^2|$ and $\epsilon > |G_i|$.

Theorem 2. *If* $\prod_a^b (1 + G^2) = 1$ *and* $\Pi_a^b (1 + G)$ *exists then if* $\epsilon > 0$ *there is a subdivision* D *of* $[a, b]$ *such that if* $D' = \{x_i\}_{i=0}^n$ *is a refinement of* D, *then* $|\prod_{x_{i-1}}^{x_i} (1 + G) - 1| < \epsilon$ *for each* $0 < i \leqq n$.

Proof. Suppose theorem is false. Then there is an $\epsilon > 0$ such that if D is a subdivision of $[a, b]$ there is a refinement $D' = \{x_i\}_{i=0}^n$ of D such that $|\prod_{x_{i-1}}^{x_i} (1 + G) - 1| \geqq \epsilon$ for some $0 < i \leqq n$. From Theorem 1 and Theorem F there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D and $0 < i \leqq n$, then $|G(x_{i-1}, x_i)| < \epsilon/2$ and $|\prod_{x_{i-1}}^{x_i} (1 + G) - [1 + G(x_{i-1}, x_i)]| < \epsilon/2$.

There is a refinement D' of D and an x_i in D' such that

$$\left| \prod_{x_{i-1}}^{x_i} (1 + G) - 1 \right| \geqq \epsilon.$$

Therefore,

$$\begin{aligned}
\epsilon &\leqq \left| \prod_{x_{i-1}}^{x_i} (1 + G) - [1 + G(x_{i-1}, x_i)] + [1 + G(x_{i-1}, x_i)] - 1 \right| \\
&\leqq \left| \prod_{x_{i-1}}^{x_i} (1 + G) - [1 + G(x_{i-1}, x_i)] \right| \\
&\quad + |[1 + G(x_{i-1}, x_i)] - 1| \\
&< \frac{\epsilon}{2} + |G(x_{i-1}, x_i)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2},
\end{aligned}$$

and $\epsilon < \epsilon$ so that the assumption is false and the theorem is true.

THEOREM 3. *The following two statements are equivalent*:
(1) $\prod_a^b (1 + G^2)$ *exists and is* 1, *and*
(2) $\int_a^b G^2$ *exists and is* 0 *and* G^2 *is product bounded on* $[a, b]$.

PROOF.
A. Suppose (1) is true and $\epsilon > 0$. Then there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D, then

$$\epsilon > \left| \prod_{D'} (1 + G_i^2) - 1 \right|$$

$$= \left| \prod_{D'} (1 + G_i^2) - \prod_{D'} 1 \right|$$

$$= \left| \sum_{D'} \left[\prod_{j=1}^{i-1} 1 \right] [1 + G_i^2 - 1] \left[\prod_{j=i+1}^{n} (1 + G_j^2) \right] \right|$$

$$\geqq \sum_{D'} G_i^2.$$

Thus $\int_a^b G^2 = 0$.

Also note that if $0 < p < q \leqq n$ then $\prod_{i=p}^q (1 + G^2) \leqq \prod_{D'} (1 + G^2) < 1 + \epsilon$, so that G^2 is product bounded on $[a, b]$.

B. Suppose (2) is true and $\epsilon > 0$. Then there is a subdivison D of $[a, b]$ and a number M such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D, then

(1) $$\text{if } 0 < p < q \leqq n, \text{ then } \prod_{i=p}^{q} (1 + G^2) < M,$$

and

(2) $$\sum_{D'} G^2 < \frac{\epsilon}{M}.$$

Let $D' = \{x_i\}_{i=0}^n$ be a refinement of D, then

$$\left| \prod_{D'} (1 + G_i^2) - 1 \right| =$$

$$\left| \prod_{D'} (1 + G_i^2) - \prod_{D'} 1 \right| = \sum_{D'} \left[\prod_{j=1}^{i-1} 1 \right] [1 + G_i^2 - 1] \left[\prod_{j=i+1}^{n} (1 + G^2) \right]$$

$$< M \sum_{D'} G_i^2$$

$$< \epsilon .$$

Hence $\prod_a^b(1 + G^2)$ exists and is 1.

