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INFINITE GROUPS WITH NORMALITY CONDITIONS ON INFINITE SUBGROUPS

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I. **Introduction.** In the papers [2, §§ 2 and 3], and [3] S. N. Černikov studies infinite groups G with the property that every infinite subgroup H of G, $H \neq G$, is distinct from its normalizer in G. Groups with this property are called *IN* groups. Another way of stating the *IN* property is "every infinite subgroup of G is ascendant in G".

IN groups are of two types:

(i) N-groups (i.e. groups with the normalizer condition) or (ii) *IN* groups with a self normalizing (necessarily finite) proper subgroup.

Obviously, the classes (i) and (ii) exhaust all *IN* groups. In [2, Theorems 2.1 and 2.2], Černikov gives a detailed description of the structure of locally finite groups of type (ii). In the later paper [3], it is shown that non-torsion *IN* groups are N groups.

In this paper, we give an exposition of these results. Our proofs are quite different and, in most cases, considerably shorter than those given in [2] and [3]. The papers [2] and [3], especially [2], make use of several results not easily accessible to non-Russian mathematicians. We have attempted here to use techniques that, for the most part, can be found in the books [14], or [18].

Also, we have put the results of [2] in a somewhat wider context. This has been achieved by replacing the notion of an ascendant subgroup by a serial subgroup. Thus, Černikov's Theorems 2.2 and 3.4 of [2] are consequences of our Theorem D (§ III). We also indicate how to construct groups satisfying Černikov's structure theorem for groups of type (ii) (Theorem F in § V).

Finally in § VI, we give a shorter proof of the results of [3], and note some related problems that remain unsolved.

II. **Notation on Basic Concepts.** The class of nilpotent groups is denoted by $\mathfrak{N}$, while the class of finite groups is given by $\mathfrak{F}$. If Σ is any class of groups, $L\Sigma$ is the class of groups having a local system of Σ subgroups [18; p. 5]. Thus $L\mathfrak{N}$ and $L\mathfrak{F}$ denote locally nilpotent and locally finite groups respectively.

H is a serial subgroup of G (H ser G) if there is a series of G that contains H as a member [18; p. 9]. The notions of ascendant (H asc G),

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descendant (H desc G) and subnormal (H sn G) subgroups are specializations of serial subgroups. We let $\mathfrak{N}(s)$, $\mathfrak{N}(a)$, $\mathfrak{N}(d)$, $\mathfrak{N}(\mathrm{sn})$ denote the classes of all groups in which all subgroups are respectively serial, ascendant, descendant or subnormal. Similarly, $I\mathfrak{N}(s)$, $I\mathfrak{N}(a)$, $I\mathfrak{N}(d)$, and $I\mathfrak{N}(\mathrm{sn})$ denote the classes of *infinite* groups in which all infinite subgroups are serial, ascendant, descendant or subnormal. We note that several of the above classes are denoted by other symbols in the literature; e.g. $\mathfrak{N}(s) = \tilde{N}$ [**14**; p. 221], $\mathfrak{N}(a) = N =$ normalizer condition [**14**; p. 220], $\mathfrak{N}(sn) = N_0$, [**15**; p. 319].

2.1. **Serial Subgroups.** The following characterization of serial subgroups in $L\mathfrak{F}$ groups plays an essential role in what follows.

2.1.1 [**11**] or [**16**]. If $G \in L\mathfrak{F}$, H ser G if and only if for every finite subgroup L of G, $H \cap L$ sn L.

We call a class Σ of groups a radical class if

(i) $L(\Sigma) \subset \Sigma$,

(ii) subgroups of Σ groups are Σ groups,

(iii) in any group, the join $\Sigma(G)$ of all the normal Σ subgroups of G is a Σ group.

The following fact is implicit in the work of Plotkin [**17**], and is not difficult to prove directly.

2.1.2. If Σ is a radical class, $G \in L\mathfrak{F}$ and H is a serial Σ subgroup of G, then $H^G \in \Sigma$.

Classes Σ for which we use 2.1.2. include Π groups, Π a set of primes, and $L\mathfrak{N}$ groups.

