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FIXED-POINT THEOREMS FOR MAPPINGS WITH NON-CONVEX DOMAIN AND RANGE

PETER KUHFITTIG

ABSTRACT. The main purpose of this paper is to obtain two fixed-point theorems under conditions weaker than the usual condition of convexity: if H is a subset of a linear space and K a subset of H, then H is defined to be "star-shaped with respect to K" if $x \in K$ and $y \in H$ implies that $(1 - t)x + ty \in H$ $(0 < t < 1)$.

Suppose X is a Banach space, H a closed subset of X, and K a closed (respectively compact and star-shaped) subset of H. Suppose further that H is star-shaped with respect to K. If $T : K \to H$ is a contractive (respectively nonexpansive) mapping, and if $T : \partial_H K \to K$, then T has a fixed point in K.

One of the most common assumptions in fixed-point theory, especially for nonexpansive mappings from a set C to itself, is that C be convex. In an attempt to relax this assumption, Dotson [1] has recently obtained a fixed-point theorem for nonexpansive self-mappings of compact star-shaped sets in Banach space. Generalizing in another direction, Kirk [2] has replaced the self-mappings by a wider class: if X is a reflexive Banach space, H a closed convex subset of X, and K a bounded, closed convex subset of H with normal structure, then $T : K \to H$, T nonexpansive, has a fixed point in K, under the natural assumption that the relative boundary of K is mapped back into K. The main purpose of this paper is to obtain, under certain conditions, two fixed-point theorems for $T : K \to H$, where H is "star-shaped with respect to K" (defined below), again assuming that the relative boundary of K (denoted by $\partial_H K$) is mapped into K.

DEFINITION 1. A subset H of a linear space is said to be *star-shaped* if there exists at least one point $c \in H$ (called a *star-center*), such that, if $x \in H$ and $0 < t < 1$, then $(1 - t)c + tx \in H$.

DEFINITION 2. Suppose H is a subset of a linear space and K a subset of H. We shall call *H star-shaped with respect to K* if every $x \in K$ is a star-center of H.

The first theorem is proved by an iteration procedure similar to that used in proving the contraction mapping principle of Banach.

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THEOREM 1. *Suppose X is a Banach space, H is a closed subset of X, and K a closed subset of H. Suppose further that H is star-shaped with respect to K. If $T : K \to H$ is a contraction mapping, and if $T : \partial_H K \to K$, then T has a (unique) fixed point in K.*

PROOF. To prove this theorem we construct a sequence in H as follows: let $x_0 \in K$ and $x_1' = Tx_0$. If $x_1' \in K$, let $x_1 = x_1'$, but if $x_1' \notin K$, choose

$$x_1 \in \partial_H K \cap \{(1-t)x_0 + tx_1' : 0 \leqq t \leqq 1\},$$

which is nonempty since the set

$$\{(1-t)x_0 + tx_1' : 0 \leqq t \leqq 1\}$$

is connected. In either case $x_1 \in K$. Next we let $x_2' = Tx_1$ and set $x_2 = x_2'$ if $x_2' \in K$; otherwise choose

$$x_2 \in \partial_H K \cap \{(1-t)x_1 + tx_2' : 0 \leqq t \leqq 1\}.$$

Proceeding inductively we obtain the sequence $\{x_n'\}$ in H as follows:
(1) $x_{n+1}' = Tx_n$
(2) $x_{n+1} = x_{n+1}'$ if $x_{n+1}' \in K$, or, if $x_{n+1}' \notin K$, choose
(3) $x_{n+1} \in \partial_H K \cap \{(1-t)x_n + tx_{n+1}' : 0 \leqq t \leqq 1\}$.
To show that $\{x_n'\}$ is a Cauchy sequence, consider

$$\|x_n' - x_{n+1}'\| = \|Tx_{n-1} - Tx_n\| \leqq \lambda\|x_{n-1} - x_n\|,$$

where λ $(0 < \lambda < 1)$ is the Lipschitz constant of T. First we observe that if $x_{n-1}' \neq x_{n-1}$, then $x_{n-1} \in \partial_H K$, so that $x_n' = Tx_{n-1} \in K$, and $x_n' = x_n$. Similarly, $x_{n-2}' = x_{n-2}$. Now if $x_{n-1}' = x_{n-1}$ and $x_n' \neq x_n$, $\|x_{n-1} - x_n\| \leqq \|x_{n-1} - x_n'\| = \|x_{n-1}' - x_n'\|$ by 3; but if $x_{n-1}' \neq x_{n-1}$ and $x_n' = x_n$, then

$$\begin{aligned}\|x_{n-1} - x_n\| &\leqq \|x_{n-1} - x_{n-1}'\| + \|x_{n-1}' - x_n'\| \\ &= \|x_{n-1} - x_{n-1}'\| + \|Tx_{n-2} - Tx_{n-1}\| \\ &\leqq \|x_{n-1} - x_{n-1}'\| + \lambda\|x_{n-2} - x_{n-1}\| \\ &< \|x_{n-2} - x_{n-1}'\| \qquad \text{(by 3)} \\ &= \|x_{n-2}' - x_{n-1}'\|.\end{aligned}$$