Theorem 4 establishes a relationship between the equivalence of sum integrals and the equivalence of product integrals for functions from $R \times R$ to R.

THEOREM 4. *If each of F and G is a function from* $R \times R$ to R *then the following two statements are equivalent:*

(1) $\int_a^b G = \int_a^b F$ *and* $\int_a^b G^2 = \int_a^b F^2 = 0$, *and*

(2) $\prod_a^b (1 + G) = \prod_a^b (1 + F) \neq 0$ *and* $\prod_a^b (1 + F^2) = \prod_a^b (1 + G^2) = 1$.

PROOF.

A. Suppose (1) is true. From Theorem C, $\prod_a^b (1 + F)$ exists, $\prod_a^b (1 + G)$ exists, and $\ln \prod_a^b (1 + F) = \int_a^b F = \int_a^b G = \ln \prod_a^b(1 + G)$. Hence, $\prod_a^b (1 + F) = \prod_a^b (1 + G)$.

Since $\int_a^b F^2 = \int_a^b G^2 = 0$, $\int_a^b F^4 = \int_a^b G^4 = 0$ so that we have, again from Theorem C, that $\prod_a^b (1 + F^2)$ exists, $\prod_a^b (1 + G^2)$ exists, and

$$\ln \prod_a^b (1 + F^2) = \int_a^b F^2 = 0 = \int_a^b G^2 = \ln \prod_a^b (1 + G^2).$$

Hence, $\prod_a^b (1 + F^2) = \prod_a^b (1 + G^2) = 1$.

Therefore if (1) is true then (2) is true.

B. Suppose (2) is true. Since $\prod_a^b (1 + F^2) = \prod_a^b (1 + G^2) = 1$, from Theorem 3, $\int_a^b G^2 = \int_a^b F^2 = 0$. Since $\int_a^b G^2 = 0$ and $\prod_a^b (1 + G) \neq 0$, from Theorem C, $\prod_a^b (1 + G)$ exists and is $\exp(\int_a^b G)$, and similarly, $\prod_a^b (1 + F) = \exp(\int_a^b F)$. Since $\prod_a^b(1 + G) = \prod_a^b (1 + F)$, then $\int_a^b G = \int_a^b F$.

The following example is of a function G having the property that $\int_{-1}^1 G$ exists and $\int_{-1}^1 G^2$ is non-zero. Thus, Theorem 3 of [2] can not be used to determine the value of $\prod_{-1}^1 (1 + G)$ nor can it be used to determine whether or not $\prod_{-1}^1 (1 + G)$ even exists.

EXAMPLE. Suppose g is defined by

$$g(x) = \begin{cases} (n+2)\left[\dfrac{n+3}{(n+1)(n+2)} - x\right], & \text{if } 0 < x \leqq 1 \text{ and } x \text{ is in} \\ & \left[\dfrac{1}{n+1}, \dfrac{n+3}{(n+1)(n+2)}\right], \\ & \text{for some positive integer } n, \\ \dfrac{(n+1)(n+2)}{2}\left[x - \dfrac{n+3}{(n+1)(n+2)}\right], & \text{if } 0 < x \leqq 1 \text{ and } x \text{ is in} \\ & \left[\dfrac{n+3}{(n+1)(n+2)}, \dfrac{1}{n}\right], \\ & \text{for some positive integer } n, \\ 0, & \text{if } x = 0 \end{cases}$$

and G is defined by

$$G(x, y) = \begin{cases} g(y) - g(x), & \text{if } 0 \leqq x < y < 1 \\ \sin \dfrac{\pi x}{4}, & \text{if } 0 < x < y = 1 \\ 0, & \text{if } -1 \leqq x < y \leqq 0 \end{cases}$$