2.2. **Černikov Groups.** A group G is a Černikov group (or an extremal group) if G is a finite extension of an abelian group with the minimum condition on subgroups (= min).

If G is an infinite Černikov group and $d(G)$ is the intersection of the subgroups of finite index in G, then $d(G)$ has finite index in G [**14**; p. 230], and $d(G)$ is a periodic divisible abelian group with min (i.e., $d(G)$ is a direct sum of a finite number of quasi-cyclic groups).

If H is an arbitrary abelian group, p a prime and n a positive integer, $H(p^n)$ denotes the subgroup of H generated by the elements of order p^n.

We require the following facts relating to Černikov groups.

2.2.1 [**1**], or [**18**; p. 84]. If Y is a periodic group of automorphisms of a Černikov group, then Y is a Černikov group.

2.2.2 [**18**; p. 84]. If H is a divisible abelian group and F is a finite group of automorphisms of H then $H = C_H(F)[H, F]$.

2.2.3 [**18**; p. 230]. A Černikov p-group G is hypercentral. Consequently $G \in \mathfrak{N}(a)$.

2.2.4 [**18**; p. 69]. If H and K are Černikov groups, and G is an extension of H by K, then G is a Černikov group. Also, (and quite simply) subgroups of Černikov groups are Černikov groups.

2.3. **Some Remarks on Injective Modules.** Our presentation here is simplified by appealing to recent results of Hartley and McDougall [**12**]. These results concern modules for integral group rings of certain groups.

We recall that if the abelian group M is a module for the group G, then M may be viewed, in a natural way, as an $R = ZG$ module; here ZG is the integral group ring of G. Specifically, let $m \in M$, $g \in G$ and denote by m^g the image of m under g. Let $\alpha = \Sigma n_g g \in ZG$, where the g's are elements of G, $n_g \in Z$, and all but a finite number of the n_g's $=$ 0. We define $m\alpha$ by $m\alpha = \Sigma n_g(m^g)$. It is easy to check that this product makes M an R module.

The following is a simplified version of Lemma 2.3 of [**12**].

2.3.1. Suppose M is an abelian p group (p a prime) and that M is a module for the finite p' group Q. Let $R = ZQ$. Then M is an injective R module if and only if M is a divisible group.

We also need.

2.3.2 [**12**; Lemma 2.1]. Let R be a ring with 1 and V be an R module (unitary). Suppose that $V = \oplus \{V_\lambda \mid \lambda \in \Lambda\}$ (module direct sum) and that Λ is finite. Let $\bar{V}$ be an R-injective hull of V. Then $\bar{V} = \oplus \{\bar{V}_\lambda \mid \lambda \in \Lambda\}$ where $\bar{V}_\lambda$ is an injective hull of V_λ.

The necessary definitions and theorems on injective modules may be found in the book [**6**; pp. 385–392]. In particular, we assume both the existence and the various characterizations of injective hulls of modules.

2.4. **A Schur-Zassenhaus Property.** The following version of the Schur-Zassenhaus property appears in the early work of Černikov [**4**]. A proof is also indicated in [**7**; p. 22].

2.4.1. Suppose G is a $L\mathfrak{F}$, locally solvable and countable group. Then for every prime p, there is a Sylow p subgroup P and a Sylow p' subgroup Q such that $G = PQ$.

We also require the following fact, which is of a much simpler nature.

2.4.2. Let $G \in L\mathfrak{F}$ have a normal Sylow p subgroup P of finite in dex: then P has a complement K. Any such complement is a Sylow p subgroup of G. Further, any Sylow p' subgroup L of G is conjugate t K.

Proof. Let T be a transversal of P in G. Then $H = \langle T \rangle$ is finite an $G = PH$. Further, $P \cap H$ is a normal Sylow p subgroup of H. It nov follows from the Schur-Zassenhaus-Feit-Thompson theorem for finit groups [20; p. 224] that $P \cap H$ has a complement K in H. Evidently K is a p' group, and $G = PH = PK$. Thus K is a complement of P. A easy argument shows that K is a Sylow p' subgroup of G.