So if $n = 2, 3, \cdots$,

$$\|x_n' - x_{n+1}'\| \leqq \lambda\|x_{n-1}' - x_n'\|$$

or

$$\|x_n' - x_{n+1}'\| \leqq \lambda\|x_{n-2}' - x_{n-1}'\|.$$

Letting $M = \max \{\|x_0' - x_1'\|, \|x_1' - x_2'\|\}$, it is easily shown by induction that $\|x_n' - x_{n+1}'\| \leqq \lambda^{(n-1)/2}M$. Consequently,

$$\begin{aligned}
&\|x_n' - x_{n+m}'\| \\
&= \|x_n' - x_{n+1}' + x_{n+1}' - \cdots - x_{n+m-2}' + x_{n+m-2}' \\
&\quad - x_{n+m-1}' + x_{n+m-1}' - x_{n+m}'\| \\
&\leqq \|x_n' - x_{n+1}'\| + \cdots + \|x_{n+m-2}' - x_{n+m-1}'\| \\
&\quad + \|x_{n+m-1}' - x_{n+m}'\| \\
&\leqq (\lambda^{(n-1)/2} + \cdots + \lambda^{(n+m-3)/2} + \lambda^{(n+m-2)/2})M \\
&= \lambda^{(n-1)/2}(1 + \cdots + \lambda^{(m-2)/2} + \lambda^{(m-1)/2})M \\
&\leqq \lambda^{(n-1)/2}/(1 - \lambda^{1/2})M.
\end{aligned}$$

Hence $\{x_n'\}$ is a Cauchy sequence converging to some $x \in H$, since H is closed.

To obtain a convergent sequence in K, recall that whenever $x_n' \neq x_n$, we have that $x_{n+1}' = x_{n+1}$, implying the existence of an infinite subsequence $\{x_{n_i}\}$ in K for which $x_{n_i}' = x_{n_i}$. Since K is closed in H, it follows that $x \in K$.

Finally, for the sequence $\{x'_{n_i+1}\}$

$$\|x'_{n_i+1} - Tx\| = \|Tx_{n_i} - Tx\| \leqq \lambda\|x_{n_i} - x\| \to 0$$

as $i \to \infty$, so that $\{x'_{n_i+1}\} \to Tx$. But $\{x'_{n_i+1}\} \to x$ as $i \to \infty$, implying that $Tx = x$.

Following a procedure similar to Dotson's we obtain

THEOREM 2. *Suppose X is a Banach space, H a closed subset of X, and K a compact star-shaped subset of H. Suppose further that H is star-shaped with respect to K. If $T : K \to H$ is nonexpansive, and if $T : \partial_H K \to H$, then T has a fixed point in K.*

PROOF. Let c be a star-center of K and let $t_n = n/(n + 1)$ for $n = 1, 2, \cdots$. Define a mapping $T_n : K \to H$ by $T_n x = (1 - t_n)c + t_n Tx$ for all $x \in K$. Since $T : K \to H$, T_n is well-defined. If $x \in \partial_H K$, $Tx \in K$, and $(1 - t_n)c + t_n Tx \in K$ for all n; hence T_n maps the relative boundary of K into K. Furthermore, for each n and all $x, y \in K$

$$\|T_n x - T_n y\| = t_n\|Tx - Ty\| \leqq t_n\|x - y\|,$$

so that T_n is a contraction mapping. Since K is closed, T_n has a unique fixed point $x_n \in K$ by Theorem 1. That K is compact implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging to some $x \in K$. Being nonexpansive, T is continuous, so that $\lim_{i\to\infty} Tx_{n_i} = Tx$. Each x_{n_i} is a fixed point of T , and so

$$\lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} T_{n_i} x_{n_i}$$
$$= \lim_{i\to\infty} [(1 - t_{n_i})c + Tx_{ni}] = Tx.$$

Since $\lim_{i\to\infty} x_{n_i} = x$, it follows that $Tx = x$.