Then (1) $\int_{-1}^{1} G$ exists and is $\sqrt{2} + 1/\sqrt{2}$, since

$$\begin{aligned} \int_{-1}^{1} G &= \int_{-1}^{0} G + \int_{0}^{1} G \\ &= 0 + \sum_{D} G_i \\ &= \sum_{D-\{x_n\}} G_i + G(x_{n-1}, x_n) \\ &= g(x_{n-1}) - g(0) + \sin \frac{\pi x_{n-1}}{4} \\ &= \frac{(2)(3)}{2}\left[x_{n-1} - \frac{4}{(2)(3)}\right] + \sin \frac{\pi x_{n-1}}{4} \\ &\to \frac{\sqrt{2}+1}{\sqrt{2}} \text{ as } x_{n-1} \to 1, \end{aligned}$$

(2) $\int_{-1}^{1} G^2$ is not 0, and
(3) G is not of bounded variation on $[-1, 1]$.

Hence the hypothesis of Theorem 3 in [2] is not satisfied. However, the hypothesis of Theorem 5 of the present paper is satisfied and hence $\prod_{-1}^{1} (1 + G)$ exists and is $\epsilon/\sqrt{2}$.

Furthermore, if $\int_a^b G^2$ is required to be 0 and $\int_a^b G$ exists, then G may still fail to be of bounded variation on $[a, b]$. So the necessity of (2) and (4) in Theorem 5 can be seen.

THEOREM 5. *Suppose G is a function from $R \times R$ to R defined on $[a, b]$ such that*

(1) *$\int_a^b G$ exists,*

(2) *there is a subdivision $D = \{x_i\}_{i=0}^n$ of $[a, b]$ such that if $[p, q]$ is a subset of (x_{i-1}, x_i) for some $0 < i \leqq n$, then $\int_p^q G^2 = 0$,*

(3)
$$H(x,y) = \begin{cases} G(x,y), & \text{if } x \neq x_i, \quad i = 0, 1, \cdots n, \\ & \quad y \neq x_i,\ i = 1, 2, \cdots, n \\ G(y^-, y), & \text{if } x = x_i, \quad i = 0, 1, \cdots n-1. \\ G(x, x^+), & \text{if } y = x_i, \quad i = 1, 2, \cdots n. \end{cases}$$

and (4) *for each x_i, $i = 0, 1, \cdots n$, in D, there is a segment (c_i, d_i) containing x_i such that G is of bounded variation on $[c_i, d_i]$.*

Then, $\prod_a^b (1 + G)$ exists and is

$$\left\{\exp\left(\int_a^b H\right)\right\}\left\{\prod_{i=1}^{n} [1 + G(x_i^-, x_i)]\right\}\left\{\prod_{i=0}^{n-1} [1 + G(x_i, x_i^+)]\right\}.$$

It should be noted that if for some $0 < i \leqq n$, $\int_{x_{i-1}}^{x_i} G^2 = 0$, then $G(x_i^-, x_i) = 0$ and $G(x_{i-1}, x_{i-1}^+) = 0$.

In order to establish Theorem 5 we need the following seven lemmas.

LEMMA 1. *Suppose* (1), (2), *and* (4) *of the hypothesis of Theorem* 5 *are satisfied, $\epsilon > 0$, and $0 < i \leqq n$, then there is an x in (x_{i-1}, x_i) and a y in (x_{i-1}, x_i) such that if $x_{i-1} < p < q \leqq x$ and $y \leqq r < s < x_i$, then* (A) *$|G(p, q)| < \epsilon$ and* (B) *$|G(r, s)| < \epsilon$.*

PROOF. Suppose the conclusion is false, then for each x in (x_{i-1}, x_i) there is a p and a q such that $x_{i-1} < p < q \leqq x$ and $|G(p, q)| \geqq \epsilon$.

