ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 1, Winter 1977

SWEEPING MEASURES FROM THE POLYDISC TO THE TORUS

HOWARD LEWIS PENN[1]

Let $A(\Delta^n)$ be the polydisc algebra consisting of functions continuous on the closed polydisc, Δ^n, and analytic on the interior. The distinguished boundary of the polydisc is the n-dimensional torus, T^n. This is a compact connected Abelian group. Its dual is Z^n, the cross product of n copies of the integers. Let $Z^n{}_+$ be the set of all $\alpha \in Z^n$ with $\alpha_i \in Z_+$ for $1 \leqq i \leqq n$. Let $A(T^n)$ be the algebra of continuous functions on T^n whose Fourier series vanish off $Z^n{}_+$. Theorem 2.2.1 in *Function Theory in Polydiscs* by Walter Rudin [4] shows that $A(\Delta^n)$ and $A(T^n)$ are in one-to-one correspondence. The element in $A(T^n)$ corresponding to $f \in A(\Delta^n)$ is denoted by f^*.

Let μ be a measure on Δ^n. The measure, μ, can be considered as a linear functional on $A(\Delta^n)$ and hence on $A(T^n)$. The linear functional can be extended to a linear functional on $C(T^n)$. This linear functional gives a measure, σ, defined on T^n such that

$$\int_{\Delta^n} f d\mu = \int_{T^n} f^* d\sigma$$

for every $f \in A(\Delta^n)$. The theorem below gives a construction method of finding the measure σ. In one variable a similar construction was given by Rubel and Shields [3].

THEOREM. *Let μ be a measure defined on Δ^n then a measure σ* can be constructed on T^n such that $\int_{\Delta^n} f d\mu = \int_{T^n} f^* d\sigma$ *for all* $f \in A(\Delta^n)$.

PROOF. Let μ be a measure on Δ^n. Then $\mu = \mu_T + \mu_0 + \mu_I$, where μ_T is the restriction of μ to T^n, μ_0 is the restriction of μ to the open polydisc, U^n and μ_I is the restriction of μ to the indistinguished boundary.

Consider first μ_0. Look at

$$\int_{U^n} f(z)\, d\mu_0(z) = \int_{U^n} \int_{T^n} f^*(t) P(r, \theta - t)\, dm(t)\, d\mu_0(r, \theta)$$

Received by the editors on September 23, 1975.

(1) The work in this paper was completed as part of the author's doctoral thesis at the University of Michigan.

where $P(r, \theta - t)$ is the Poisson kernel and r and θ are multivariables such that $z_i = r_i e^{i\theta}$. Now $f^*(t)P(r, \theta - t)$ is an integral function with respect to t. $\int_{T^n} f^*(t)P(r, \theta - t)\, dm(t)$ is an integral function with respect to $\mu(r, \theta)$ since it is just $f(z)$. Therefore we may apply Fubini's Theorem.

This gives us

$$\int_{T^n} f^*(t) \left\{ \int_{U^n} P(r, \theta - t)\, d\mu_0(r, \theta) \right\} dm(t) = \int_{U^n} f(z)\, d\mu_0(z).$$

In particular, if we let $f = 1$ we see that the quantity inside the brackets is an integral function. Let

$$\sigma_0(t) = \int_{U^n} P(r, \theta - t)\, d\mu_0(r, \theta) m(t);$$

then σ_0 is absolutely continuous with respect to m and $\int_{T^n} f\, d\sigma_0 = \int_{U^n} f\, d\mu_0$ for all $f \in A(T^n)$.

