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EXISTENCE-UNIQUENESS FOR TWO-POINT BOUNDARY VALUE PROBLEMS FOR nth ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Our main result is an existence and uniqueness theorem for the $(n - 1, 1)$-boundary value problem for $y^{(n)} = f(x, y, \cdots, y^{(n-1)})$ under suitable conditions on f.

1. **Introduction.** Let L be the classical nth order linear differential operator defined by

$$Ly \equiv y^{(n)} + s_{n-1}(x)y^{(n-1)} + \cdots + s_0(x)y$$

where the coefficients are assumed to be continuous. The main result (Theorem 3.1) is a uniqueness and existence theorem for two-point boundary value problems associated with the nonlinear differential equation

$$Ly = f(x, y, \cdots, y^{(n-1)}) \tag{1}$$

where f is continuous and satisfies the one-sided Lipschitz conditions

$$\begin{aligned} f(x, y_0, \cdots, z_i, \cdots, y_{n-1}) - f(x, y_0, \cdots, w_i, \cdots, y_{n-1}) \\ \geqq p_i(x)[z_i - w_i] \end{aligned} \tag{2}$$

for $x \in [a, b]$, $z_i \geqq w_i$, $i = 0, \cdots, n - 1$. The technique of proof is the well known shooting method. To use this method some elementary comparison theorems for disfocal linear equations are proved in § 2. At the end of § 3 a result is given for third order nonlinear differential equations.

In [9], the author proves existence-uniqueness theorems for the $(p, q)-$, $p + q = n$, boundary value problems (BVP's) for the nonlinear differential equation (1) where $f = f(x, y)$. The problem seems to become much more difficult when f also depends on $y', \cdots, y^{(n-1)}$. For results of this type for $n = 3$ see [4] and [5], and for $n = 4$ see [10].

2. **Comparison of Disfocal Linear Equations.** In this section we will be concerned with proving comparison theorems between

$$Ly = p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y \tag{3}$$

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and the comparison equations

$$(4) \qquad Ly = q_{n-1}(x)y^{(n-1)} + \cdots + q_0(x)y,$$

$$(5) \qquad Ly = r_{n-1}(x)y^{(n-1)} + \cdots + r_0(x)y,$$

where the coefficients in these three equations are assumed to be continuous.

We will say that $Ly = 0$ is disfocal on an interval I provided the Cauchy function $u_{n-1}(x, s)$ for $Ly = 0$ satisfies

$$u_{n-1}^{(i)}(x, s) > 0$$

for $i = 0, \cdots, n-1$, $x > s$ with $x, s \in I$. We will say that $Ly = 0$ is left disfocal on I provided

$$u_{n-1}^{(i)}(x, s) \neq 0$$

for $i = 0, \cdots, n-1$, $x < s$ with $s, x \in I$. In [8] it is proved (John Barrett indicated to me that this is folklore) by using results in [7] and [1, p. 506] that if

$$s_i(x) \leqq 0 \qquad \text{on } I$$

for $i = 0, \cdots, n-1$, then $Ly = 0$ is disfocal on I; whereas, if

$$(-1)^{n+i}s_i(x) \leqq 0 \qquad \text{on } I$$

for $i = 0, \cdots, n-1$, then $Ly = 0$ is left disfocal on I. For more examples see Hartman [3].

For the sake of completeness we prove the following simple result concerning differential inequalities.

Lemma 2.1. *Assume* (4) *is disfocal {left disfocal} on I and $u(x)$ is a solution of $Ly \geqq q_{n-1}(x)y^{(n-1)} + \cdots + q_0(x)y$ on I. If $x_0 \in I$ and $y(x)$ is the solution of* (4) *satisfying the same initial conditions as $u(x)$ at x_0, then*

$$u^{(i)}(x) \geqq y^{(i)}(x),\ x \geqq x_0 \text{ in } I$$

$$\{(-1)^{n+i}u^{(i)}(x) \geqq (-1)^{n+i}y^{(i)}(x),\ x \leqq x_0 \text{ in } I\}$$

for $i = 0, \cdots, n-1$.