Since G is of bounded variation on $[c_{i-1}, d_{i-1}]$, there is a number M and a subdivision $D_1 = \{r_i\}_{i=0}^z$ of $[x_{i-1}, d_i]$ such that if $D' = \{p_i\}_{i=0}^m$ is a refinement of D_1, then $\sum_{D'} |G(p_{i-1}, p_i)| < M$. Let Q be a positive integer such that $Q > 2M/\epsilon$. Let $\{[\alpha_i, \beta_i]\}_{i=1}^{Q}$ denote a sequence of intervals such that for each $0 < i \leqq Q$,

(1) $x_{i-1} < \alpha_i < \beta_i \leqq d_i$,

(2) $\beta_{i+1} < \alpha_i$,

and (3) $|G(\alpha_i, \beta_i)| \geqq \frac{\epsilon}{2}$.

Let $D' = D_1 + \sum_{i=1}^{Q} \{\alpha_i\} + \sum_{i=1}^{Q} \{\beta_i\} = \{p_i\}_{i=0}^{m}$ denote a subdivision of $[x_{i-1}, d_i]$ which is a refinement of D_1. Then,

$$\begin{aligned} M &> \sum_{D'} |G(p_{i-1}, p_i)| \\ &\geqq \sum_{i=1}^{Q} |G(\alpha_i, \beta_i)| \\ &\geqq \frac{\epsilon}{2} \sum_{i=1}^{Q} 1 \\ &\geqq \frac{\epsilon}{2} Q \\ &> \frac{\epsilon}{2} \cdot \frac{2M}{\epsilon} \\ &= M, \text{ which is a contradiction.} \end{aligned}$$

Hence (A) is true.

A similar argument may be used to show (B) is true.

The following lemma is stated for convenience.

Lemma 2. *If* $|A - B| < \epsilon$ *and* $|c - D| < \epsilon$, *then* $|Ac - BD| < \epsilon[|c| + |B|]$.

Lemma 3. *Suppose* (1), (2), *and* (4) *of the hypothesis of Theorem* 5 *are satisfied and* $0 < i \leqq n$, *then* $G(x_{i-1}, x_{i-1}^{+})$ *and* $G(x_i^{-}, x_i)$ *exist.*

Proof. Let $\epsilon > 0$, then from Lemma 1 there is an $x > x_{i-1}$ such that if $x_{i-1} < p < q \leqq x$, then $|G(p, q)| < \epsilon/2$ and from Theorem E there is a subdivision D_1 of $[x_{i-1}, x_i]$ such that if $D' = \{p_j\}_{j=0}^{m}$ is a refinement of D_1 and $D_j = \{y_k\}_{k=0}^{\Omega}$ is a subdivision of $[p_{j-1}, p_j]$, for some $0 < j \leqq m$, then $|\sum_{D_j} G_k - G(p_{j-1}, p_j)| < \epsilon/3$.

Let $x_{i-1} < z < x$ and $x_{i-1} < z < p_1$ and let s and t be numbers such that $x_{i-1} < s < z$ and $x_{i-1} < t < z$. Hence $D_1 + \{z\}$ is a refinement of D_1 and $\{x_{i-1}, s, t, p_1\}$ is a subdivision of $[p_0, z]$ so that

$$|G(x_{i-1}, s) + G(s, t) - G(x_{i-1}, t)| < \epsilon/2$$

and $|G(x_{i-1}, s) - G(x_{i-1}, t)| < \epsilon/2 + |G(s, t)| < \epsilon$.

Therefore $\lim_{s \to x_{i-1}^+} G(x_{i-1}, s) = G(x_{i-1}, x_{i-1}+)$ exists. A similar argument shows that $G(x_i^-, x_i)$ exists.

LEMMA 4. *If the hypothesis of Theorem* 5 *is satisfied then* $\int_a^b H^2 = 0$.

PROOF. Let $1 > \epsilon > 0$ and $0 < i \leqq n$.