We will now replace μ_I by a measure supported on T^n. We break I up into $2^n - 2$ subsets which are determined by which coordinates are equal to 1 in modulus. One such subset, U_j, would look like $\Delta x T x T x \Delta x \Delta x T x \cdots x T$. We will denote an element of U_j by (z_j, e^{it_j}), where z_j is a vector having coordinates which are equal to one in modulus. If we fix t_j, then the points of U_j having these coordinates form the interior of a polydisc of dimension less than n. The boundary of this polydisc, i.e., those points in T^n with the proper coordinates equal to t_j, shall be denoted by T_t. The restriction of μ_I to U_j shall be called μ_j. We shall replace μ_j by a measure supported on T^n which represents the same linear functional on $A(T^n)$ as μ_j. Let $f^* \in A(T^n)$. Let f be the extension of f^* to Δ^n and consider $f(z_j, e^{it_j})$. Look at

$$\int_{U_j} f(z_j, e^{it_j})\, d\mu_j.$$

But $f(z_j, e^{it_j}) = \int_{T_j} P(r, \theta - \alpha) f^*(e^{i\alpha}, e^{it_j})\, dm_t(\alpha)$, where $P(r, \theta - \alpha)$ is the Poisson kernel and m_t is the Lebesgue measure on T_t. Therefore,

$$\int_{U_j} f\, d\mu_j = \int_{U_j} \left\{ \int_{T_t} P(r, \theta - \alpha) f(e^{i\alpha}, e^{it_j})\, dm_t(\alpha) \right\} d\mu_j.$$

The formula on the right is meaningful for all continuous functions on T^n. It defines a linear function of $C(T^n)$, and hence is a measure, which we shall call σ_j. So we have

$$\int_{T^n} g\, d\sigma_j = \int_{U_j} \left\{ \int_{T_t} P(r, \theta - \alpha) g(e^{i\alpha}, e^{it_j})\, dm_t(\alpha) \right\} d\mu_j.$$

Therefore, $\int_{T^n} f^*\, d\sigma_j = \int_{U_j} f\, d\mu_j$ for all $f \in A(\Delta^n)$. Let $\sigma = \mu_T$

$+ \sigma_0 + \Sigma \sigma_i$. It is clear by the construction that $\int_{\Delta^n} f \, d\mu = \int_{T^n} f^* \, d\sigma$.

The following definitions involve the theory of uniform algebras [2].

DEFINITION. A compact set K is said to be a peak interpolation set (PI) with respect to the algebra, $\mathcal{A}$, if given any $f \in C(K)$ there exists $g \in A$ such that $g = f$ on K and $|g(x)| < \|f\|_K$ for all $x \in X \backslash K$, where X is the compact connected Hausdorff space on which the algebra is defined.

DEFINITION. The set K is called a peak set (P) if there exists $f \in \mathcal{A}$ such that $f(x) = 1$ for $x \in K$ and $|f(x)| < 1$ for $x \in X \backslash K$.

DEFINITION. The set K is a null set (N) if $|\mu|(K) = 0$ for every complex measure, μ, on X such that $\int g \, d\mu = 0$ for all $g \in \mathcal{A}$. In other words K has total variation zero for any measure which annihilates $\mathcal{A}$.

Since the algebra $A(\Delta^n)$ can be defined either on Δ^n or on T^n, there are two possible definitions for each of the properties. They will be called respectively PI_Δ, PI_T, P_Δ, P_T, N_Δ, and N_T. Theorem 6.1.2 in Rudin [4] shows the equivalence of 5 properties, including N_T, PI_Δ, and P_Δ. The following example shows that P_Δ and P_T are not equivalent.

EXAMPLE. Let $K = \{(z, w) \in T^2 : z = 1\}$. The set K has the property P_T. Let $f(z) = (1 + z)/2$. Then $f = 1$ on K and $|f(z)| < 1$ for $z \in T^2 \backslash K$. However, K does not have the property P_Δ. Any function having the value one on K and belonging to the 2-disc algebra will also have the value one at any point of the polydisc with z coordinate equal to one. Indeed, the function, f, being a member of the 2-disc algebra, has a restriction to $\{(z, w) : z = 1\}$ which is a member of the disc algebra. This restricted function has boundary values equal to one and, hence, is equal to one throughout the disc.

The proof that PI_T implies N_Δ is accomplished by showing PI_T implies PI_Δ and then using a theorem due to Bishop [1] which shows that for any uniform algebra on a compact Hausdorff space PI and N are equivalent. This clearly also shows that N_T and N_Δ are equivalent. The previous theorem can be used to give a direct proof that N_T and N_Δ are equivalent.