Let L be another Sylow p' subgroup of G. Then $LP/P \simeq L/L \cap$ $\simeq L$, so that L is finite. Since $G \in L\mathfrak{F}$, $Y = \langle L, K \rangle$ is finite. Ob serve also that Y has the normal Sylow p-subgroup $Y \cap P$. Agai using the Schur-Zassenhaus-Feit-Thompson theorem [20; p. 227] w deduce that L and K are conjugate in Y.

2.5. **A Remark on Locally Solvable Groups.** We note the theore of Mal'cev which states that every locally solvable group is an SI grou This result is proved in the book [19; p. 97] and also in [14; p. 183] From this and the Corollary of [16; p. 348], we deduce

2.5.1. If G is a locally solvable group and $1 \neq x \in G$, then $(x^G)' <$ x^G.

III. **The Structure of Locally Finite Groups in $I\mathfrak{N}(s) - \mathfrak{N}(s)$.** W need several lemmas before proceeding with the main result.

Lemma 3.1. $\mathfrak{N}(s) \cap L\mathfrak{F} = (L\mathfrak{N}) \cap L\mathfrak{F}$.

Proof. We need only note that $L\mathfrak{N} \subset \mathfrak{N}(s)$ [14; p. 222].

Lemma 3.2. *If $G \in I\mathfrak{N}(s) \cap L\mathfrak{F}$, Σ is a radical class, and H is maximal Σ subgroup with infinite normalizer, then $H \lhd G$.*

Proof. From the hypotheses, H ser G. Now use 2.1.2.

Lemma 3.3. *If $G \in I\mathfrak{N}(s) \cap L\mathfrak{F}$ and for some finite normal subgrou F, $G/F \in \mathfrak{N}(s)$, then $G \in \mathfrak{N}(s)$.*

Proof. By Lemma 3.1, it suffices to show that $G \in L\mathfrak{N}$. We do th by showing that all Sylow p subgroups are normal. In view of Lemm 3.2, we need only show that finite Sylow p subgroups have infini normalizers. Let K be a finite Sylow p-subgroup of G. Then KF/F is Sylow p subgroup of G/F and hence $KF \lhd G$. This implies that $N($ has finite index in G. By Lemma 3.2, $K \lhd G$.

Lemma 3.4. *If $G \in I\mathfrak{N}(s) \cap L\mathfrak{F}$ and some collection $\Gamma \neq \emptyset$ of infinite normal subgroups has the property $\cap \Gamma = 1$, then $G \in \mathfrak{N}(s)$.*

Proof. Let x and y be elements of coprime order in G, then (using Lemma 3.1) $[x, y] \in H$ for each $H \in \Gamma$. Thus $[x, y] \in \cap\Gamma = 1$ so that x and y commute. It follows that $G \in L\mathfrak{N}$.

Lemma 3.5. *If $G \in I\mathfrak{N}(s) \cap L\mathfrak{F}$ and $G = P]\,R$ ($]$ = split extension), where P is a p group and R is a finite p' group, then for all infinite subgroups $Y \leqq P$ with $Y \lhd G$, we have $YR \lhd G$ and $[P, R] \leqq Y$.*

Proof. By Lemma 2.4.2, YR/Y is a Sylow p' subgroup of G/Y. Since, $G/Y \in \mathfrak{N}(s) \cap L\mathfrak{F} \leqq L\mathfrak{N}$, both results follow.

Theorem A. *If $G \in (I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$, then G has an infinite Sylow p subgroup P such that $G = P]\,K$, where $N(K)$ is a finite group and K is a non-identity nilpotent group.*

Proof. Let A be the intersection of all the infinite normal subgroups of G. A is infinite by Lemma 3.3 and 3.4. Thus A has an infinite abelian subgroup Y ([10] or [18; p. 95]). It follows that $A = Y^G$ is $L\mathfrak{N}$ (2.1.2). Thus, $A = \Pi\{Q_i \mid i \in I\}$, where Q_i is a non-identity Sylow q_i subgroup of A. For each i, $Q_i \lhd G$. Thus, if $|I| > 1$, all Q_i's are finite, and since A is infinite, we conclude that $|I|$ is infinite. But this yields a proper infinite subgroup of A that is normal in G. Hence $|I| = 1$, and A is a p group.

