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# HAUSDORFF AND H-COMPACT ARE NOT COMPLEMENTARY

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In a talk to the American Mathematical Society in 1973, R. E. Larson [3] introduced the concept of complementary topological properties and exhibited several examples (If $P$ and $Q$ are topological properties such that $P$ is preserved under strengthening and $Q$ is preserved under weakening, then $P$ and $Q$ are *complementary* if a topology is minimal $P$ if and only if it is maximal $Q$). Larson conjectured that Hausdorff and $H$-compact are complementary.

A topological space is *H-compact* if the following conditions hold:

H(i) Every open filter has a cluster point;

H(ii) If an open filter has a unique cluster point, then it converges.

Hausdorff is preserved by strengthening and $H$-compact is preserved by weakening [3].

It is well known that a Hausdorff space is minimal Hausdorff if and only if it is $H$-compact [1], and Larson [3] has shown that minimal Hausdorff spaces are maximal $H$-compact. In this paper we shall give an example of a maximal $H$-compact space which is not Hausdorff and thus show that Larson's conjecture is false. The example we use was first exhibited by H. Tong [7] as an example of a maximal compact space which is not Hausdorff.

Let $X = \{a, b\} \cup N \times N$ where $N$ is the set of natural numbers, $E$ be the set of even natural numbers and $0 = N - E$. The topology $\tau$ on $X$ consists of those sets $U$ such that

(1) $a \notin U$ and $b \notin U$,

(2) $a \in U$ implies there is a family of finite subsets $A_m \subseteq N$ for $m \in E$ such that $\bigcup\{\{m\} \times (N - A_m) : m \in E\} \subseteq U$,
and

(3) $b \in U$ implies there is a family of finite subsets $B_m \subseteq N$ for $m \in O$ such that $\bigcup\{\{m\} \times (N - B_m) : m \in O\} \subseteq U$, and there is a finite subset $B \subseteq N$ such that $(N - B) \times N \subseteq U$.

Since the space $(X, \tau)$ is compact, it is H(i) and H(ii) [5]; and thus $H$-compact.

Since $(X, \tau)$ is maximal compact, every compact set is closed [6]. Thus for $\tau' \supset \tau$, there is $G \in \tau' - \tau$ such that $G' = X - G$ is not com-

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pact. Since $G \notin \tau$, either $a \in G$ or $b \in G$. If $a \in G$ and $G$ does not contain any $\tau$-neighborhood of $a$, then $G'$ contains an infinite number of points $A$ in some even column. $A$ is $\tau$-open and $\tau'$-open and discrete and not H(i). Thus there is a $\tau$-open filter $F$ with $a$ as the unique $\tau$-cluster point but no $\tau'$-cluster points; so $(X, \tau')$ is not H(i).

If $b \in G$ and $G$ does not contain any $\tau$-neighborhood of $b$, then there are an infinite number of columns with at least one point in $G'$ or an odd column which has an infinite number of points in $G'$. This set $B$ of points is not H(i) and, as for $a$, there is a $\tau$-open filter $F'$ with $b$ as the unique $\tau$-cluster point, but no $\tau'$-cluster point. Therefore $(X, \tau)$ is maximal H(i) and thus maximal $H$-compact but not Hausdorff, and Larson's conjecture is false.

Since a Hausdorff space is minimal Hausdorff if and only if it is H(ii) [1], the question to be asked is "Are Hausdorff and H(ii) complementary?" H(ii) is preserved under weakening [3]. The question is answered negatively by using the preceding example and showing that $(X, \tau)$ is maximal H(ii).

If $\tau' \supset \tau$, then $(X, \tau')$ is not H(i), and so there is a $\tau'$-open filter $\mathcal{O}$ with no $\tau'$-cluster points. For $(m, n) \in X$,

$\mathcal{O}^* = \{\{(m, n)\} \cup O \mid O \in \mathcal{O}\}$ is a $\tau'$-open filter with unique cluster point $(m, n)$ and does not converge since $\{(m, n)\} \notin \mathcal{O}^*$. Thus $(X, \tau)$ is maximal H(ii) and not Hausdorff.

The question concerning the existence of a complementary property for Hausdorff is therefore unanswered. It has been shown [4] that Hausdorff spaces which are maximal H(i) (i.e., maximal $H$-closed spaces) are $H$-closed spaces that are r.o. maximal (called submaximal in [2]); however, maximal H(i) spaces which are not Hausdorff have not been characterized.

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