Since G is of bounded variation on $[c_i, d_i]$, and on $[c_{i-1}, d_{i-1}]$, x_i is in (c_i, d_i), and x_{i-1} is in (c_{i-1}, d_{i-1}), there is a number M, a subdivision D_2 of $[x_{i-1}, d_{i-1}]$, and a subdivision D_3 of $[c_i, x_i]$ such that if $D' = \{p_i\}_{i=0}^m$ is a refinement of D_2 or D_3 then $\sum_{D'} |G(p_{i-1}, p_i)| < M$.

From Lemma 1 there is an x in (x_{i-1}, d_{i-1}) and a y in (c_i, x_i) such that if $x_{i-1} < p < q \leqq x$ and $y < r < s \leqq x_i$ then $|G(p, q)| < \epsilon/4M$, $|G(r, s)| < \epsilon/4M$, $|G(p, q)| < \epsilon/3$, and $|G(r, s)| < \epsilon/3$. Since $[x, y]$ is a subset of (x_{i-1}, x_i), $\int_x^y G^2 = 0$, and since $H(p, q) = G(p, q)$ for each p in $[x, y]$, q in $[x, y]$, then $\int_x^y H^2 = 0$.

Since $\int_x^y H^2 = 0$ there is a subdivision D_1 of $[x, y]$ such that if $D' = \{p_i\}_{i=0}^m$ is a refinement of D_1, then $|\sum_{D'} [H(p_{i-1}, p_i)]^2| < \epsilon/4$. Let $D_4 = D_1 + D_2 + D_3$ and $D' = \{p_i\}_{i=0}^m$ be a refinement of D_4. Then

$$|\textstyle\sum_{D'} [H(p_{i-1}, p_i)]^2|$$

$$= \sum_{D' \cdot [x_{i-1}, x]} [H(p_{i-1}, p_i)]^2 + \sum_{D' \cdot [x, y]} [H(p_{i-1}, p_i)]^2 + \sum_{D' \cdot [y, x_i]} [H(p_{i-1}, p_i)]^2$$

$$= [H(p_0, p_1)]^2 + \sum_{D' [p_1, x]} [H(p_{i-1}, p_i)]^2 + \sum_{D' \cdot [x, y]} [H(p_{i-1}, p_i)]^2 + \sum_{D' \cdot [y, p_{m-1}]} [H(p_{i-1}, p_i)]^2 + [H(p_{m-1}, p_m)]^2$$

$$< [H(p_0, p_1)]^2 + \sum_{D' \cdot [p_1, x]} [H(p_{i-1}, p_i)]^2 + \frac{\epsilon}{4} + \sum_{D' \cdot [y, p_{m-1}]} [H(p_{i-1}, p_i)]^2 + [H(p_{m-1}, p_m)]^2$$

$$= |G(p_1^-, p_1)|^2 + \sum_{D'\cdot[p_1,x]} |G(p_{i-1}, p_i)|^2 + \frac{\epsilon}{4}$$

$$+ \sum_{D'\cdot[y,p_{m-1}]} |G(p_{i-1}, p_i)|^2 + |G(p_{m-1}, p_{m-1}^+)|^2$$

$$< \left(\frac{\epsilon}{3}\right)^2 + \frac{\epsilon}{4M} \sum_{D'\cdot[p_1,x]} |G(p_{i-1}, p_i)| + \frac{\epsilon}{4}$$

$$+ \frac{\epsilon}{4M} \sum_{D'\cdot[y,p_{m-1}]} |G(p_{i-1}, p_i)| + \left(\frac{\epsilon}{3}\right)^2$$

$$< \frac{\epsilon}{9} + \frac{\epsilon}{4M} \cdot M + \frac{\epsilon}{4} + \frac{\epsilon}{4M} \cdot M + \frac{\epsilon}{9}$$

$$< \epsilon.$$

Therefore $\int_{x_{i-1}}^{x_i} H^2 = 0$ for $0 < i \leqq n$. Hence $\int_a^b H^2 = 0$.