Proof. We will prove only the first statement of the theorem, the second statement being similar. For $x \geqq x_0$ in I set

$$h(x) = Lu(x) - q_{n-1}(x)u^{(n-1)}(x) - \cdots - q_0(x)u(x) \geqq 0.$$

Let $u_{n-1}(x, s; 4)$ be the Cauchy function for (4) at $x = s$, then, since $y(x)$ satisfies the same initial conditions as $u(x)$ at x_0,

$$u(x) = y(x) + \int_{x_0}^{x} u_{n-1}(x, s; 4)h(s)\, ds.$$

It follows that

$$u^{(i)}(x) \geqq y^{(i)}(x) \qquad x \geqq x_0 \text{ in } I.$$

Let $u_{n-1}(x, x_0; 3)$, $u_{n-1}(x, x_0; 4)$ be the Cauchy functions at x_0 for equations (3) and (4) respectively. Lemma 2.1 gives us the following comparison theorem for (3) and (4).

LEMMA 2.2. *If* (4) *is disfocal {left disfocal} on I and $q_i(x) \leqq p_i(x)$ $\{(-1)^{n+i}q_i(x) \leqq (-1)^{n+i}p_i(x)\}$ on I, $i = 0, \cdots, n-1$, then* (3) *is disfocal {left disfocal} on I and if $x_0 \in I$, then*

$$u_{n-1}^{(i)}(x, x_0; 3) \geqq u_{n-1}^{(i)}(x, x_0; 4) > 0, x > x_0 \textit{ in } I,$$

and

$$\{(-1)^{n+i+1}u_{n-1}^{(i)}(x, x_0; 3)$$
$$\geqq (-1)^{n+i+1}u_{n-1}^{(i)}(x, x_0; 4) > 0, x < x_0 \textit{ in } I\}$$

for $i = 0, \cdots, n-1$.

PROOF. Let $u(x) = u_{n-1}(x, x_0; 3)$ and $y(x) = u_{n-1}(x, x_0; 4)$, then the first inequality holds, by Lemma 2.1, therefore (3) is disfocal. The other claim of the lemma is proved similarly.

REMARK 2.3. Here we will briefly comment on several results for $n = 3$. Since, for $n = 3$, (2, 1)-disconjugacy and (1, 2)-disconjugacy of (3) on I implies (3) is disconjugate on I we have the following result from Lemma 2.2.

PROPOSITION. *If $n = 3$, equation* (4) *is disfocal on I, equation* (5) *is left disfocal on I with $q_0(x) \leqq p_0(x) \leqq r_0(x)$, $q_2(x) \leqq p_2(x) \leqq r_2(x)$, $p_1(x) \geqq r_1(x)$, and $p_1(x) \geqq q_1(x)$ on I, then* (1) *is disconjugate on I.*

Comparing this result to Proposition 4.1 in Hartman [3], the author noticed the following generalization whose proof is a straightforward application of Theorem 1.1_3 [2].

PROPOSITION. *Assume $n = 3$ and there are solutions y_1 and y_2 of* (5) *and* (4) *respectively such that $y_1(x) > 0$, $y_1'(x) < 0$, $y_1''(x) \geqq 0$, $y_2(x) > 0$, $y_2'(x) > 0$, $y_2''(x) \geqq 0$ on I. If $q_0(x) \leqq p_0(x) \leqq r_0(x)$, $q_2(x) \leqq p_2(x) \leqq r_2(x)$, $p_1(x) \geqq r_1(x)$, and $p_1(x) \geqq q_1(x)$ on I, then* (1) *is disconjugate on I.*

If $I = [a, b]$ the first sentence of this last proposition can be replaced by $u_2^{(i)}(x, a; 5) > 0$ on $(a, b]$, $(-1)^i u_2^{(i)}(x, b; 4) > 0$ on $[a, b)$, $i = 0, 1, 2$. Furthermore, if in this last proposition $q_2(x) = r_2(x)$ on I, the assumptions $y_1''(x) \geqq 0$, $y_2''(x) \geqq 0$ on I can be deleted. Finally the assumptions $y_1'(x) < 0$, $y_2'(x) > 0$ on I can be replaced by $y_1'(x) \leqq 0, y_2'(x) \leqq 0$, $[y_1'(x)]^2 + [y_2'(x)]^2 > 0$ on I.

3. **Main Results on Existence-Uniqueness.** In this section we are concerned with the nonlinear differential equation (1) where we assume throughout that f is continuous on $[a, b] \times R^n$ and solutions of initial value problems for (1) extend to $[a, b]$. See [**6**] for a Nagumo condition for (1). We now state and prove our main result.