Let P be a Sylow p-subgroup of G containing A; since P is infinite, we have $P \lhd G$. We now show that G/P is finite. Suppose that G/P is infinite. Let x and y be p' elements of G and $U \leqq G$ such that $x, y \in U$ and U/P is countably infinite. Let T be a transversal of P in U and $T_1 = \langle T\rangle$. Now $P \cap T_1$ is a normal Sylow p subgroup of T_1, and T_1 is countable. By 2.4.1, there is a p' group $Q \leqq T_1$ such that $T_1 = (P \cap T_1)Q$. Then $U = T_1P = Q(P \cap T_1)P = QP$. Evidently, Q is an infinite p' subgroup of U. By 2.1.2, Q^U is a p' group. Thus $U = Q^U \times P$ and Q^U is a normal Sylow p' subgroup of U. Thus, $x, y \in Q^U$, so that $\langle x, y\rangle$ is a p' group. We conclude that $S = \{x \in G \mid (|x|, p) = 1\}$ is a p' subgroup of G. S is clearly infinite. Lemma 3.2 now yields $S \lhd G$, and consequently we have $G = P \times S$. This contradicts Lemma 3.4. Thus, G/P is finite. Further, since P is infinite, G/P $\mathfrak{N}(s)$; hence, G/P is nilpotent.

We are now permitted to apply 2.4.2 to produce a complement K of P. If $N(K)$ were infinite, we would have $K \lhd G$ by Lemma 3.2; this implies $G = PK \in L\mathfrak{N}$, a contradiction.

Theorem B. *If $G \in (I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$, then G is a Černikov group.*

Proof. Write $G = P]K$, where P and K are as in Theorem A, a let A be the intersection of the infinite normal subgroups of G. Th $A \leqq P$ and A is infinite (Lemma 3.4). We show that A' is finite. If is infinite, then $A = A'$. Since G is locally solvable, $x \in A$ impl $x^G < A$ (2.5.1). Thus for each $x \in A$, x^G is finite, and $G/C(x^G)$ is fini It follows that $A/(A \cap C(x^G))$ is finite and this implies that $A \leqq C(x$ Hence $A' = 1$, a contradiction. Thus A' is finite.

Let $G_1 = G/A'$ and $A_1 = A/A'$. By Lemma 3.3, $G_1 \in (I\mathfrak{N}(s)$ $\mathfrak{N}(s)) \cap L\mathfrak{F}$; further A_1 contains no proper normal infinite subgrou of G_1. Now $A_1(p) \lhd G_1$. If $A_1(p)$ is infinite, then $A_1(p) = A_1$. Th A_1 is residually finite and consequently A_1K_1 is residually finite, wh $K_1 = KA'/A'$. By Lemma 3.4, $A_1K_1 \in \mathfrak{N}(s)$. Thus $A_1 \leqq C(K_1)$ and Lemma 3.2, $K_1 \lhd G_1$; also K_1 is a nilpotent p' group and $G_1 = P/A' \times$ Thus G_1, as a direct product of two $L\mathfrak{N}$ groups, is in $L\mathfrak{N} \cap L\mathfrak{F}$ $\mathfrak{N}(s)$, a contradiction. We conclude that $A_1(p)$ is finite.

Since $A_1(p)$ is finite A_1 and consequently A, has min. Further, $C(A$ $\leqq C(K)$. Note that K is a Sylow p' subgroup of G (2.4.2), and that nilpotent (Theorem A). Suppose for the moment that $N(K)$ is infini Lemma 3.2 then forces $K \lhd G$. At this stage, we have $G = P \times$ Then G, as the direct product of two $L\mathfrak{N}$ groups, is $L\mathfrak{N}$. This viola the assumption that $G \notin \mathfrak{N}(s)$. We conclude then that $N(K)$, and he $C(K)$, is finite. The finiteness of $C(K)$ implies that $C(AK)$ is finite. Lemma 3.5, $N(AK) = G$. By 2.2.1, $G/C(AK)$ is a Černikov group. N G is an extension of a finite group by a Černikov group. It now follo from 2.2.4 that G is a Černikov group.