THEOREM. *A compact set, K, has the property N_Δ if and only if it has the property N_T.*

PROOF. The fact that $N_\Delta \Rightarrow N_T$ is trivial. Assume K is an N_T set. We first show that $m(K) = 0$, where m is Lebesgue measure on the torus. Look at the annihilating measure formed in the following manner. Let T' be the subcircle of T^n formed by the collection of points in T^n which have all coordinates equal. Let m' be the Lebesgue measure of T'. Then m' is a representing measure for evaluation at the point $(0, 0, \cdots, 0)$. Consider the measure $dm - dm'$. Since both dm and dm' are representing measures for the same point in the maximal ideal space, their difference is an annihilating measure and therefore has no mass on K. But dm and dm' are mutually singular and therefore neither dm nor dm' can have any mass K.

Let μ be a measure on Δ^n which annihilates $A(\Delta^n)$. Then as in the previous theorem $\mu = \mu_T + \mu_0 + \mu_I$. We will replace μ by $\sigma = \mu_T + \sigma_0 + \sigma_I$. The measure σ, is an annihilating measure on T^n. Therefore $|\sigma|(K) = 0$ because K is a N_T set. Therefore $|\mu_T|(K) = 0$ and hence $|\mu|(K) = 0$. This will prove the theorem.

Recall that $\sigma_0(t) = \int_{U^n} P(r, \theta - t)\, d\mu_0(r, \theta) m(t)$, and that σ_0 is absolutely continuous with respect to the measure, m. Therefore $|\sigma_0|(K) = 0$. Now recall that μ_I was divided into the sum of measures, μ_j, which have disjoint support. Each of these was replaced by a measure σ_j. We wish to show that $|\sigma_j|(K) = 0$. We will do this by showing that $\sigma_j(K') = 0$ for all $K' \subset K$.

$$* \qquad \sigma_j(K') = \int_{Uj} \left\{ \int_{T_t} P(r, \theta - \alpha) \chi_{K'}(e^{i\alpha}, e^{it_j})\, dm_t(\alpha) \right\} d\mu_j,$$

where χ_K' is the characteristic function of the set K'. Fix t_j. I claim that $m_t(K' \cap T_j) = 0$. Consider the measure τ defined by

$$\int f d\tau = \int_{T_t} f(e^{i\alpha}) g(e^{i\alpha})\, dm_t(\alpha)$$

where

$$g(e^{i\alpha}) = (e^{i\alpha_{j1}}) \cdot (e^{i\alpha_{j2}}) \cdot \cdots \cdot (e^{i\alpha_{js}})$$

where the α_{j1}'s are the indices of α. τ is an annihilating measure for $A(T^n)$. But τ and m_t are mutually absolutely continuous. Therefore, the inner integral in * is always zero. Therefore $\sigma_j(K') = 0$, and hence $|\sigma_j|(K) = 0$. Now let

$$\sigma = \mu_T + \sum_{0}^{2^n - 2} \sigma_j.$$

Then σ is an annihilating measure of $A(T^n)$ supported on T^n because

$$\int_{T^n} f\,d\sigma = \int_{T^n} d\mu_T + \sum_0^{2^n-2} \int_{T^n} f\,d\sigma_j$$

$$= \int_{T^n} f\,d\mu_T + \sum_0^{2^n-2} \int_{U_j} f\,d\mu_j = \int_{\Delta^n} f\,d\mu = 0.$$

Therefore, $|\sigma|(K) = 0$ because K is an N_T set. So $|\mu_T|(K) = 0$ since μ_T is the sum of a finite number of measures, each with no mass on K. Therefore K is an N_Δ set.

References

1. Erret Bishop, *A minimal boundary for function algebras,* Pac. J. Math. 9 (1959), 629–642.

2. Theodore W. Gamelin, *Uniform Algebras,* Prentice-Hall, Englewood Cliffs, N.J. (1969).

3. L. A. Rubel and A. L. Shields, *The space of bounded abalytic functions on a region,* Ann. Inst. Fourier, Grenoble **16**, 1(1966), 235–277.

4. Walter Rudin, *Function Theory in Polydiscs,* Benjamin, New York (1969).

UNITED STATES NAVAL ACADEMY, ANNAPOLIS, MARYLAND 21402