LEMMA 5. *Suppose the hypothesis of Theorem* 5 *is satisfied and* $0 < i \leqq n$, *then* $\int_{x_{i-1}}^{x_i} H$ *exists.*

PROOF. Let $\epsilon > 0$, then from Theorem B, there is a subdivision D_1 of $[x_{i-1}, x_i]$ such that if $D' = \{p_j\}_{j=0}^m$ is a refinement of D_1 then $\sum_{j=1}^m |\int_{p_{j-1}}^{p_j} G - G(p_{j-1}, p_j)| < \epsilon/7$.

From Lemma 3, $G(x_i^-, x_i)$ exists, $G(x_{i-1}, x_{i-1}^+)$ exists and there is an x in $[x_{i-1}, x_i]$ and a y in $[x_{i-1}, x_i]$ such that if $x_{i-1} < p \leqq x$ and $y \leqq q < x_i$, then

(1) $|G(x_{i-1}, p) - G(x_{i-1}, x_{i-1}^+)| < \epsilon/7$

and (2) $|G(q, x_i) - G(x_i^-, x_i)| < \epsilon/7$.

From Lemma 4, $\int_{x_{i-1}}^{x_i} H^2 = 0$ so that there is a number r in (x_{i-1}, x_i) and a number s in (x_{i-1}, x_i) such that if $x_{i-1} < p \leqq r$ and $s \leqq q < x_i$ then $|H(x_{i-1}, p)| < \epsilon/7$ and $|H(q, x_i)| < \epsilon/7$.

Let $D_2 = D_1 + \{x\} + \{y\} + \{r\} + \{s\}$ and $D' = \{p_j\}_{j=0}^m$ be a refinement of D_2. Then

$$\left| \sum_{D'} H_j - \int_{x_{i-1}}^{x_i} G - G(x_i^-, x_i) - G(x_{i-1}, x_{i-1}^+) \right|$$

$$\leqq |H(p_0, p_1)| + |H(p_{m-1}, p_m)| + \left| \sum_{D'-\{p_0\}-\{p_m\}} H_j \right.$$

$$\left. - \int_{p_1}^{p_{m-1}} G \right| + \left| \int_{x_{i-1}}^{p_1} G - G(x_{i-1}, p_1) \right|$$

$$+ \left| \int_{p_{m-1}}^{x_i} G - G(p_{m-1}, x_i) \right|$$
$$+ |G(x_{i-1}, p_1) - G(x_{i-1}, x_{i-1}^+)|$$
$$+ |G(p_{m-1}, x_i) - G(x_i^-, x_i)|$$
$$< \frac{\epsilon}{7} + \frac{\epsilon}{7} + \frac{\epsilon}{7} + \frac{\epsilon}{7} + \frac{\epsilon}{7} + \frac{\epsilon}{7} + \frac{\epsilon}{7}.$$

Hence, $\int_{x_{i-1}}^{x_i} H$ exists.

LEMMA 6. *Suppose the hypothesis of Theorem 5 is satisfied and $0 < i \leqq n$, and $\epsilon > 0$, then there is an x in (x_{i-1}, x_i) and a y in (x_{i-1}, x_i) such that if $x_{i-1} < r \leqq x$ and $y \leqq s \leqq x_i$, then*

$$|[1 + H(x_{i-1}, r)]^{-1}| < \epsilon + 1 \text{ and } |[1 + H(s, x_i)]^{-1}| < \epsilon + 1.$$

PROOF. Lemma 6 follows directly from Lemma 4.

LEMMA 7. *Suppose the hypothesis of Theorem 5 is satisfied and $0 < i \leqq n$, then there is a number M and a subdivision D_1 of $[x_{i-1}, x_i]$ such that if $D' = \{p_j\}_{j=0}^m$ is a refinement of D_1 then*

$$\left| \prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j) \right| < M.$$

PROOF. From Lemmas 4 and 5 and Theorem C, $\prod_{x_{i-1}}^{x_i} (1 + H)$ exists. Lemma 7 then follows immediately from the definition of H and Lemma 6.