Theorem C. *Let $G \in (I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$ and write $G = P]K$ in Theorem A. If $x \in K$ normalizes some proper infinite subgroup $d(P)$, then $x \in C(P)$.*

Proof. Let $x \in K$, and H be an infinite proper subgroup of d such that $H^{\langle x \rangle} = H$. We now show that $C(x) \cap d(P)$ is infinite.

If $C(x) \cap d(P)$ is finite, then $[d(P), \langle x \rangle]$ has finite index in $d(P)$ 2.2.2. But we also have $[d(P), \langle x \rangle] < H$ by Lemma 3.5 (applied to $I\mathfrak{N}(s)$ group $d(P)\langle x \rangle$). Since H has infinite index in $d(P)$, we hav contradiction. Hence, $C(x)$ is infinite and by Lemma 3.2, we h $\langle x \rangle \lhd P\langle x \rangle$. Thus $x \in C(P)$.

Theorem D. *An infinite locally finite group G is in $I\mathfrak{N}(s) - \mathfrak{N}(s$ and only if G has an infinite normal Sylow p subgroup P such that*

(i) *P is a Černikov p group,*

(ii) *$G = P]K$ where $K \neq 1$, and $G/d(P)$ is a finite nilpotent group,*

(iii) *if $x \in K$ normalizes an infinite proper subgroup of $d(P)$, t $x \in C(P)$,*

(iv) *there is an $x \in K$ such that $[x, P] \neq 1$.*

Proof. The necessity of conditions (i)–(iv) is incorporated in Theorems A–C. We show here that in any group G with properties (i)–(iv), every infinite subgroup is ascendant.

Let H be an infinite subgroup of G. If $d(P) \leqq H$, then H sn G since $G/d(P)$ is nilpotent. Suppose $d(P) \not\leqq H$. Then $H = (H \cap P)]L$, $H \cap P$ being a normal Sylow p subgroup of H and L a p' group (2.4.2). By 2.4.2, L is conjugate to a subgroup L^t of K.

Now, $H^t \cap d(P)$ is a proper infinite subgroup of $d(P)$ and is normalized by L^t. By condition (iii) $L^t \leqq C(P)$ and so $L \leqq C(P)$. We now have $H = (H \cap P) \times L$ asc $P \times L$ (2.2.3). Since $P \times L$ sn G, we have H asc G.

The condition (iv) insures that $G \notin \mathfrak{N}(s)$.

Later in § IV, we give a somewhat different statement of Theorem D (Theorem D$'$). To do this, we shall need

Corollary 3.1. *Let $G \in (I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$. Then, in the notation of Theorem* D, *$d(P) = [P, K]$ and $[d(P), K] = d(P)$.*

Proof. It suffices to prove $[d(P), K] = d(P)$. Suppose $[d(P), K] < d(P)$. Using 2.2.2, we see that $C_{d(P)}(K)$ is infinite, as $d(P)/[d(P), K]$ is infinite. Then, Theorem D (iii) yields $K \leqq C(P)$ which violates part (iv) of Theorem D. Thus, $[d(P), K] = d(P)$.

As a consequence of Theorem D, we have

Corollary 3.2. $(I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F} = (I\mathfrak{N}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F}$.

Proof. Suppose $G \in (I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$. By Theorem D, G has min. This implies that $G \in I\mathfrak{N}(a)$. Evidently, $G \notin \mathfrak{N}(a)$. This yields $(I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F} \leqq (I\mathfrak{N}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F}$. For the other inclusion, let $G \in (I\mathfrak{N}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F}$. Clearly $G \in I\mathfrak{N}(s)$. Further it is not hard to show the existence of a finite subgroup F of G with $N(F) = F$. In view of 2.1.1, F is not a serial subgroup. This establishes $(I\mathfrak{N}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F} \leqq (I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$ and the corollary is proved.