THEOREM 3.1. *Assume* (2) *holds and* (3) *is disfocal on* $[a, b]$, *then the* $(n-1, 1)$-BVP (1)

$$y^{(i)}(a) = A_i \qquad i = 0, \cdots, n-2$$

$$y(b) = B_1$$

has a solution. If, in addition, solutions of initial value problems for (1) *are unique, then the above* $(n-1, 1)$-BVP *has a unique solution.*

PROOF. For $k = 1, 2, \cdots$, define the integral means $f_k \equiv f_k(x, y_0, \cdots, y_{n-1})$ on $[a, b] \times R^n$ by

$$f_k = \frac{k^n}{2^n} \int_{y_{n-1}-1/k}^{y_{n-1}+1/k} \cdots \int_{y_0-1/k}^{y_0+1/k} f(x, s_0, \cdots, s_{n-1})\, ds_0 \cdots ds_{n-1}.$$

Note that $f_k \to f$ uniformly on compact subsets of $[a, b] \times R^n$, the functions f_k, $\partial f_k/\partial y_i$, $i = 0, \cdots, n-1$, $k = 1, 2, \cdots$ are continuous on $[a, b] \times R^n$ and

$$\frac{\partial f_k}{\partial y_i} \geqq p_i(x)$$

on $[a, b] \times R^n, i = 0, \cdots, n-1, k = 1, 2, \cdots$.
Let $y_k(x, m)$ be the solution of the IVP

$$Ly = f_k(x, y, \cdots, y^{(n-1)})$$

$$y^{(i)}(a) = A_i \qquad i = 0, \cdots, n-2 \tag{6}$$

$$y^{(n-1)}(a) = m. \tag{7}$$

Assume $m > 0$, then by Theorem V-3.1 [1], there is an $\overline{m} \in (0, m)$ such that

$$y_k(b, m) - y_k(b, 0) = m \frac{\partial y_k}{\partial m}(b, \bar{m})$$
$$= m\, z_k(b, \bar{m})$$

where $z_k(x, \bar{m})$ is the solution of the IVP

$$Lz = \sum_{i=0}^{n-1} \frac{\partial f_k}{\partial y_i}(x, y_k(x, \bar{m}), \cdots, y_k^{(n-1)}(x, \bar{m}))z^{(i)}$$
$$z^{(i)}(a) = 0 \qquad i = 0, \cdots, n-2$$
$$z^{(n-1)}(a) = 1.$$

By use of Lemma 2.2 we get that

$$y_k(b, m) - y_k(b, 0) \geqq m\, u_{n-1}(b, a; 3)$$

where $u_{n-1}(x, a; 3)$ is the Cauchy function at a for (3). It follows by successive use of Kamke's convergence theorem [1, Theorem II-3.2] that for each $m > 0$,

$$y(b, m) - y(b, 0) \geqq m\, u_{n-1}(b, a; 3),$$

where $y(x, m)$ is a fixed solution of (1), (6), (7) and $y(x, 0)$ is a solution of (1), (6), (7) with $m = 0$ (at this stage of the proof we are not assuming uniqueness of IVP's). Since $u_{n-1}(b, a; 3) > 0$,

$$\lim_{m \to \infty} y(b, m) = \infty.$$

Similarly for each $m < 0$ there is a solution $y(x, m)$ of (1), (6), (7) such that

$$\lim_{m \to -\infty} y(b, m) = -\infty.$$

Since $\{y(b, m) : m \in R$ and $y(x, m)$ is now any solution of (1), (6), (7)$\}$ is connected (see, e.g., [10]), we have proved the first statement of the theorem.

Now assume that solutions of initial value problems for (1) are unique. Assume some $(n-1, 1)$-BVP has distinct solutions, then without loss of generality assume there are points $m_2 > m_1$ (since solutions of IVP's are unique) such that

$$y(b, m_2) = y(b, m_1). \tag{8}$$

It follows from (8) that there is an $x_1 > a$ such that

$$y^{(i)}(x, m_2) > y^{(i)}(x, m_1)$$

for $a < x < x_1$, $i = 0, \cdots, n-1$, and $y^{(n-1)}(x_1, m_2) = y^{(n-1)}(x_1, m_1)$.
$= y^{(n-1)}(x_1, m_1)$.

Set

$$w(x) = \frac{y(x, m_2) - y(x, m_1)}{m_2 - m_1}.$$

By the one sided Lipschitz condition on f is is easy to see that

$$Lw(x) \geqq p_{n-1}(x) w^{(n-1)}(x) + \cdots + p_0(x) w(x)$$

on $[a, b]$.