PROOF OF THEOREM 5. Suppose $0 < i \leqq n$ and $\epsilon > 0$.

From Lemma 3, $G(x_{i-1}, x_{i-1}^+)$ and $G(x_i^-, x_i)$ exists and there is a subdivision D_1 of $[x_{i-1}, x_i]$ and a number $M > \epsilon$ such that if $D' = \{p_j\}_{j=0}^m$ is a refinement of D_1, then, if $0 < j \leqq m$,

(1) $|1 + G(p_0, p_1)| < M$ and $|1 + G(p_{m-1}, p_m)| < M$

(2) $|1 + G(p_m^-, p_m)| < M$

(3) $|1 + G(p_0, p_0^+)| < M$

(4) $|G(p_0, p_1) - G(p_0, p_0^+)| < 7/32M^2$

(5) $|G(p_{m-1}, p_m) - G(p_m^-, p_m)| < 7/32M^2$

(6) $\left| \exp \left(\int_{x_{i-1}}^{x_i} H \right) \right| < M$

(7) $\left| \prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j) \right| < M.$

From Lemma 1, $H(x_{i-1}, x^+_{i-1}) = 0$, $H(x_i^-, x_i) = 0$, and there is a number x_i' in $[x_{i-1}, x_i]$ and a number x_i'' in $[x_{i-1}, x_i]$ such that if $x_{i-1} < p < q \leqq x_i'$ and $x_i'' \leqq r < s < x_i$ then $|H(p, q)| < \epsilon/8M$ and $|H(r, s)| < \epsilon/8M$. From Lemma 4, $\int_{x_{i-1}}^{x_i} H^2 = 0$ and from Lemma 5, $\int_{x_{i-1}}^{x_i} H$ exists, so that from Theorem C, $\prod_{x_{i-1}}^{x_i} (1 + H)$ exists and is $\exp(\int_{x_{i-1}}^{x_i} H)$. Hence there is a subdivision D_2 of $[x_{i-1}, x_i]$ such that if $D' = \{p_i\}_{i=0}^m$ is a refinement of D_2, then $|\prod_{D'} (1 + H_i) - \prod_{x_{i-1}}^{x_i} (1 + H)| < \epsilon/8M^2$. Let $D_3 = D_1 + \{x_i'\} + \{x_i''\} + D_2$ and let $D' = \{p_j\}_{j=0}^m$ be a refinement of D_3. Let k_0, k_m, and W denote numbers such that

$$(1)\quad H(x_{i-1}, p_1) + k_0 = H(p_0, p_1) + k_0 = J(p_0, p_0{}^+) = 0,$$

$$(2)\quad H(p_{m-1}, p_m) + k_m = H(p_{m-1}, x_i) + k_m = H(p_m{}^-, p_m) = 0,$$

$$(3)\quad |k_0| < \frac{\epsilon}{16M^3},\ |k_m| < \frac{\epsilon}{16M^3},\ |k_0| < 1, \text{and}$$

$$(4)\quad W = \exp\left(\int_{x_{i-1}}^{x_i} H\right)[1 + G(x_i^-, x_i)]\,[1 + G(x_{i-1}, x^+_{i-1})].$$