Also of interest is

Corollary 3.3. *An uncountable $I\mathfrak{N}(s) \cap L\mathfrak{F}$ ($I\mathfrak{N}(a) \cap L\mathfrak{F}$) group is an $\mathfrak{N}(s)$ ($\mathfrak{N}(a)$) group.*

Corollary 3.3 is proved by observing that the groups occurring in Theorem D are countable.

IV. Strongly Irreducible Automorphisms of Divisible Abelian Groups.

DEFINITION. Let P be a divisible abelian p group of finite rank (i. the p-rank of the abelian group P, [19; p. 35]) and A a periodic gro of automorphisms of P. A is called strongly irreducible ($S-I$) if each $1 \neq a \in A$, and each infinite subgroup $H \leqq P$, $H^{\langle \alpha \rangle} = P$.

These are the types of automorphisms that arise in Theorem D (i In the section, we prove

THEOREM E. *Let P be a divisible abelian p-group of finite rank, a A be a nilpotent p' group of $S-I$ automorphisms of P. Then*
(i) *A is cyclic,*
(ii) *if $1 \neq a \in A$, then $\langle a \rangle$ acts irreducibly on $P(p)$,*
(iii) *if $n =$ rank (P), and q is the smallest prime dividing $|A|$, th $n =$ the order of $p(\bmod q)$; and so $n \mid q-1$.*

Theorem E will be used to give a different formulation of Theor D. Before proceeding further, we note that periodic groups of au morphisms of divisible abelian p groups of finite rank are finite [p. 84]. Thus, we assume all $S-I$ groups of automorphisms are fin Also, throughout the remainder of this section, P will denote a divisi abelian p group of finite rank.

LEMMA 4.1. *Let $1 \neq A$ be an $S-I$ subgroup of* Aut(P) *w* $(|A|, p) = 1$. *Then for each element $y \in P(p)$, $y \neq 1$, and each e ment $1 = a \in A$, we have $y^{\langle \alpha \rangle} = P(p)$.*

PROOF. Let $1 \neq a \in A$. We view P as a module for the ring R $Z\langle a \rangle$. By 2.3.1 P is an injective R module, and it is not difficult to sh that P is an R injective hull of $P(p)$.

Let $1 \neq y \in P(p)$ and $H = y^{\langle \alpha \rangle}$. Then H is an R submodule of If $H < P(p)$, Maschke's theorem yields a R submodule $0 \neq H_1 \leqq P$ such that $P(p) = H \oplus H_1$. We now use 2.3.2 to obtain $P = \overline{P(p)} =$ $\oplus \overline{H_1}$; here the $\overline{\ \ }$'s denote R injective hulls. Now $\bar{H}$ is an infinite invariant subgroup of P so that $\bar{H} = P$. This forces $\overline{H_1} = 0$ and c sequently $H_1 = 0$ (in the module notation). This contradicts the sumption $H < P(p)$. Thus $H = P(p)$ and the proof is complete.

The following easy remark will simplify some of our arguments.

LEMMA 4.2. *Let G be any abelian p group and $\alpha \in$* Aut(G) *with finite and $(|\alpha|, p) = 1$. If α centralizes all elements of order p in then $\alpha = 1$.*

PROOF. Suppose the lemma false and let $x \in G$ be of least order s that $\alpha \notin C(x)$. Since G is abelian, $[x, \alpha]^p = [x^p, \alpha]$. By the minima of $|x|$, $[x^p, \alpha] = 1$. Thus $[x, \alpha]^p = 1$ and $[x, \alpha] \in C(\alpha)$. Thus, 1 $[x, \alpha]^p = [x, \alpha^p]$. Therefore $\langle \alpha^p \rangle = \langle \alpha \rangle \leqq C(x)$; a contradiction.

LEMMA 4.3. *Let $1 \neq A$ be an $S-I$ subgroup of* Aut(P) *with* $(|A|, p) = 1$. *Let* $n =$ rank(P), *and* $\alpha: A \to GL(n, p)$ *be defined by* $a\alpha = a \mid P(p)$ *(the restriction of a to $P(p)$). Then α is an isomorphism and for each $1 \neq a \in A$, $\langle a\alpha \rangle$ is an irreducible subgroup of $GL(n, p)$.*

PROOF. Certainly α is a homomorphism. Lemma 4.2 implies ker $\alpha =$ 1. The irreducibility of $\langle a\alpha \rangle$ follows from Lemma 4.1.