If H is convex, it is star-shaped with respect to any subset:

COROLLARY 1. *Let X be a Banach space, H a closed convex subset of X, and K a compact star-shaped subset of H. If $T : K \to H$ is nonexpansive, and if $T : \partial_H K \to K$, then T has a fixed point in K.*

Another consequence of the theorem results by taking $H = X$.

COROLLARY 2. *Suppose X is a Banach space, K a compact star-shaped subset of X, and $T : K \to X$ a nonexpansive mapping such that $T : \partial_X K \to K$. Then T has a fixed point in K.*

Suppose $U = \{T_\lambda\}_{\lambda \in \Lambda}$ is a family of mappings from a set K to a set $H \supseteq K$, such that each member has at least one fixed point in K. Denote by F_λ $(\lambda \in \Lambda)$ the set of fixed points of T_λ, and let $D = \bigcup_{\lambda \in \Lambda} F_\lambda$. We shall call U *commutative on the set of fixed points* provided that $T_\lambda(D) \subseteq K$ for all $\lambda \in \Lambda$ and $T_\lambda T_\alpha x = T_\alpha T_\lambda x$ for all $\lambda, \alpha \in \Lambda$ and all $x \in D$.

Another definition required in the next theorem is the following: a Banach space is called *strictly convex* if for any pair of non-zero elements $x, y \in X$, from $\|x + y\| = \|x\| + \|y\|$ it follows that $x = ay$, $a > 0$.

THEOREM 3. *Suppose X is a strictly convex Banach space, H a closed subset of X, and K a compact and convex subset of H. Suppose further that H is star-shaped with respect to K. Let $U = \{T_\lambda\}_{\lambda \in \Lambda}$ be a family of nonexpansive mappings each member of which satisfies the hypothesis of Theorem 2. If U is commutative on the set of fixed points, then U has a common fixed point in K.*

PROOF. To justify the hypothesis, we observe that each T_λ $(\lambda \in \Lambda)$ has a fixed point in K by Theorem 2. Now if $x \in F_\lambda$ for some $\lambda \in \Lambda$, then $T_\lambda(T_\alpha x) = T_\alpha(T_\lambda x) = T_\alpha x$ for any $\alpha \in \Lambda$; hence $T_\alpha x \in F_\lambda$, so that T_α maps F_λ into itself.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be a sequence from Λ, and assume that $F = F_{\lambda_1} \cap F_{\lambda_2} \cap \cdots \cap F_{\lambda_{n-1}}$ is nonempty. Since T_{λ_n} maps F_{λ_i} $(i = 1, 2, \cdots, n - 1)$ into itself, T_{λ_n} may be viewed as a self-mapping of F.

Next we show that F_λ is convex and closed for each $\lambda \in \Lambda$. If $x, y \in F_\lambda$, there exists a line segment in K joining the two points, since K is convex. For any z on this segment

$$\begin{aligned}\|x - y\| &\leqq \|x - T_\lambda z\| + \|T_\lambda z - y\| \\ &= \|T_\lambda x - T_\lambda z\| + \|T_\lambda z - T_\lambda y\| \\ &\leqq \|x - z\| + \|z - y\| \\ &= \|x - y\|.\end{aligned}$$

Hence $\|x - y\| = \|x - T_\lambda z\| + \|T_\lambda z - y\|$, which implies that $T_\lambda z$ lies on the segment joining x and y by strict convexity. Since T_λ is nonexpansive,

$$\|x - T_\lambda z\| = \|T_\lambda x - T_\lambda z\| \leqq \|x - z\|,$$

so that $T_\lambda z$ lies on the segment joining x and z. Similarly, $T_\lambda z$ lies on the segment joining z and y, which is possible only if $T_\lambda z = z$. That F_λ is closed follows from the continuity of T_λ.

As a result, F is both convex and closed and therefore compact. Since T_{λ_n} is a self-mapping of F, it has a fixed point by the fixed-point theorem of Schauder, as well as by Theorem 2, proving that $F \cap F_{\lambda_n}$ is nonempty. Consequently, the family $\{F_\lambda\}_{\lambda \in \Lambda}$ has the finite intersection property, and since K is compact, $\bigcap_{\lambda \in \Lambda} F_\lambda$ is nonempty.

References

1. W. G. Dotson, Jr., *Fixed point theorems for non-expansive mappings on star-shaped subsets of Banach spaces,* J. London Math. Soc. (2) 4 (1972), 408–410.

2. W. A. Kirk, *Fixed point theorems for nonlinear nonexpansive and generalized contraction mappings,* Pacific J. Math. 39 (1971), 89–94.

Milwaukee School of Engineering, Milwaukee, Wisconsin 53201