Then

$$\begin{aligned}
&|\textstyle\prod_{D'} (1 + G_j) - W| \\
&= \Big| [1 + G(p_0, p_1)] \Big[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j) \Big] \\
&\quad [1 + G(p_{m-1}, p_m)] - W \Big| \\
&= |[1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)] \\
&\quad \Big[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j) \Big] [1 + H(p_0, p_1) + k_0] \\
&\quad [1 + H(p_{m-1}, p_m) + k_m] - W \Big| \\
&= \Big| [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)] \\
&\quad \Big[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j) \Big] [1 + H(p_0, p_1)]\,[1 + H(p_{m-1}, p_m)] \\
&\quad + [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)] \\
&\qquad \Big[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j) \Big] [k_0 k_m]
\end{aligned}$$

$$+ [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)]$$

$$\left[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j)\right] [1 + H(p_0, p_1)]\,[k_m]$$

$$+ [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)]$$

$$\left[\prod_{D' - [p_0 p - [pm]} (1 + G_j)\right] [1 + H(p_{m-1}, p_m)]\,[k_0] - W \Big|$$

$$< \Big| [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)] \left[\prod_{D'} (1 + H_j)\right] - W \Big|$$

$$+ \Big| [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)]$$

$$\left[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j)\right] [k_0 k_m] \Big|$$

$$+ 2 \cdot [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)]$$

$$\left[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j)\right] [k_m] \Big|$$

$$+ 2 \,|[1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)]$$

$$\left[\prod_{D' - \{p_0\} - \{p_m\}} (1 + G_j)\right] [k_0]|$$

$$< |\cdot| + M^3 \cdot (1) \cdot \frac{\epsilon}{8M^3} + M^3 \cdot \frac{\epsilon}{8M^3} + M^3 \cdot \frac{\epsilon}{8M^3}$$

$$= \frac{3\epsilon}{8} + \Big| [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)] \left[\prod_{D'} (1 + H_j)\right] - W \Big|$$

$$\leqq \frac{3\epsilon}{8} + |1 + G(p_0, p_1)| \cdot |1 + G(p_{m-1}, p_m)| \cdot$$

$$\cdot \Big| \prod_{x_{i-1}}^{x_i} (1 + H) - \prod_{D'} (1 + H_j) \Big|$$

$$+ \Big| [1 + G(p_0, p_1)]\,[1 + G(p_{m-1}, p_m)] \prod_{x_{i-1}}^{x_i} (1 + H) - W \Big|$$

$$< \frac{3\epsilon}{8} + M^2 \cdot \frac{\epsilon}{8M^2} + \Big| [1 + G(p_0, p_1)]$$

$$[1 + G(p_{m-1}, p_m)] \exp\left(\int_{x_{i-1}}^{x_i} H \right)$$

$$- \exp\left(\int_{x_{i-1}}^{x_i} H\right)[1 + G(x_i^-, x_i)][1 + G(x_{i-1}, x_{i-1}^+)] \Big|$$

$$= \frac{\epsilon}{2} + \Big| \exp\left(\int_{x_{i-1}}^{x_i} H\right)\Big| \cdot \Big| [1 + G(x_{i-1}, p_1)][1 + G(p_{m-1}, x_i)]$$

$$- [1 + G(x_i^-, x_i)][1 + G(x_{i-1}, x_{i-1}^+)] \Big|$$

$$< \frac{\epsilon}{2} + M \cdot \frac{7\epsilon}{32M^2} [|1 + G(p_{m-1}, x_i)|$$

$$+ |1 + G(x_{i-1}, x_{i-1}^+)|], \text{ (Lemma 2)}$$

$$< \frac{\epsilon}{2} + M \cdot \frac{7\epsilon}{32M^2} [M + M]$$

$$< \epsilon.$$

Hence, for $0 < i \leqq n, \prod_{x_{i-1}}^{x_i} (1 + G)$ exists and is

$$\exp\left(\int_{x_{i-1}}^{x_i} H\right)[1 + G(x_i^-, x_i)][1 + G(x_{i-1}, x_{i-1}^+)].$$

Then, from Theorem D, $\prod_a^b (1 + G)$ exists and is

$$\exp\left(\int_a^b H\right)\left\{\prod_{i=1}^{n} [1 + G(x_i^-, x_i)]\right\}\left\{\prod_{i=0}^{n-1} [1 + G(x_i, x_i^+)]\right\}.$$

Note: Results related to Theorem 5 may be found in [4, Theorem 1.7].

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