PROOF OF THEOREM E. Lemma 4.3 immediately gives (ii). Let T be an abelian subgroup of A. By Lemma 4.3, T is an irreducible subgroup of $GL(n, p)$. As is well known, abelian irreducible subgroups of $GL(n,p)$ are cyclic [8; p. 65]. Thus T is cyclic. Thus, if q is a prime, $q \neq 2$, then the Sylow q subgroup of A is cyclic [20; p. 252]. Let L be the Sylow 2 subgroup of A, and $s \in L$ be an element of order 2. Since $\langle s \rangle$ acts irreducibly on $P(p)$, we must have $r(P) = 1$. Thus, (again by Lemma 4.3) L is cyclic.

Hence, all Sylow subgroups of A are cyclic. The nilpotence of A implies now that A is cyclic. This establishes (i).

Let q be the smallest prime divisor of A and $a \in A$ be an element of order q. Then $P(p)$ is a faithful and irreducible module for $\langle a \rangle$; i.e. $P(p)$ is irreducible as a $Z_p\langle a \rangle$ module. The possible dimensions of such modules are enumerated in the book [13; II, 3.10]. In particular, we must have $n =$ order of p (mod q). Consequently, $n \mid q - 1$. This completes the proof of Theorem E.

The proof of Theorem E shows than an $S - I$ subgroup A of Aut(P) has cyclic Sylow q subgroups for $q \neq p$. Černikov has shown [2; p. 723–724] that the Sylow p subgroups of A are also cyclic. The proof of this fact makes essential use of results in the paper [5].

In view of Theorem E and Corollary 1 of Theorem D, we may now give a somewhat different form of Theorem D.

THEOREM D$'$. *$G \in (I\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$ if and only if G has an infinite normal Sylow p subgroup P such that*

(i)$'$ *P is a Černikov p group,*

(ii)$'$ *$G = P]K$ where K is a finite nilpotent p' group and $K \neq 1$,*

(iii)$'$ *$Y = [P, K]$ has finite index in P and is a divisible abelian group with min,*

(iv)$'$ *$K/C_C(P)$ is a non-trivial cyclic group of $S - I$ automorphisms of Y.*

We note the following special case of this theorem.

COROLLARY 4.1. *Suppose in the notation of Theorem D$'$ that $Y = P$ (i.e., P is divisible abelian of finite rank). Let H be an infinite subgroup* *of G, where G satisfies the conditions of Theorem D$'$. Then either*

(a) $P \leqq H$, *or*

(b) $H = (H \cap P) \times L$ *where L is conjugate to a subgroup of K.*

Consequently, H sn G.

If, in addition, $C_K(P) = 1$, *then* H sn_2 G (H sn_2 G *means* $H \lhd H^G$).

This corollary can be proved by using Theorem E in conjunction with the method of proof of Theorem D. We do not repeat these arguments here.

V. **Construction of Groups Satisfying the Conditions of Theorem D.** This task turns out to be rather easy if we make use of the results of 2.3.

THEOREM F. *Let E be an elementary abelian p group of* rank *n. Further, let A be a cyclic p$'$ subgroup of GL(n, p) such that every non-identity subgroup of A acts irreducibly on E. Viewing E as an R = ZA module, let $\bar{E}$ be an R-injective hull of E. Then $\bar{E}$ is a divisible abelian p-group of* rank *n and the group* $G = \bar{E}]A$ *satisfies the conditions of Theorem D (or Theorem D$'$). (Here the action of A on $\bar{E}$ is induced from the action of R on $\bar{E}$).*

PROOF. The existence of R injective hulls is well known [**6**; p. 389]. We first note that $\bar{E}$, as an abelian group, is divisible. This is a consequence of [**6**;, p. 393, Ex. 3]. Let Y be the Sylow p-subgroup of $\bar{E}$. Then Y is a divisible abelian group and is also on R-submodule of $\bar{E}$. Now 2.3.1 implies that Y is injective as an R-module. Since $E \leqq Y$, we deduce that $Y = \bar{E}$; thus $\bar{E}$ is a p-group.