Since (3) is disfocal on $[a, b]$ it follows from Lemma 2.1 that

$$w^{(n-1)}(x) = \frac{y^{(n-1)}(x, m_2) - y^{(n-1)}(x, m_1)}{m_2 - m_1}$$
$$\geqq u_{n-1}^{(n-1)}(x, a; 3) > 0 \qquad \text{on } [a, x_1).$$

This implies that $y^{(n-1)}(x_1, m_2) > y^{(n-1)}(x_1, m_1)$ which is a contradiction.

The proof of the following theorem is similar to the proof of Theorem 3.1 and so will be omitted.

THEOREM 3.2. *Assume*

$$\begin{aligned}(-1)^{n+i}[f(x, y_0, \cdots, z_i, \cdots, y_{n-1}) \\ - f(x, y_0, \cdots, w_i, \cdots, y_{n-1})] \\ \geqq (-1)^{n+i} p_i(x)[z_i - w_i]\end{aligned}$$

where $x \in [a, b]$, $z_i \geqq w_i$, $i = 0, \cdots, n-1$ *and* (3) *is left disfocal on* $[a, b]$. *Then the* $(1, n-1)$-BVP

$$y(a) = A_1$$
$$y^{(i)}(b) = B_i \qquad i = 0, \cdots, n-2$$

has a solution. If, in addition, solutions of initial value problems for (1) *are unique, then the above* $(1, n-1)$-BVP *has a unique solution.*

We conclude this paper by stating a result for third order differential equations. For an interesting related result see Theorem 3 [5].

THEOREM 3.3. *Assume* $n = 3$, (4) *is disfocal on* $[a, b]$, (5) *is left disfocal on* $[a, b]$ *and*

$$q_0(x)[z_0 - w_0] \leqq f(x, z_0, y_1, y_2) - f(x, w_0, y_1, y_2)$$
$$\leqq r_0(x)[z_0 - w_0],$$
$$q_1(x)[z_1 - w_1] \leqq f(x, y_0, z_1, y_2) - f(x, y_0, w_1, y_2),$$
$$r_1(x)[z_1 - w_1] \leqq f(x, y_0, z_1, y_2) - f(x, y_0, w_1, y_2),$$
$$q_2(x)[z_2 - w_2] \leqq f(x, y_0, y_1, z_2) - f(x, y_0, y_1, w_2)$$
$$\leqq r_2(x)[z_2 - w_2],$$

where $x \in [a, b]$, $y_i \in R$, $z_i \geqq w_i$, $i = 0, 1, 2$. *Then* (2, 1)- *and* (1, 2)-BVP's *for* (1) *have solutions. If, in addition, solutions of* IVP's *for* (1) *are unique then the* (2, 1)-, (1, 2)-, *and* (1, 1, 1)-BVP's *have unique solutions.*

PROOF. This follows immediately from Theorem 3.1, Theorem 3.2, and Theorem 2 [5].

REFERENCES

1. P. Hartman, *Ordinary Differential Equations,* John Wiley and Sons, Inc., New York, 1964.

2. P. Hartman, *Principal solutions of disconjugate n-th order linear differential equations,* Amer. J. Math. 41 (1969), 306-362.

3. ——, *On solutions of disconjugate differential equations,* J. Math. Anal. Appl. 46 (1974), 338-351.

4. J. Innes, *Existence and uniqueness of solutions of boundary value problems for a third order differential equation,* Ph.D. dissertation, University of Nebraska, 1974.

5. L. Jackson, *Existence and uniqueness of solutions of boundary value problems for third order differential equations,* J. Differential Equations 13 (1973), 432-437.

6. ——, *A Nagumo condition for ordinary differential equations,* Proc. Amer. Math. Soc. 57 (1976), 93-96.

7. J. Mikusinski, *Sur un probleme d'interpolation pour les integrals des equations differentielle lineaires,* Annales de la Society Polanaise de Mathematique 19 (1946), 165-205.

8. A. Peterson, *Distribution of zeros of solutions of linear differential equations of order four,* Ph.D. dissertation, University of Tennessee, 1968.

9. ——, *Comparison theorems and existence theorems for ordinary differential equations,* J. Math. Anal. Appl. 55 (1976), 773-784.

10. D. Peterson, *Uniqueness, existence, and comparison theorems for ordinary differential equations,* Ph.D. dissertation, University of Nebraska, 1973.

11. T. Sherman, *Properties of solutions of Nth order linear differential equations,* Pacific J. Math. 15 (1965), 1045-1060.

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