Suppose $\bar{E}(p) > E$, and let $y \in \bar{E}(p) - E$. Then by Maschke's theorem, $D = E + yR = E \oplus F$ for some R submodule F of D (yR is a finite elementary p-group). On the other hand, $F \cap E \neq 0$, since $\bar{E}$ is an essential extension of E. This contradiction shows that $\bar{E}(p) = E$; thus, $\bar{E}$ has rank n.

Let $1 \neq a \in A$ and suppose H is an infinite $\langle a \rangle$ invariant subgroup of G. Then $d(H)$ is an infinite divisible group and $d(H)$ is $\langle a \rangle$ invariant. Certainly $d(H) \cap E \neq 1$; this follows from the fact that $\text{rank}(E) = \text{rank}(\bar{E})$. The $\langle a \rangle -$ irreducibility of E now yields $E \leqq d(\text{H})$. Thus, $\text{rank}(d(H)) = \text{rank}(\bar{E}) = n$. The divisibility of $d(H)$ now implies that $d(H) = \bar{E}$ (otherwise $d(H)$ would be a proper direct summand of $\bar{E}$). Thus, $H = \bar{E}$ and A is an $S-I$ group of automorphisms of $\bar{E}$.

It is now a simple matter to show that all the conditions of Theorem D hold. This completes the proof of Theorem F.

We note some additional properties of the group G constructed in Theorem F.

5.1 $[\bar{E}, A] = \bar{E}$.

5.2 If H is an infinite subgroup of G, then either
(a) $\bar{E} < H$, or
(b) H is abelian and $H \text{ sn}_2 G$.

5.3 Every non-abelian infinite subgroup is normal.

The property 5.1 follows from Corollary 3.1, while 5.2 and 5.3 are consequences of Corollary 4.1.

VI. **Non-periodic $I\mathfrak{N}(a)$ Groups.** The main result here is that an element of infinite order in an $I\mathfrak{N}(a)$ group forces the group to be an $\mathfrak{N}(a)$ group. Similar results hold for the class $I\mathfrak{N}(sn)$. However, we have been unable to develop a corresponding theory for either of the classes $I\mathfrak{N}(s)$ or $I\mathfrak{N}(d)$. The key fact that enables us to prove our main result here is the local nilpotence of groups in the class $\mathfrak{N}(a)$ ([18; p. 61]). Whether or not $\mathfrak{N}(s)$ groups (or even $\mathfrak{N}(d)$) groups are locally nilpotent is an unresolved question.

THEOREM G [3; p. 29]. *If G is not periodic and $G \in I\mathfrak{N}(a)$ ($I\mathfrak{N}(\text{sn})$), then $G \in \mathfrak{N}(a)$ ($\mathfrak{N}(\text{sn})$).*

PROOF. Let x be an element of infinite order in the $I\mathfrak{N}(a)$ group G and let F be a finite subgroup of G. Then $\langle x \rangle$ asc $\langle F, x \rangle$ so that $x^F \in L\mathfrak{N}$ ([9] or [18; p. 61]). The finiteness of F insures that x^F is finitely generated. Thus, x^F is a finitely generated nilpotent group and consequently D, the center of x^F is a finitely generated infinite abelian group. D contains an infinite characteristic torsion free subgroup E. Let $S = EF$ and p be a prime not dividing $|F|$. The group S/E^{p^n} is a finite $\mathfrak{N}(a)$ (and hence nilpotent) group for each positive integer n. Thus, for each n, $[E, F] \leqq E^{p^n}$, and this gives $[E, F] \leqq \bigcap E^{p^n} = 1$.
We now have $F \lhd EF = S$ asc G which completes the proof. The same proof shows that non-periodic $I\mathfrak{N}(\text{sn})$ groups are $\mathfrak{N}(\text{sn})$.

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